INDUZIONE

Titolo nota 21/02/2006

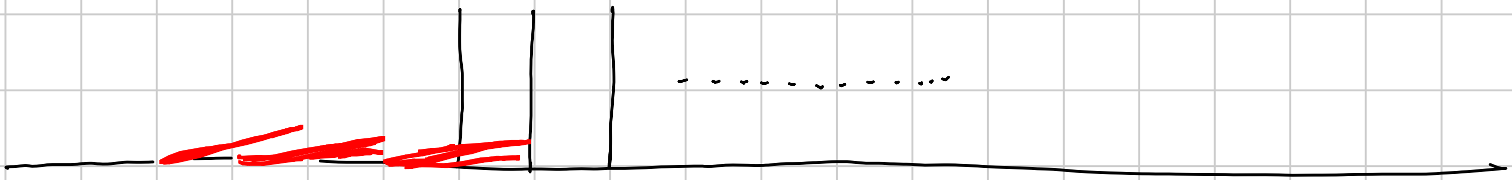
$\mathbb{N}$ insieme dei numeri naturali $= \{1, 2, 3, \dots\}$

P_n è un'affermazione che dipende da $n \in \mathbb{N}$

Sappiamo che P_1 è vera

Sappiamo che SE P_n è vera, allora anche P_{n+1} è vera, per n generico $\in \mathbb{N}$

$\Rightarrow$ P_n è vera per $\overset{\forall}{\text{ogni}}$ $n \in \mathbb{N}$



$S_n = 1 + 2 + 3 + \cdots \quad n$

$S_1 = 1$ Scelgo $n = 1$

$S_2 = 1 + 2 = 3$ $\frac{1 \cdot (1+1)}{2} = 1$

$S_3 = 1 + 2 + 3 = 6$

$\vdots$

$\underbrace{1 + 2 + \cdots n}_{S_n} + (n+1) =$

$\boxed{S_n = \frac{n(n+1)}{2}}$

$= \frac{n(n+1)}{2} + (n+1) = \frac{n(n+1) + 2(n+1)}{2} = \frac{(n+1)(n+2)}{2} = \frac{(n+1)[(n+1)+1]}{2}$

$= S_{n+1}$

$1 + 2 + 3 + \cdots + 99 + 100$

$100 + 99 + 98 + \cdots + 2 + 1$

$101 \quad 101 \quad 101 \qquad 101 \quad 101$

$2S_{100} = 100 \cdot 101$

$S_{100} = \frac{100 \cdot 101}{2}$

$$a_i = \alpha + (i-1)\cdot r$$

$$\sum_{i=1}^{n} a_i = \frac{n\,(2\alpha + (n-1)\cdot r)}{2}$$

$$S_n^{(2)} = 1^2 + 2^2 + 3^2 + 4^2 + \dots \quad n^2$$

$S_n^{(k)}$ è un polinomio in n di grado $k+1$

$$\to an^3 + bn^2 + cn + d$$

$$S_n^{(2)} = \frac{n\,(2n+1)(n+1)}{6}$$

$$S_1^{(2)} = \frac{1\cdot(2\cdot 1+1)\cdot(1+1)}{6} = \frac{1\cdot 3\cdot 2}{6} = 1$$

$$\underbrace{1^2 + \dots + n^2} + (n+1)^2 = \frac{n\,(2n+1)\,(n+1) + 6\,(n+1)^2}{6} = \frac{(n+1)\left[n(2n+1) + 6(n+1)\right]}{6} =$$

$$= \frac{(n+1)\left[2n^2+7n+6\right]}{6} = \frac{(n+1)(2n+3)(n+2)}{6} = \frac{(n+1)(2(n+1)+1)((n+1)+1)}{6}$$

