Algebra Advanced

Note Title 02/11/2019

1| $f: \mathbb{Q} \to \mathbb{Q}$ t.c.

$\forall x, y \quad f(x + y f(x^2)) = f(x) + x f(xy)$

Proviamo $f(x) = ax + b$

$$a(\cancel{x} + y(ax^2 + b)) + \cancel{b} = \cancel{ax} + \cancel{b} + x(axy + b)$$

$$a^2x^2y + aby = ax^2y + bx$$

$$ab = 0$$
$$b = 0$$
$$a^2 = a$$

$\left[\begin{array}{l} f(x) = 0 \\ f(x) = x \end{array}\right]$

$P(x,y)$: $f(x + yf(x^2)) = f(x) + x f(xy)$

$\Rightarrow \forall x, y \in \mathbb{Q} \quad P(x,y)$

$P(x,0)$: $\cancel{f(x)} = \cancel{f(x)} + x f(0)$ $\quad x f(0) = 0$

$x = 1 \rightsquigarrow f(0) = 0$

$P(1,y)$: $f(1) = c$

$\boxed{f(1 + yc) = c + f(y)}$ $(*)$

proviamo $1 + yc = y \iff y = \frac{1}{1-c} \quad (c \neq 1)$

$c \neq 1 \Rightarrow c = 0 \overset{(*)}{\rightsquigarrow} 0 = f(1) = f(y) \quad \forall y \in \mathbb{Q}$

$\boxed{f \equiv 0}$

$c = 1 \quad (*) \Rightarrow f(1 + y) = 1 + f(y)$

$\boxed{f(n + y) = n + f(y)} \quad \forall n \in \mathbb{Z}$

x intero in $P(x,y)$

$\cancel{x} + f(y f(x^2)) = \cancel{x} + x f(xy)$

$\forall x \in \mathbb{Z}$, $\forall y \in \mathbb{Q}$ $\quad f(yx^2) = x f(xy)$

$z = xy$, $\boxed{y = z/x}$

$\forall x \in \mathbb{Z} \quad \forall z \in \mathbb{Q} \quad f(xz) = x f(z)$

$z = a/b \quad , \quad a \in \mathbb{Z},\ b \in \mathbb{N}^*$

$f\left(x \cdot \frac{a}{b}\right) = x\, f\left(\frac{a}{b}\right)$

$x = b$

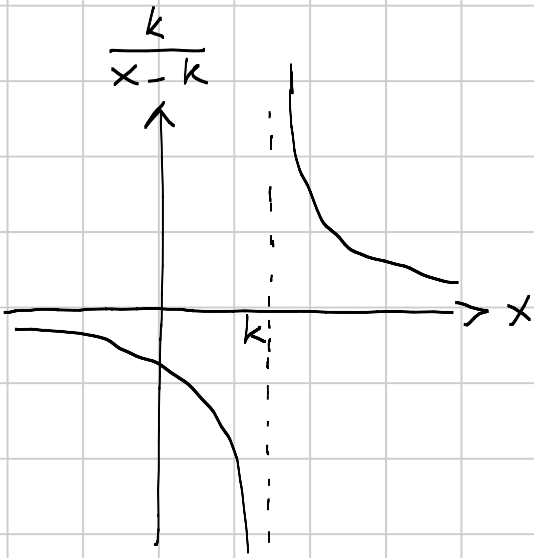
$a = f(a) = b\, f(a/b)$

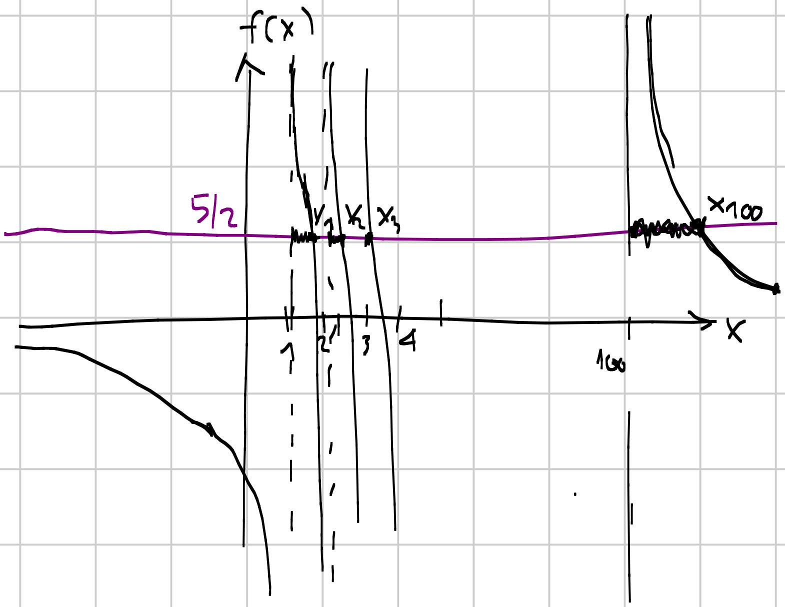
$\Rightarrow \quad f(a/b) = a/b \quad \forall\, a/b \in \mathbb{Q}$

