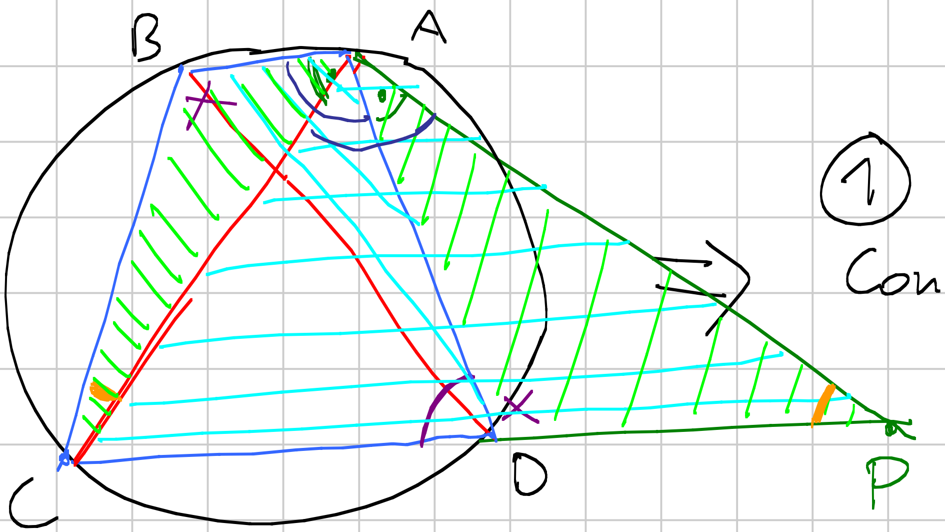
GEOMETRIA II

Titolo nota 24/02/2007

TEOREMA DI TOLOMEO

Dato un quadrilatero convesso esso è inscrivibile in una circonferenza (ciclico) $\Leftrightarrow$ $AD \cdot BC =$

$$= AB \cdot CD + BC \cdot AD$$



1 Considero $B\hat{A}C$

ABBIAMO COSTRUITO

$$P\hat{A}D = B\hat{A}C$$

$A\hat{D}P = A\hat{B}C$ perchè sono entrambi supplementari di $A\hat{D}C$

$\overset{\triangle}{ABC}$ è simile a $\overset{\triangle}{ADP}$ $\qquad \frac{AC}{AP} = \frac{AB}{AD} = \frac{BC}{DP}$

$$\boxed{DP = \frac{AD \cdot BC}{AB}}$$

(2) $\quad P\hat{A}C = D\hat{A}B \;, \quad \frac{AC}{AP} = \frac{AB}{AD}$

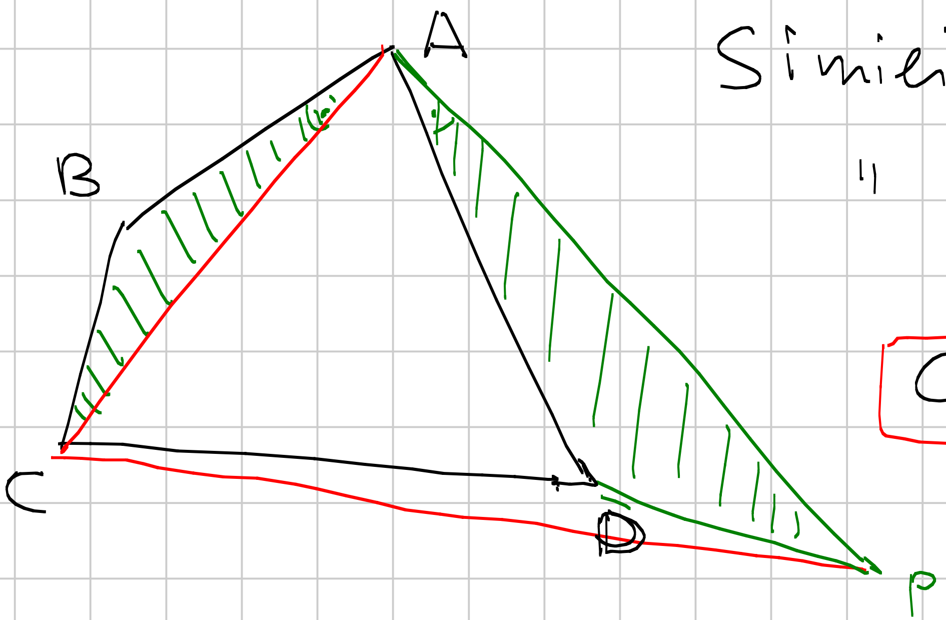
$\overset{\triangle}{ABD}$ è simile ad $\overset{\triangle}{ACP}$ $\rightarrow \quad \frac{BD}{CP} = \frac{AB}{AC} \qquad \boxed{CP = \frac{BD \cdot AC}{AB}}$