$(2n+3)(n+2)$

$$S_n^{(3)} = \left(\frac{n(n+1)}{2}\right)^2$$

$$a_i = \alpha r^{i-1} \qquad a_1 = \alpha \quad a_2 = r\alpha \quad a_3 = r^2\alpha \quad \cdots\cdots$$

$$S_n^G = \sum_{i=1}^{n} a_i = \alpha \frac{r^n - 1}{r-1}$$

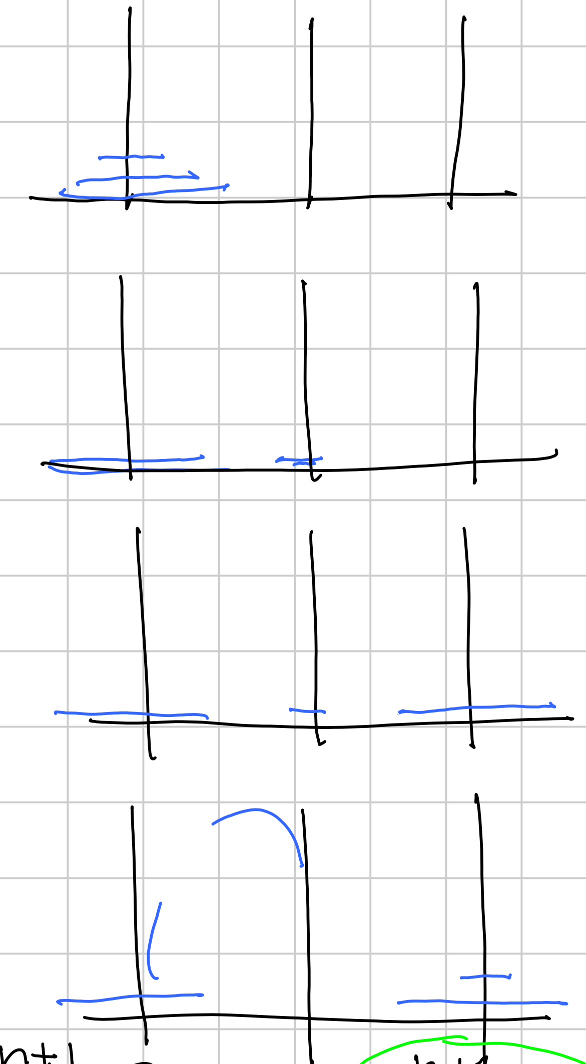
$$S_1^G = \alpha \cdot \frac{r^1 - 1}{r-1} = \alpha$$

$$S_{n+1}^G = \alpha \frac{r^n - 1}{r-1} + \alpha r^n =$$

$$= \alpha \frac{r^n - 1 + r^n(r-1)}{r-1} = \frac{r^{n+1} \cancel{+r^n} \cancel{-r^n} - 1}{r-1} =$$

$$= \alpha \frac{r^{n+1} - 1}{r-1}$$

TORRE DI HANOI



$$\begin{cases} N_1 = 1 \\ N_n = 2N_{n-1} + 1 \end{cases}$$

$$N_2 = 2N_1 + 1 = 2 \cdot 1 + 1 = 3$$

$$N_3 = 2 \cdot 3 + 1 = 7$$

$$N_4 = 2 \cdot 7 + 1 = 15$$

$$N_n = 2^n - 1$$

$$N_{n+1} = 2(2^n - 1) + 1 = 2^{n+1} - 2 + 1 = 2^{n+1} - 1 =$$

$$f(x) = \frac{x}{x+1} \qquad f(f(x)) = \frac{\frac{x}{x+1}}{\frac{x}{x+1}+1} = \frac{\frac{x}{x+1}}{\frac{x+x+1}{x+1}} =$$

$$= \frac{x}{\cancel{x+1}} \cdot \frac{\cancel{x+1}}{2x+1} = \frac{x}{2x+1}$$

$$f(\underbrace{f(f(x))}) = \frac{\frac{x}{2x+1}}{\frac{x}{2x+1}+1} = \frac{x}{\cancel{2x+1}} \cdot \frac{\cancel{2x+1}}{3x+1} = \frac{x}{3x+1}$$

$$\underbrace{f(f(f(\cdots f}_{n \text{ volte}}(x))\cdots)) \doteq f^{(n)}(x) = \frac{x}{nx+1}$$

$$f^{(n+1)}(x) = \frac{\frac{x}{nx+1}}{\frac{x}{nx+1}+1} = \frac{x}{\cancel{nx+1}} \cdot \frac{\cancel{nx+1}}{nx+x+1} = \frac{x}{(n+1)x+1}$$

DISCESA INFINITA

$a_1, a_2, a_3 \ldots a_n \qquad a_i \in \mathbb{N}$

$a_i \leq a_j \quad \forall\, i \geq j$ $\Longrightarrow$ a_n è costante da un certo indice in poi