$\square$

2| $\underbrace{\sum_{k=1}^{100} \frac{k}{x-k}}_{f(x)} \geq 5/2$ ⇝ unione di intervalli

somma delle lunghezze $= 2020$





$\frac{k}{x-k} < 1/100$

$L = \sum_{i=1}^{100} (x_i - i) = \sum_{i=1}^{100} x_i - 5050$

deg $P_k = 99$

$P_k(x)$

$\sum_{k=1}^{100} \frac{k}{x-k} = 5/2 \xrightarrow{\text{molt. per } \prod (x-i)} \sum_{k=1}^{100} k \cdot \prod_{j=1,\ j\neq k}^{100} (x-j) = 5/2 \prod_{j=1}^{100} (x-j)$

$\cdot 2/5$

$\prod_{j=1}^{100} (x-j) - \frac{2}{5} \sum_{k=1}^{100} k \cdot P_k(x) = 0$

termine di deg 100 (coeff 1)

$\frac{2}{5} \sum_{k=1}^{100} k = 2020$

il coeff di x^{99} è la somma delle radici (per -1)

quindi è $-\sum_{j=1}^{100} j = -5050$

Quindi il coeff. di x^{99} in tutto il LHS è $-5050 - 2020 = -7070$

ovvero, la somma delle radici è 7070.

$$L = \sum_{i=1}^{100} (x_i - i) = \sum_{i=1}^{100} x_i - 5050 = 2020.$$

$$\sum_{i=1}^{100} x_i = 7070$$

□

3| α la più grande radice reale di $x^3 - 17x^2 + 1 = 0$ ($P(x)$)

β, γ le altre 2

$\lfloor \alpha^{2018} \rfloor \mod 17$

Prodotto delle radici $= -1$

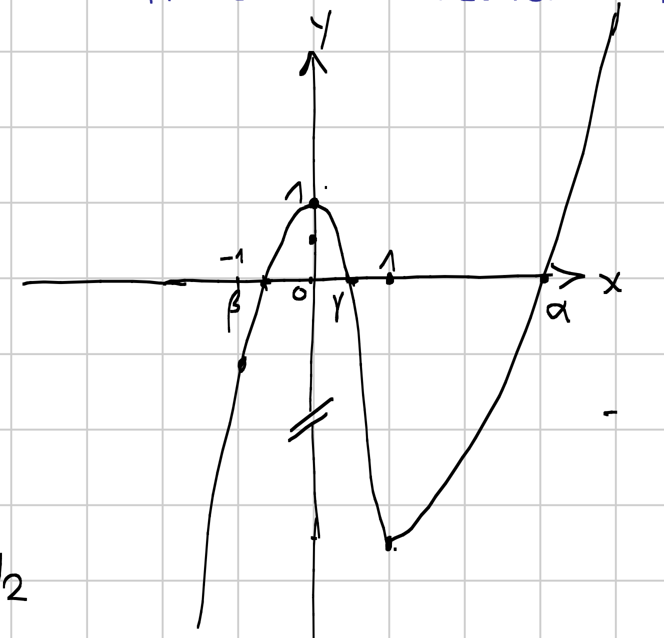
$$\alpha\beta\gamma = -1$$

$$\underbrace{|\alpha|}_{>1} \cdot \underbrace{|\beta|}_{<1} \cdot \underbrace{|\gamma|}_{<1} = 1$$

se $P(-1/2) < 0$ ✓

$P(1/2) < 0$ ✓ allora

$$-\frac{1}{2} < \beta, \gamma < 1/2$$



$$A_{2018} = \alpha^{2018} + \beta^{2018} + \gamma^{2018} < \alpha^{2018} + 1/2$$

$$\beta^{2018} + \gamma^{2018} < 1/2$$

se fosse intera $A_{2018} - 1/2 < \alpha^{2018} < A_{2018} \leq \alpha^{2018} + 1/2$