(3) $\quad \boxed{CP = CD + DP}$

$$\frac{BD \cdot AC}{AB} = CD + \frac{AD \cdot BC}{AB}$$

$\Leftarrow$ Diseguaglianza triangolare

$$\boxed{BD \cdot AC = CD \cdot AB + AD \cdot BC}$$



Similitudine fra $\overset{\triangle}{ABC}$ e $\overset{\triangle}{ADP}$

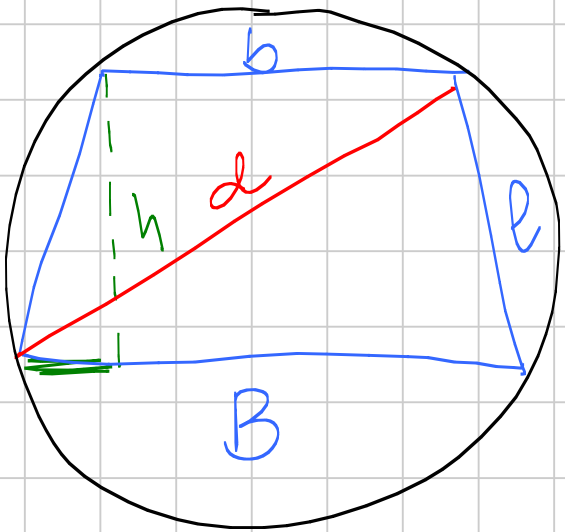
" fra $\overset{\triangle}{ABD}$ e $\overset{\triangle}{ACP}$

$$\boxed{CP \le CD + DP}$$

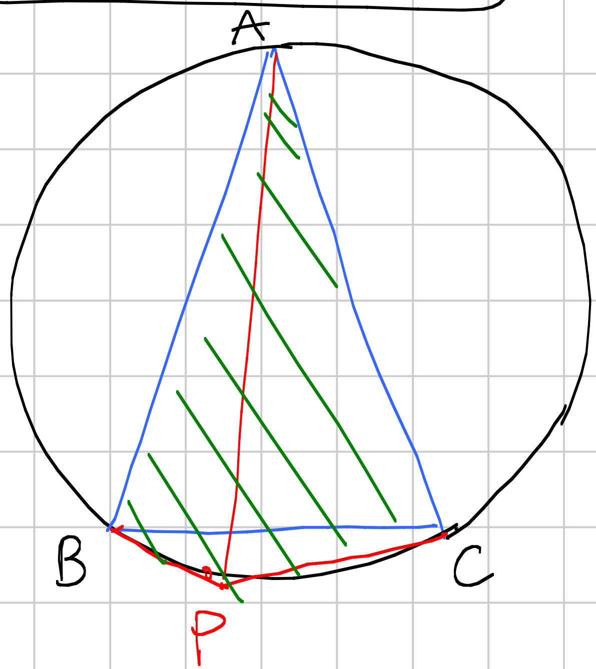
$$\frac{BD \cdot AC}{AB} \le CD + \frac{AD \cdot BC}{AB}$$

