TdN EGMO 2019

Note Title 01/11/2019

Problema 1

$$x^3+3x+14=2p^n$$

$$(x+2)(x^2-2x+7)=2p^n$$

$$\begin{cases} x+2=2p^\alpha \\ x^2-2x+7=p^\beta \\ \alpha+\beta=n \end{cases} \quad \text{o} \quad \begin{cases} x+2=p^\alpha \\ x^2-2x+7=2p^\beta \\ \alpha+\beta=n \end{cases}$$

$\rightsquigarrow$ mod p^α $\quad x \equiv -2$ $\qquad x^2-2x+7 \geq x+2$

$$x^2-3x+5=\left(x-\frac{3}{2}\right)^2+\left(5-\frac{9}{4}\right) \geq 0$$

> 0

$\beta < \alpha \quad \Rightarrow 2p^\beta \leq p^\alpha$

$\beta \geq \alpha$

$0 \equiv (-2)^2+4+7=15 \qquad p^\alpha \mid 15$

$p^\alpha=3, \quad p^\beta=5$

- $x=2p^\alpha-2 \qquad \underbrace{4p^{2\alpha}-c_1p^\alpha+c_2}=p^\beta$

$$p^{2\alpha}<3p^{2\alpha}+\left[p^{2\alpha}-c_1p^\alpha+c_2\right]<p^{2\alpha+1}$$

$$p^{2\alpha+1}=4p^{2\alpha}+c\ldots$$

$$p^{2\alpha} \mid -c_1p^\alpha+c_2$$

$x=p^\alpha-2$

$$\rightsquigarrow \quad p^{2\alpha}\underbrace{-c_1p^\alpha+c_2}_{<0}=p^\beta$$

$\beta<2\alpha$

$$p^{2\alpha}-c_1p^\alpha+c_2>p^{2\alpha-1} \qquad p^\alpha \gg 1$$

Problem 2

$p_j \mid M+j \quad , j=1,\dots,n \qquad M,n \in \mathbb{Z}_+, \ M > n^{n-1}$

$M+1 \ , \dots \ M+i \ , \dots \ M+j \ \dots \ M+n$

$p_i \xleftarrow[\neq ?]{} , p_j$

- $d \mid (M+i, M+j) \ \rightarrow \ d \mid M+i-M-j = i-j < n$

$(M+i, M+j) \mid (n-1)!$

$$a_i := \frac{M+i}{((n-1)!, M+i)}$$

$\rightarrow \ a_i > 1, \ (a_i, a_j) = 1.$

$((n-1)!, M+i) \leq (n-1)! \leq n^{n-1} \quad M+i > n^{n-1}$

$a_i > 1$

- $(M+i, M+j) \mid (n-1)!$

$$\left(\frac{M+i}{((n-1)!, M+i)}, \frac{M+j}{((n-1)!, M+j)} \right) \mid \frac{(M+i, M+j)}{\gcd(((n-1)!, M+i), ((n-1)!, M+j))} = 1$$

$p_i \mid a_i \qquad p_i \neq p_j$

- $p_i \mid M+i, \quad p_i > n$

$p_i^k \nmid M+i \quad , \ p_i^k > n$

$\rightarrow \ \nexists \ \bar{k} \ \text{t.c.} \ p_i^{\bar{k}} \mid M+j \ , \ p_i^{\bar{k}} > n.$

$$M+i = \prod_{i=1}^{k} \underbrace{p_i^{\alpha_i}}_{\leq n} \qquad p_i \leq n \qquad k \leq n-1$$

$\leq n^{n-1}$

assurdo

$\exists\, p_i$ t.c. $\exists\, k_i$ t.c. $p_i^{k_i} > n$, $p_i^{k_i} \mid M+i$.

$p_i' = \dots$

$p_i = p_j$, $\quad p_i^{k_i} \mid M+i$, $\quad p_j^{k_j} \mid M+j$

$p_i^{\min(k_i,k_j)} > n \qquad p_i^{\min(k_i,k_j)} \mid i-j < n$

Problema 3:

$$f^{(i)}(n) = \underbrace{f\cdots(f}_{i \text{ volte}}(n)\cdot)\cdot\cdot)$$

$$f^{(f(n))}(n) = \frac{n^2}{f^{(2)}(n)} \qquad \forall n \in \mathbb{N}$$

Quali sono i possibili valori di $f(1000)$?

