Egmo Camp 2018 - Algebra

Note Title 1/21/2018

Problema 1: Sia $(a_n)_{n\in\mathbb{N}}$ la successione definita da $a_0=a_1=a_2=1$ e

$$a_{n+3}=\frac{n!+a_{n+1}a_{n+2}}{a_n} \quad \forall n\geq 0$$

Tesi: a_n intero $\forall n$.

Soluzione: Calcoliamo un po' di termini

$a_0 = a_1 = a_2 = 1$

$$a_3 = \frac{0! + a_1 \cdot a_2}{a_0} = \frac{1 + 1 \cdot 1}{1} = 2$$

$$a_4 = \frac{1! + a_2 \cdot a_3}{a_1} = \frac{1 + 2 \cdot 1}{1} = 3$$

$$a_5 = \frac{2! + a_3 \cdot a_4}{a_2} = \frac{2 + 2 \cdot 3}{1} = 8$$

$a_6 = \ldots = 15 = 5 \cdot 3$

Non è che

$a_7 = 48 = \boxed{8} \cdot 6$

$a_{n+2} = a_n \cdot (n+1)$

$a_8 = 105 = 7 \cdot 15$

D2 $a_{n+2} = (n+1)\,a_n$

$a_{n+2} = (n+1)\cdot(n-1)\cdot a_{n-2} = (n+1)(n-1)(n-3)\cdot a_{n-4}\dots$

$n = 2K$ Notazione

$a_{2K+2} = (2K+1)(2K-1)\cdot\ldots\cdot 3\cdot 1 := (2K+1)!!$

$a_{2K+1} = (2K)(2K-2)\cdot\ldots\cdot 4\cdot 2 := 2K!!$

Verifica: $a_0, a_1, a_2 = 1$

$a_{n+3} = (n+2)!!$?

$a_{n+3} = \dfrac{n! + a_{n+1}\cdot a_{n+2}}{a_n}$

$(n+2)!! = \dfrac{n! + (n!!)\cdot(n+1)!!}{(n-1)!!}$

$(n+2)!!(n-1)!! = n! + n!!\cdot(n+1)!!$

$\cancel{(}(n+2)!! = (n+2)\cdot n!!$

$\underline{(n+2)n!!\cdot(n-1)!!} = n! + \underline{(n+1)\cdot n!!(n-1)!!}$

$$n!! \cdot (n-1)!! \boxed{=} n! = n(n-1)\cdot \ldots$$

$$n\cdot(n-2)\cdot(n-4)\cdot(n-6)\cdot\ldots \qquad (n-1)(n-3)(n-5)\ldots$$

PROBLEMA 3.

$$f:\mathbb{R}\to\mathbb{R}$$

$$f(x+yf(x)) = f(f(x)) + xf(y)$$

Soluzione:

• $x=0$. $\quad f(yf(0)) = f(f(0))$

Se $f(0)\neq 0$ $\quad z = yf(0)$. z varia in tutto $\mathbb{R}$

$f(z) = f(f(0)) \Rightarrow f$ è costante.

Sostituendo $f(x)=K$ $\forall x\in\mathbb{R}$

otteniamo che $K=0$. $f(x)=0$ Verifica

Quindi abbiamo a $f(0)=0$

Poniamo $y=0$.

Vediamo $f(x)=f(f(x))+xf(0)=f(f(x))$

Domanda: $\exists\, \alpha\neq 0$ tali che $f(\alpha)=0$?

Sostituiamo α ad x.

$f(\alpha+yf(\alpha))=f(f(\alpha))+\alpha f(y)$

$f(\alpha)=f(0)+\alpha f(y) \Rightarrow$

$\Rightarrow \alpha f(y)=0$. $\rightarrow f(y)=0 \ \forall y \rightarrow$ Soluzione Verifica!

$\rightarrow \tilde{y}$ t.c. $f(\tilde{y})\neq 0$

Sostituendo $\tilde{y}$ a y nell'eq. prec e dividendo per $f(\tilde{y})$ otteniamo $\alpha=0$.

Una via $\rightarrow$ iniettività?

Un'altra via $\rightarrow$ Provare ad "ammazzare" termini. Uguagliamo

Cerchiamo quindi y tali che

$$x + y f(x) = f(x).$$

$$y = \frac{f(x) - x}{f(x)} \qquad (f(x) \neq 0,\ x \neq 0)$$

Sostituendo

$$f\left(\cancel{x} + \cancel{\frac{(f(x) - x)}{f(x)}} \cdot f(x)\right) = \cancel{f(f(x))} + x f\left(\frac{f(x) - x}{f(x)}\right)$$

$$0 = x f\left(\frac{f(x) - x}{f(x)}\right) \quad \forall x \neq 0$$

$$0 = f\left(\underbrace{\frac{f(x) - x}{f(x)}}\right) \quad \forall x \neq 0$$

$$\frac{f(x) - x}{f(x)} = 0 \quad \forall x \neq 0$$

Inizio

$$f(x) = x \quad \forall x \neq 0 \qquad f(0) = 0$$

$$f(x) = x \quad \forall x \in \mathbb{R} \qquad \begin{cases} f(x) = x \\ f(x) = 0 \end{cases}$$

VERIFICARE

Commento:

- Le soluzioni di un'eq. funz. spesso sono $ax+b$, x^2, costanti...