$$\lfloor \alpha^{2018} \rfloor = A_{2018} - 1$$

A_{2018} è una funz. simmetrica di α, β, γ e si scrive come somme e prodotti dei coeff. del polinomio P

$$\alpha+\beta+\gamma, \quad \alpha\beta+\beta\gamma+\alpha\gamma, \quad \alpha\beta\gamma$$

$$Q = x^n - (a_{n-1}x^{n-1} + \dots + a_1 x + a_0)$$

$\longrightarrow$ radici $\alpha_1, \dots, \alpha_n$

$$b_k = c_1 \cdot \alpha_1^k + \dots + c_n \alpha_n^k$$

$$\begin{cases} b_0 = \dots \\ \vdots \\ b_{n-1} = \dots \\ b_k = a_{n-1} b_{k-1} + \dots + a_0 b_{k-n} \end{cases}$$

$A_n = \alpha^n + \beta^n + \gamma^n$ è una successione per ricorrenza!

viene dal polinomio monico che ha come radici $\alpha, \beta, \gamma \longrightarrow P$

$P(x) = x^3 - 17x^2 + 1$

mod 17 : $A_{n+3} \equiv -A_n \pmod{17}$

$A_{n+3} - 17 A_{n+2} + A_n = 0 \quad (*)$

α^n rispetta

$\alpha^{n+3} - 17\alpha^{n+2} + \alpha^n = \alpha^n P(\alpha) = 0$

α^n rispetta $(*)$

lo stesso per β^n, γ^n

Abbiamo spiegato (forse) perché A_n rispetta $(*)$

$A_{2018} \mod 17$

$A_0 = \alpha^0 + \beta^0 + \gamma^0 = 3$

$A_1 = \alpha + \beta + \gamma = 17 \equiv 0 \pmod{17}$

$A_2 = \alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) \equiv 0 \pmod{17}$

$(\alpha+\beta+\gamma)^2 = 17^2$, $\quad 2(\alpha\beta+\beta\gamma+\gamma\alpha) = 2 \cdot 0$

A_0	A_1	A_2	A_3	A_4	A_5	A_6	...	A_{2018}
3	0	0	−3	0	0	3		

(periodo $A_0 \ldots A_5$: 6)

$2018 = 2 + 2016 = 2 + 336 \cdot 6$

$A_{2018} \equiv A_2 \equiv 0$

$\lfloor \alpha^{2018} \rfloor \equiv A_{2018} - 1 \equiv 16 \pmod{17}$

4| $a, b, c \geq 0$ $\quad \sqrt[3]{\dfrac{a^3+b^3+c^3}{3}} + 8\sqrt[3]{abc} \leq 3(a+b+c)$ $\square$

"succumbed quickly to power means"

— Evan Chen

$M_3(a,b,c) + 8 M_0(a,b,c) \leq 9 M_1(a,b,c)$

La $\sqrt[3]{\ }$ compare in varie medie p-esime

- M_3 (cubica)
- M_0 (geometrica) per caso
- $M_{1/3}(a,b,c)$

$$= \left(\frac{a^{1/3}+b^{1/3}+c^{1/3}}{3}\right)^3 = \left(\frac{\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}}{3}\right)^3$$

$$\sqrt[3]{\frac{a^3+b^3+c^3}{3}} + 8\sqrt[3]{abc} \le 3(a+b+c)$$

$$q = \frac{a^3+b^3+c^3}{3} \qquad p = abc$$

$$\left(\frac{\sqrt[3]{q}+8\sqrt[3]{p}}{9}\right)^3 \le \frac{q+8p}{9}$$

$$[M_{1/3} \le M_1]$$

$$LHS = \sqrt[3]{q}+8\sqrt[3]{p} \le 9\sqrt[3]{\frac{q+8p}{9}} \overset{?}{\le} 3(a+b+c)$$

$$3\sqrt[3]{\frac{\frac{a^3+b^3+c^3}{3}+8abc}{9}} \le a+b+c$$

$$\Updownarrow$$

$$27\,\frac{(a^3+b^3+c^3)/3+8abc}{9} \le (a+b+c)^3$$

$$= a^3+b^3+c^3+24abc \qquad = a^3+b^3+c^3+3\sum_{sym} a^2b+6abc$$

$$\Updownarrow$$

$$\cancel{18}\,6\,abc \le \cancel{3}\,\frac{(a^2b+ab^2+b^2c+bc^2+c^2a+ca^2)}{6}$$

Ma ora

$$\frac{a^2b+ab^2+b^2c+bc^2+c^2a+a^2c}{6} \ge \sqrt[6]{a^6b^6c^6} = abc$$

$\square$