$$BD \cdot AC \le AB \cdot CD + AD \cdot BC$$

Se $BD \cdot AC = AB \cdot CD + AD \cdot BC$, allora C, D e P devono essere allineati. Ma allora $C\hat{D}P = 180^\circ$ quindi, poiché $A\hat{D}P = A\hat{B}C$, anche la somma fra $A\hat{D}C$ e $A\hat{B}C$ è 180°, e quindi ABCD è ciclico



$$d^2 = b \cdot B + \ell^2$$



ABC isoscele

$AB = AC$

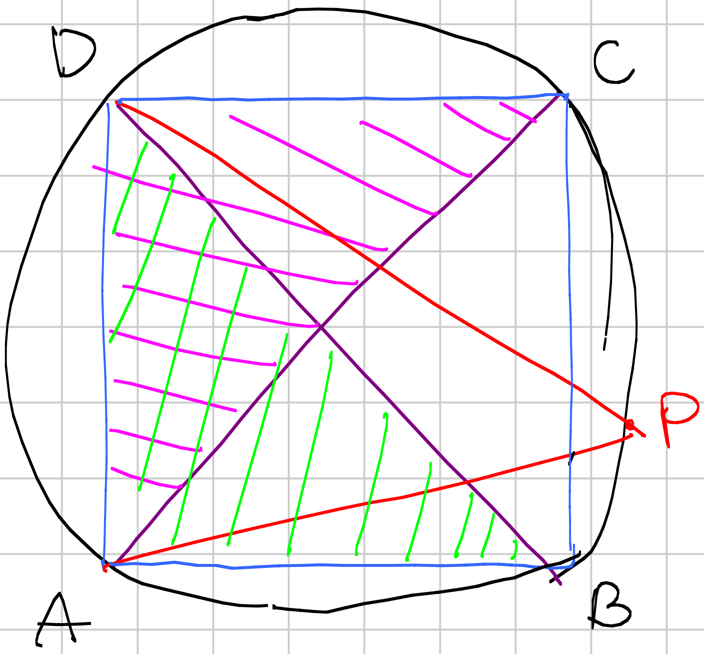
$P \in \overset{\frown}{BC}$

$$\frac{PA}{PB+PC} = \frac{AC}{BC}$$

$$PA \cdot BC = BP \cdot AC + PC \cdot AC$$

$$PA \cdot BC = AC\,(BP + PC)$$

$$\frac{PA}{BP+PC} = \frac{AC}{BC}$$

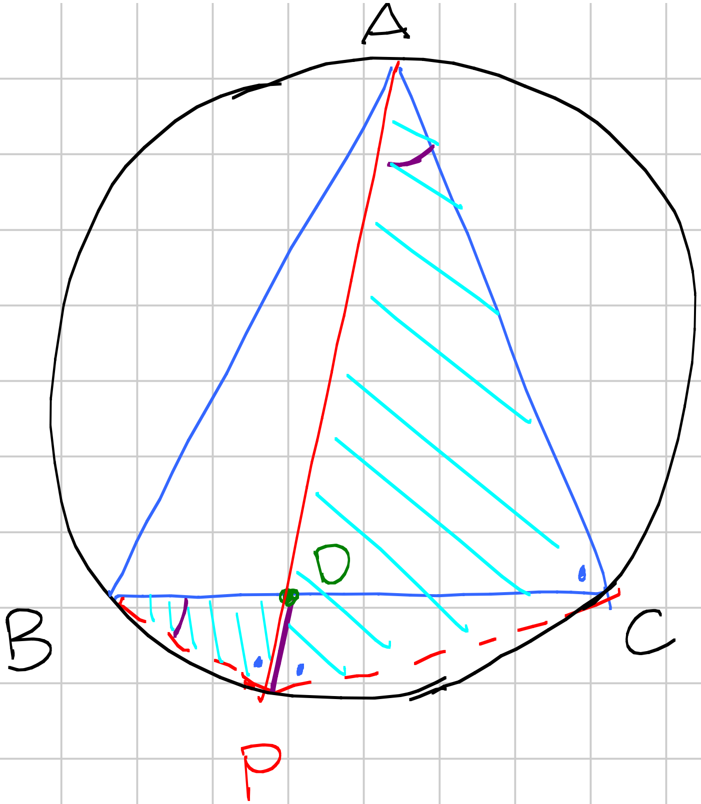


$$\frac{PA+PC}{PB+PD} = \frac{PD}{PA}$$ devo dim.

$\overset{\triangle}{ABD}$ è isoscele $\quad \dfrac{PA}{PB+PD} = \boxed{\dfrac{AD}{DB}}$