Ed è utile perché indovinare $\to$ punto $\to$ Può dare idee per continuare

- $f(x+y) = f(x) + f(y)$ (Eq. Cauchy)

$f: \mathbb{Q} \to \mathbb{Q}$

$f(x) = \lambda x$ con $\lambda \in \mathbb{Q}$

$f: \mathbb{R} \to \mathbb{R}$

Solo sotto certe ipotesi $f(x) = \lambda x$ $\forall x \in \mathbb{R}$.

- monotonia
- limitatezza

Problema 2

$a_1, a_2, \dots a_n \in \mathbb{R}$ $\quad -1 < a_i < 1 \quad i = 1, \dots n$

$a_1 + a_2 + \dots + a_n = 0$

$(*)\ a_1^2 + a_2^2 + \dots + a_n^2 = 40$

$a_i^2 < 1$

$\to n > 40$

$a_1 = a_2 = \dots = a_{21} = \sqrt{\frac{20}{21}}$

$n = 42$ $\quad 42 a_1^2 = 40 \to a_1^2 = \frac{40}{42} = \frac{20}{21}$ $\quad a_{22} = a_{23} = \dots = a_{42} = -\sqrt{\frac{20}{21}}$

. 41 ? $\quad \bullet\ a_i \neq 0$

$\bullet\ a_1 \geq a_2 \geq \dots \geq a_m > 0 > a_{m+1} \geq \dots \geq a_{41}$

$$\underbrace{|a_1 + a_2 + \dots + a_m|}_{m \text{ termini}} = \underbrace{|a_{m+1} + a_{m+2} + \dots + a_{41}|}_{41-m \text{ termini}} = k > 0$$

$\min\{m, 41-m\} \leq 20 \qquad k < m \qquad \rightsquigarrow k < 20$

$\qquad\qquad\qquad\qquad\qquad\quad\ k < 41-m$

$\bullet\ a^2 \leq |a|$

$$a_1^2 + a_2^2 + \dots + a_m^2 \leq |a_1| + |a_2| + \dots + |a_m| = |a_1 + \dots + a_m| = k < 20$$

$$a_{m+1}^2 + a_{m+2}^2 + \dots + a_{41}^2 \leq |a_{m+1}| + |a_{m+2}| + \dots + |a_{41}| = |a_{m+1} + \dots + a_{41}| = k < 20$$

$$a_1^2 + a_2^2 + \dots + a_m^2 + a_{m+1}^2 + \dots + a_{41}^2 < 40 \qquad \text{Assurdo!}$$

Problema 4

$a, b, c \in \mathbb{R}^+, \quad a+b+c = 1 \qquad \dfrac{7+2b}{1+a} + \dfrac{7+2c}{1+b} + \dfrac{7+2a}{1+c} \geq \dfrac{69}{4}$

$$LHS = 5 \cdot \underbrace{\left(\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c}\right)}_{A} + 2 \cdot \underbrace{\left(\frac{1+b}{1+a} + \frac{1+c}{1+b} + \frac{1+a}{1+c}\right)}_{B} = \text{⊛}$$

$$\boxed{x, y, z > 0 \qquad \frac{x+y+z}{3} \geq \sqrt[3]{xyz}}$$

$$B \geq 3 \sqrt[3]{\frac{1+b}{1+a} \cdot \frac{1+c}{1+b} \cdot \frac{1+a}{1+c}} = 3$$

Remind:

AM - HM

$x, y, z \in \mathbb{R}^+ \qquad \dfrac{x+y+z}{3} \geq \dfrac{3}{\frac{1}{x} + \frac{1}{y} + \frac{1}{z}}$

$$\underbrace{\frac{1}{1+a}}_{x} + \underbrace{\frac{1}{1+b}}_{y} + \underbrace{\frac{1}{1+c}}_{z} \geq \frac{9}{1+a+1+b+1+c} = \frac{9}{3+a+b+c} = \frac{9}{4}$$

$$\text{⊛} \geq 5 \cdot \frac{9}{4} + 2 \cdot 3 = \frac{45}{4} + 6 = \frac{69}{4}$$

Bunching

x, y, z $\qquad \sum_{cyc} x^2 y = x^2y + y^2z + z^2x$

$$\sum_{sym} x^2 y = x^2y + y^2z + z^2x + xy^2 + yz^2 + zx^2$$

$$\sum_{sym} x^{a_1} \cdot y^{a_2} \cdot z^{a_3} \geq \sum_{sym} x^{b_1} \cdot y^{b_2} \cdot z^{b_3}$$

- $a_1 + a_2 + a_3 = b_1 + b_2 + b_3$
- $a_1 \geq b_1$
- $a_1 + a_2 \geq b_1 + b_2$

$$\sum_{sym} x^3 \geq \sum_{sym} xyz$$

$$\sum_{sym} x^5 y^3 z \geq \sum_{sym} x^4 y^4 z \geq \sum_{sym} x^4 y^3 z^2$$

$$\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} \geq \frac{9}{4} \qquad a+b+c=1$$

$$\frac{a+b+c}{2a+b+c} + \frac{a+b+c}{a+2b+c} + \frac{a+b+c}{a+b+2c} \geq \frac{9}{4}$$

$$4 \cdot \sum_{cyc} (a+b+c)(2a+b+c)(a+2b+c) \geq 9\,(2a+b+c)(a+2b+c)(a+b+2c)$$

$(\cdot \cdot \cdot)$ conti

$$\sum_{sym} a^3 + \sum_{sym} a^2 b \geq 2 \sum_{sym} abc$$

- $\sum_{sym} a^3 \geq \sum_{sym} abc$
- $\sum_{sym} a^2 b \geq \sum_{sym} abc$

$\blacksquare$