$n=1$: $\quad f(f(1)) = 1 \qquad a = f(1) \to f(a) = f(f(1)) = 1$

$n = a \to 1 = f(a) = f^{(f(a))}(a) = \dfrac{a^2}{f(f(a))} = \dfrac{a^2}{a} = a$

$\to \boxed{f(1) = 1}$

k t.c. $f(k) = 1 \implies k = 1 \to 1$

- $n = p \qquad f^{(f(p))}(p) = \dfrac{p^2}{f(f(p))} \qquad f(f(p)) \in \{1, p, p^2\}$

$f(f(p)) = 1 \to f(p) = 1 \to p = 1$

$f(f(p)) = p^2 \to f^{(f(p))}(p) = 1 \to p = 1$

$$\boxed{f(f(p)) = p}$$

- $f(m) = f(n) = d \Rightarrow f^{(d)}(m) = \dfrac{m^2}{f(d)}$

$\boxed{\text{INIETTIVA}}$

$$\left.\begin{array}{l} f^{(d-1)}(f(m)) \\ \quad \| \\ f^{(d-1)}(f(n)) \end{array}\right\} m = n$$

$$f^{(d)}(n) = \frac{n^2}{f(d)}$$

- $f(n) = n \quad \forall\, n$ dispari

P.B: $n = 1$ già vista

P.I: supponiamo che $\forall\, d$ dispari $d < n$: $f(d) = d$
e dimostriamo che $f(n) = n$ n dispari

$$\boxed{f^{(2)}(n) \cdot f^{(f(n))}(n) = n^2 \qquad n \text{ dispari}}$$

$n \qquad n$

$d_1 = f(f(n))$ dispari $< n$: $f(n) = d_1 < n \rightarrow n = d_1$ (ass)

$$f(f(n)) = n$$

$$f^{(f(n))}(n) = n$$

$m = f(n)$: $f(m) = n$

$$f^{(n+1)}(n) = f^{(f(f(n)))}(f(n)) = \frac{(f(n))^2}{f(\underbrace{f(f(n))}_{n})} = \frac{(f(n))^{\cancel{2}}}{\cancel{f(n)}}$$

$$\Rightarrow f(n) = f^{(n+1)}(n) = n$$

dis p

pari

$f(1000)$ — INIETTIVA

— $f(d) = d \quad \forall\, d$ dispari

$$f(1000) = 2k \rightarrow f(n) = \begin{cases} n & n \neq 1000, 2k \\ 2k & n = 1000 \\ 1000 & n = 2k \end{cases}$$

$n = 2k$: $f(2k) = 1000 \qquad f(f(2k)) = 2k$

$$2k = f^{(2000)}(2k) \overset{?}{=} \frac{(2k)^2}{2k} = 2k \quad ✓$$

Problema 4.

$$\boxed{f(n)} = \sum_{\substack{d \leq n \\ (d,n) \neq 1}} d \quad : \forall n \forall p \quad f(n+p) \neq f(n)$$
$$\forall p > 7$$

$$\sum_{\substack{d \leq n \\ (d,n)=1}} d$$

$(d,n) = 1$

$(n-d, n) = (-d, n) = 1$

$$2\sum_{\substack{d \leq n \\ (d,n)=1}} d \begin{cases} 1 + d_1 + d_2 \dots + n-1 \\ n-1 + n-d_1 + n-d_2 \dots + d_n \; 1 \end{cases}$$

$n \quad n \quad n \quad n \quad = n \cdot \phi(n)$

$$\sum_{\substack{d \leq n \\ (d,n)=1}} d = \frac{n}{2} \cdot \phi(n)$$

$$f(n) = \frac{n(n+1)}{2} - \frac{n\,\phi(n)}{2} = \frac{n}{2}\left(n+1 - \phi(n)\right)$$