$\overset{\triangle}{ADC}$ è isoscele $\quad \dfrac{PD}{PA+PC} = \boxed{\dfrac{DC}{AC}}$

$$\frac{PA}{PB+PD} = \frac{PD}{PA+PC} \rightarrow \frac{PA+PC}{PB+PD} = \frac{PD}{PA}$$



$$\frac{1}{PD} = \frac{1}{PB} + \frac{1}{PC}$$ devo dim.

$\overset{\triangle}{APC}$ è simile a $\overset{\triangle}{BPD}$

$$\frac{PA}{PB} = \frac{PC}{PD}$$

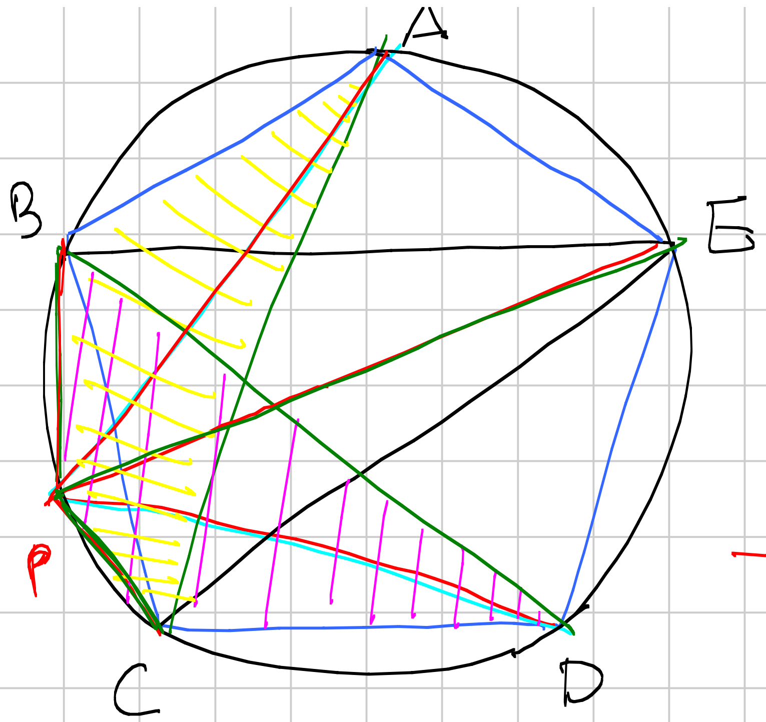
$$PA \cdot PD = PC \cdot PB$$

$$AP = BP + PC$$

$$(BP+PC) \cdot PD = PC \cdot PB$$

$$\frac{\cancel{BP} \cdot \cancel{PD}}{\cancel{PB} \cdot PC \cdot \cancel{PD}} + \frac{\cancel{PC} \cdot \cancel{PD}}{PB \cdot \cancel{PC} \cdot \cancel{PD}} = \frac{\cancel{PC} \cdot \cancel{PB}}{\cancel{PB} \cdot \cancel{PC} \cdot PD}$$