(*) $x^3 + 2y^3 = 4z^3$ soluzioni intere non negative

PARI PARI PARI

$x=0 \quad y=0 \quad z=0$

$x = 2x'$ ($\uparrow \in \mathbb{N}$) $\qquad 8x'^3 + 2y^3 = 4z^3 \qquad 4x'^3 + y^3 = 2z^3$

PARI

$y = 2y' \qquad 4x'^3 + 8y'^3 = 2z^3 \qquad 2x'^3 + 4y'^3 = z^3$

PARI

$z = 2z' \qquad 2x'^3 + 4y'^3 = 8z'^3 \qquad x'^3 + 2y'^3 = 4z'^3$

Se (x, y, z) è soluzione di (*), anche $\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right)$ è soluzione

$x = y = 0 = z$

DISEGUAGLIANZA DI BERNOULLI

$$(1+x)^n \geqslant 1+nx \qquad \forall x > -1 \leftarrow$$
$$\forall n \in \mathbb{N}_0$$

$$(1+x)^0 \qquad 1+0\cdot x$$

$$1 \geqslant 1 \quad \text{OK}$$

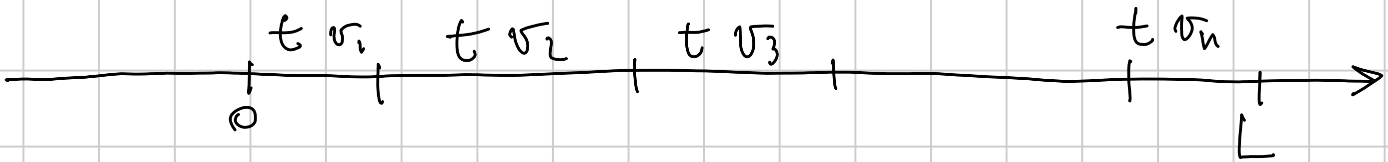
$$(1+x)^{n+1} = (1+x)^n(1+x) \geqslant (1+nx)(1+x) = 1+nx+x+nx^2 =$$

$$= 1+(n+1)x+\underbrace{nx^2}_{\geqslant 0} \geqslant 1+(n+1)x$$

$$(1+x)^{n+1} \geqslant 1+(n+1)x$$

DISUGUAGLIANZE FRA LE MEDIE

$$AM = \frac{a_1+a_2+\dots+a_n}{n} \qquad a_i>0$$

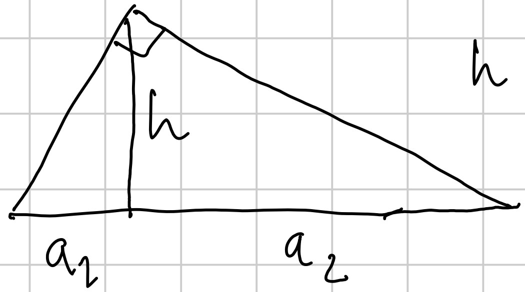


$$L = v_1\cdot t + v_2\cdot t + \dots + v_n t = t\,(v_1+v_2+\dots+v_n)$$

$$T = \underbrace{t+t+t+t\dots+t}_{n \text{ volte}} \quad nt$$

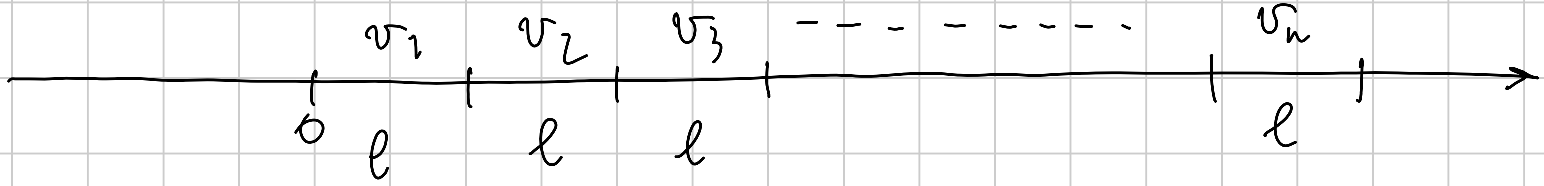
$$\bar{v} = \frac{L}{T} = \frac{v_1+\dots+v_n}{n}$$