Riassunto : $\phi(n) = \#\{ d < n \mid (d,n) = 1 \}$

$\phi(p) = p - 1 \quad , \quad \phi(p^\alpha) = p^{\alpha-1}(p-1)$

$$\phi(p_1^{\alpha_1} \cdots p_k^{\alpha_k}) = \prod p_i^{\alpha_i - 1}(p_i - 1)$$

$$= \prod p_i^{\alpha_i}\left(1 - \frac{1}{p_i}\right) \quad (*)$$

$$(m,n) = 1 \iff \boxed{\phi(mn) = \phi(m)\phi(n)}$$

$$\frac{\phi(n)}{n} = \prod_{p \mid n}\left(1 - \frac{1}{p}\right)$$

$$f(n) = \frac{n}{2}\left(n+1 - \phi(n)\right)$$

$\exists\, n\ \exists\, p > 7:\quad f(n+p) = f(n)$

$$\Rightarrow (n+p)(n+p+1 - \phi(n+p)) = n(n+1 - \phi(n))$$

$(n,p) = 1 \Rightarrow \quad n+p \mid n(n+1 - \phi(n))$

$(n+p, n) = 1 \longrightarrow \quad n+p \mid n+1 - \phi(n) < n+p$ an.

$\Rightarrow$ $n = pk$ $\quad (k\cancel{p} + \cancel{p}^{1})(kp+p+1 - \phi(p(k+1))) = k\cancel{p}(kp+1 - \phi(kp))$

$$\boxed{\phi(p(k+1)) + k\underbrace{\boxed{(\phi(p(k+1)) - \phi(pk))}}_{r} = 2kp + p + 1}$$

$\Rightarrow p-1 \mid r \qquad kr = \alpha . k(p-1) \qquad \alpha \geq 1$

$\alpha k(p-1) < 2kp + p + 1 \qquad p > 7$

$\boxed{\alpha < 3}$ $\begin{cases} 3k(p-1) > 2kp + p + 1 \\ kp - 3k \overset{?}{>} p + 1 \iff (p-3)(k-1) > 4 \end{cases}$

$\hookrightarrow k = 1$ a meno

$$\phi(p(k+1)) < p(k+1) < (p+1)(k+1)$$

$k\alpha(p-1)$

$$= kr = 2kp + p + 1 - \phi(p(k+1)) \quad \}\alpha = 1$$

$$> \cancel{2}kp + \cancel{p} \cancel{+1} - \cancel{pk} - k \cancel{-1} \cancel{-1} = k(p-1)$$

$$\Rightarrow \alpha = 2$$

$$\phi(p(k+1)) - \phi(pk) = 2(p-1)$$

$$\phi(p(k+1)) = 2kp + p + 1 - 2k(p-1)$$

$$= 2k + p + 1$$

$$\phi(pk) = 2k - p + 3$$

$p \mid k \Rightarrow p \mid \phi(pk) = 2k - p + 3$

$p \mid p, \; p \mid 2k \Rightarrow p \mid 3$ an.

$p \mid k+1 \rightarrow p \mid 2(k+2) - 2 + p + 1 \equiv -1 \; (p)$

$$\phi(k+1) - \phi(k) = \boxed{2}$$

$\phi(n) \equiv 2 \; (4) \qquad n = 2^a p^m \qquad a = 0, 1$

$$\prod p_i^{\alpha_i - 1}(p_i - 1)$$

Uno fra $\phi(k)$ e $\phi(k+1)$ non è div per 4

$\rightarrow$ uno dei due (ad esempio $\phi(k)$) =

$k = \{ 2p^m, \underline{p^m}, 4 \}$

$$\phi(k) = q^m \longrightarrow \phi(k) = \frac{2k - p + 3}{p - 1}$$

$$1 - \frac{1}{q} = \frac{\phi(k)}{k} = \frac{2k - p + 3}{k(p-1)} = \frac{2}{p-1} - \underbrace{\frac{p-3}{k(p-1)}}_{>0} < \frac{2}{p-1}$$

$$\frac{q-1}{q} < \frac{2}{p-1} \iff (p-1)(q-1) < 2q$$

$$q(\cancel{p-2}) < \frac{p-1}{p-2} < 2$$