$$\frac{1}{PC} + \frac{1}{PB} = \frac{1}{PD}$$



Voglio mostrare che

$$PB + PC + PE = PA + PD$$

T su ABPC

$$\rightarrow \quad AP \cdot BC = PB \cdot AC + PC \cdot AB$$

T su BPCD

$$\rightarrow \quad PD \cdot BC = PC \cdot BD + CD \cdot BP$$

$\overset{\triangle}{BEC}$ è isoscele

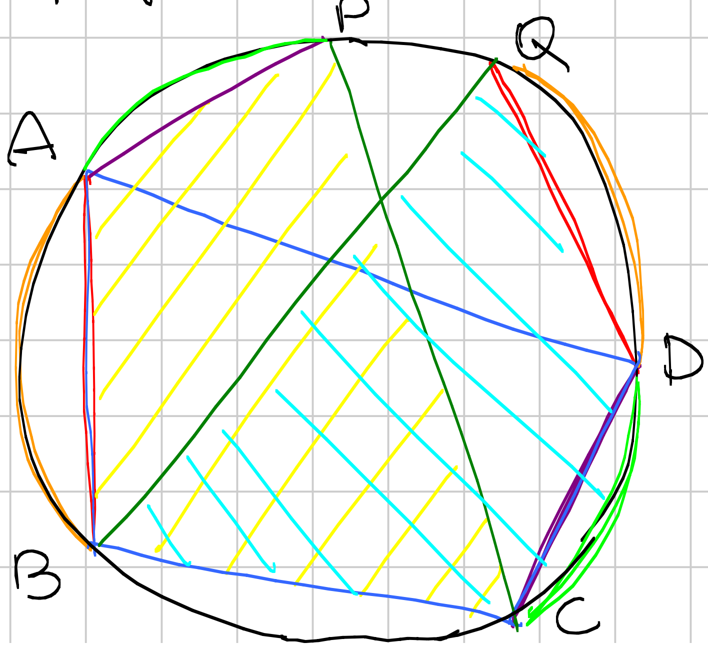
$$(AP + PD) \cdot BC = (PB + PC) \cdot AC + (PC + BP) \cdot AB$$

$$\frac{CE}{BC} = \frac{PE}{PB + PC} \quad \leftarrow \quad AC = CE = \frac{PE \cdot BC}{PB + PC}$$

$$(AP+PD)\cdot \cancel{BC} = \cancel{(PB+PC)}\cdot \frac{PE\cdot \cancel{BC}}{\cancel{PB+PC}} + (PC+BP)\cdot \cancel{BC}$$

$$\boxed{AP+PD = PE+PC+PB}$$

Dato un quadrilatero ciclico ABCD, calcolare il rapporto fra le lunghezze delle diagonali $\frac{AC}{BD}$ in funzione delle lunghezze dei lati.



$AP = DC$

$QD = AB$

T su ABCP

$AC \cdot BP = AP\cdot BC + AB\cdot PC$

T. su $BCQD$ $\qquad BD \cdot QC = DC \cdot QB + BC \cdot QD$

$\overset{\frown}{PAB} = \overset{\frown}{QDC} \rightarrow \underline{PB = QC}$

$\overset{\frown}{PBC} = \overset{\frown}{DBA} \rightarrow PC = AD$

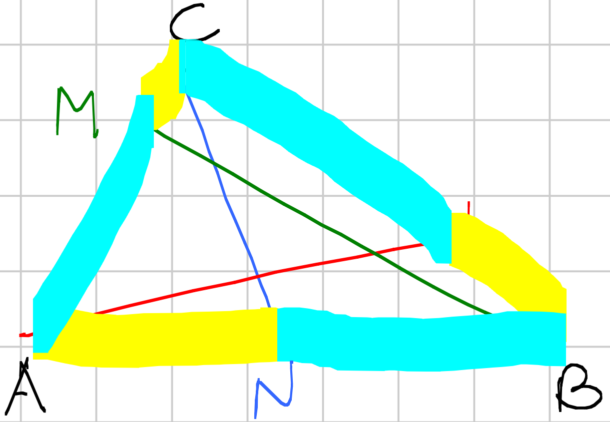
$\overset{\frown}{QCB} = \overset{\frown}{ACD} \rightarrow QB = AD$