$$GM = \sqrt[n]{a_1 \cdot a_2 \cdots a_n}$$



$$h = \sqrt{a_1 \cdot a_2}$$

$$HM = \frac{1}{\frac{\frac{1}{a_1} + \frac{1}{a_2} + \cdot \frac{1}{a_n}}{n}} = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdot \frac{1}{a_n}}$$



$$L = n l$$

$$T = \underbrace{\frac{l}{v_2}}_{t_1} + \underbrace{\frac{l}{v_2}}_{t_2} + \cdots + \underbrace{\frac{l}{v_n}}_{t_n} = l\left(\frac{1}{v_2} + \frac{1}{v_2} + \cdots + \frac{1}{v_n}\right)$$

$$\bar{v} = \frac{L}{T} = \frac{n}{\frac{1}{v_1} + \frac{1}{v_2} + \cdots \frac{1}{v_n}}$$

$$QM = \sqrt{\frac{a_1^2 + a_2^2 + \dots + a_n^2}{n}}$$

$$\boxed{HM \leq GM \leq AM \leq QM}$$

$n = 2$

$$\frac{a_1 + a_2}{2} \leq \sqrt{\frac{a_1^2 + a_2^2}{2}} \qquad \frac{(a_1 + a_2)^2}{4} \leq \frac{a_1^2 + a_2^2}{2}$$

$$(a_1 + a_2)^2 \leq 2(a_1^2 + a_2^2) \qquad a_1^2 + 2a_1a_2 + a_2^2 \leq 2a_1^2 + 2a_2^2$$

$$a_1^2 - 2a_1a_2 + a_2^2 \geq 0 \qquad (a_1 - a_2)^2 \geq 0 \text{ vero}$$

$$\sqrt{a_1a_2} \leq \frac{a_1 + a_2}{2} \qquad a_1a_2 \leq \frac{(a_1 + a_2)^2}{4}$$

$$4a_1a_2 \leq a_1^2 + 2a_1a_2 + a_2^2 \qquad (a_1 - a_2)^2 \geq 0 \text{ vera}$$

$$\frac{2}{\frac{1}{a_1}+\frac{1}{a_2}} = \frac{2a_1a_2}{a_1+a_2} \leq \sqrt{a_1a_2}$$

$$2a_1a_2 \leq (a_1+a_2)\sqrt{a_1a_2}$$

$$2\sqrt{a_1a_2} \leq (a_1+a_2)$$

$$GM \leq AM$$

$$\sqrt{a_1a_2} \leq \frac{a_1+a_2}{2}$$

Quando vale e' = ?

$$a_1 = a_2 = a_3 \ldots a_n$$

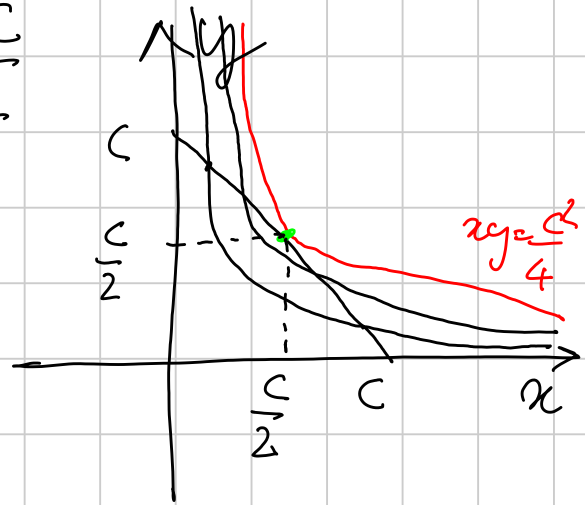
$$\Updownarrow$$

$$HM = GM = AM = QM$$

Massimizzare il prodotto fra due numeri che hanno somma fissata

$x \quad y$ se $x+y=C$, allora $xy \leq \frac{C^2}{4}$

$\sqrt{xy} = \sqrt{\max}$ $\qquad x=y=\frac{C}{2}$



$$\sqrt{xy} \leq \frac{x+y}{2}$$

$$xy \leq \left(\frac{x+y}{2}\right)^2 = \frac{C^2}{4}$$

Massimizzare x^2y con $x+y=1$

$x \cdot x \cdot y$ $\qquad \frac{x}{2}+\frac{x}{2}+y=1$

$$x^2y = 4 \cdot \frac{x}{2} \cdot \frac{x}{2} \cdot y$$

$\frac{x}{2}=\frac{1}{3}$ $\qquad x=\frac{2}{3}$

$$\sqrt[3]{\frac{x}{2} \cdot \frac{x}{2} \cdot y} \leq \frac{\frac{x}{2}+\frac{x}{2}+y}{3}$$