$$\frac{AC \cdot \cancel{BP}}{BD \cdot \cancel{QC}} = \frac{\underbrace{AP}_{DC} \cdot BC + AB \cdot \underbrace{PC}_{=AD}}{DC \cdot \underbrace{QB}_{=AD} + BC \cdot \underbrace{QD}_{=AB}}$$

$$\boxed{\frac{AC}{BD} = \frac{DC \cdot BC + AB \cdot AD}{DC \cdot AD + BC \cdot AB}}$$

TEOREMA DI CEVA

Concorrenza di segmenti in un triangolo



CEVIANA

CN, BM e AL sono concorrenti (si incontrano in un punto)

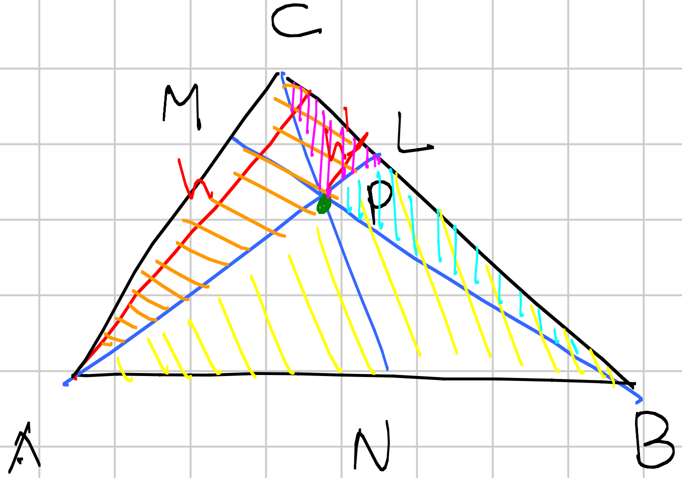
$\Updownarrow$

$$AM \cdot NB \cdot LC = AN \cdot CB \cdot CM$$

oppure

$$\frac{AN}{NB} \cdot \frac{BL}{LC} \cdot \frac{CM}{MA} = 1$$

Se le ceviane sono concorrenti, allora $\frac{AN}{NB}\cdot\frac{BL}{LC}\cdot\frac{CM}{MA}=1$



$$\frac{BL}{LC}=\frac{S(ALB)}{S(ALC)}=\frac{S(PBL)}{S(PCL)}$$

$\overset{\triangle}{PBC}$

$$=\frac{S(ALB)-S(PBL)}{S(ALC)-S(PCL)}=$$

$S(ALC)=CL\cdot h$

$S(ALB)=LB\cdot h$

$$=\frac{S(APB)}{S(APC)}$$

$$\frac{a}{b}=\frac{c}{d}=\frac{a\mp c}{b\mp d}$$

$ad=bc$ $\qquad c(b\mp d)=d(a\mp c)$

$$cb\mp\cancel{cd}=ad\mp\cancel{cd}$$

$$\frac{BL}{LC} = \frac{S(\cancel{APB})}{S(\cancel{APC})}$$

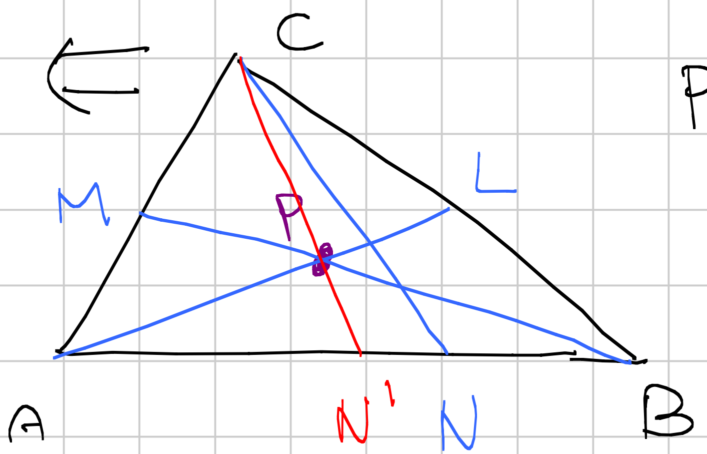
$$\frac{CM}{MA} = \frac{S(\cancel{CPB})}{S(\cancel{APB})}$$

$$\frac{AN}{NB} = \frac{S(\cancel{CPA})}{S(\cancel{CPB})}$$

$$\frac{BL}{LC} \cdot \frac{CM}{MA} \cdot \frac{AN}{NB} = 1$$

$\Leftarrow$ Se $\frac{BL}{LC} \cdot \frac{CM}{MA} \cdot \frac{AN}{NB} = 1$, allora AL, BM e CN sono concorrenti

($\Rightarrow$ già dim.)



Per le ceviane AL, BM e CN' vale il ⟹ di CEVA

$$\frac{AN'}{N'B}\cdot\frac{BL}{LC}\cdot\frac{CM}{MA}=1$$

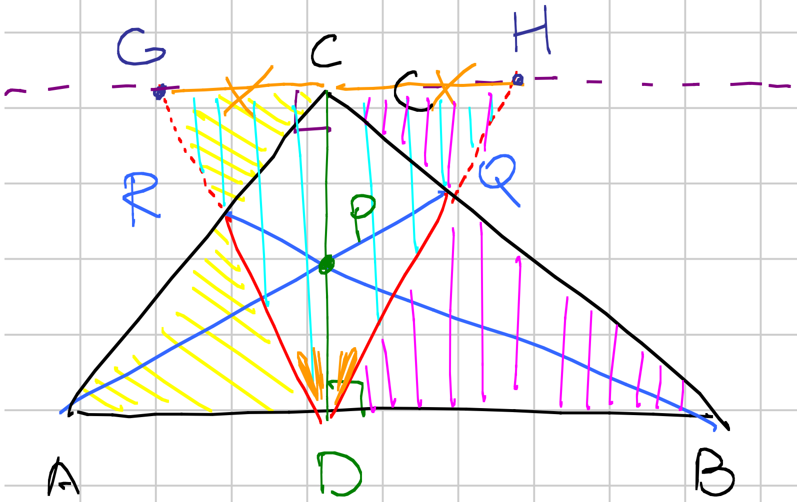
per ipotesi vale

$$\frac{AN}{NB}\cdot\frac{BL}{LC}\cdot\frac{CM}{MA}=1$$

$$\Rightarrow \frac{AN'}{N'B}=\frac{AN}{NB}$$

$$\Downarrow$$

$$N \equiv N'$$



$\forall P \in CD$, si ha che

$$R\hat{D}C = Q\hat{D}C$$

(DC è bisettrice di $Q\hat{D}R$)

$\overset{\triangle}{GCR}$ è simile a $\overset{\triangle}{ADR}$

$\overset{\triangle}{HCQ}$ è simile a $\overset{\triangle}{DQB}$

$$\frac{CR}{RA} = \frac{GC}{AD} \qquad \frac{BQ}{CQ} = \frac{DB}{CH}$$

$$\frac{CR}{RA} \cdot \frac{AD}{DB} \cdot \frac{BQ}{QC} = 1 \qquad \frac{GC}{AD} \cdot \frac{\cancel{AD}}{\cancel{DB}} \cdot \frac{\cancel{DB}}{CH} = 1$$

$$GC = CH \qquad C\hat{G}D = H\hat{C}D$$