$y=\frac{1}{3}$

$$\frac{x^2y}{4} \leq \frac{1}{27} \qquad x^2y \leq \frac{4}{27}$$

$$\left(\frac{2}{3}\right)^2 \cdot \frac{1}{3} = \frac{4}{27}$$

RADICI DI POLINOMI

Un polinomio di grado n in una variabile a coefficienti complessi ha esattamente n radici complesse se vengono contate con la loro molteplicità

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = 0$$

$$\lambda_1 \dots \lambda_n \in \mathbb{C} \quad n \text{ soluzioni}$$

Voglio generare un polinomio che ha per soluzioni

$$\frac{1}{\lambda_1} \quad \frac{1}{\lambda_2} \quad \dots \quad \frac{1}{\lambda_n} \qquad (\lambda_i \neq 0)$$

$$\frac{a_n \lambda_1^n}{\lambda_1^n} + \frac{a_{n-1} \lambda_1^{n-1}}{\lambda_1^n} + \dots + \frac{a_1 \lambda_1}{\lambda_1^n} + \frac{a_0}{\lambda_1^n} = 0$$

$$a_0\left(\frac{1}{\lambda_1}\right)^n + a_1\left(\frac{1}{\lambda_1}\right)^{n-1} + \cdots + a_{n-1}\frac{1}{\lambda_1} + a_n = 0$$

$x^2 - 5x + 6 = 0$ $\qquad \lambda_1 = 2 \qquad \lambda_2 = 3$

$6x^2 - 5x + 1$ $\qquad \lambda_1 = \frac{1}{2} \qquad \lambda_2 = \frac{1}{3}$

$$a_2x^2 + a_1x + a_0 = a_2(x-\lambda_1)(x-\lambda_2)$$

$$x^2 - Sx + P = 0$$

$$\lambda_1+\lambda_2 = S = -\frac{a_1}{a_2} \qquad \Big| \qquad \lambda_1\lambda_2 = P = \frac{a_0}{a_2}$$

$$a_0 = a_2 \cdot \lambda_1\lambda_2$$

$$a_1 = -a_2(\lambda_1+\lambda_2)$$

$$a_3x^3 + a_2x^2 + a_1x + a_0 = 0$$

$$a_3(x-\lambda_1)(x-\lambda_2)(x-\lambda_3)$$

$$P = \lambda_1\lambda_2\lambda_3 = -\frac{a_0}{a_3}$$

$$Q = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = \frac{a_1}{a_3}$$

$$S = \lambda_1+\lambda_2+\lambda_3 = -\frac{a_2}{a_3}$$

$$a_3 = 1$$

$$x^3 - Sx^2 + Qx - P = 0$$

Per il grado n $\quad \lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdots \lambda_n = \frac{a_0}{a_n}(-1)^n$

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = -\frac{a_{n-1}}{a_n}$$

$$\sum_{i<j} \lambda_i \lambda_j = \frac{a_{n-2}}{a_n}$$

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \cdots \lambda_n^2 = ?$$

$$(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \cdots \lambda_n^2) = (\lambda_1 + \lambda_2 + \cdots \lambda_n)^2 - 2\sum_{i<j} \lambda_i \lambda_j$$

$$= \left(\frac{a_{n-1}}{a_n}\right)^2 - 2\frac{a_{n-2}}{a_n}$$

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \cdots + \frac{1}{\lambda_n} = -\frac{a_1}{a_0} \qquad \left(\frac{1}{\lambda_1}\right)^2 + \left(\frac{1}{\lambda_2}\right)^2 + \cdots + \left(\frac{1}{\lambda_n}\right)^2 = \left(\frac{a_1}{a_0}\right)^2 - \frac{2a_2}{a_0}$$

$$1 + x + x^2 + \cdots\cdots + x^{2006}$$

può avere radici tutte reali ?

$$0 < \left(\frac{1}{\lambda_1}\right)^2 + \left(\frac{1}{\lambda_2}\right)^2 + \cdots + \left(\frac{1}{\lambda_{2006}}\right)^2 = \left(\frac{a_1}{a_0}\right)^2 - 2\frac{a_2}{a_0} = 1 - 2 < 0$$