ALGEBRA BASIC

$x^n - 1 = (x-1)(x^{n-1} + \dots + 1)$

Note Title 02/11/2019

Complessi

$X + 3 = 0$ ⇝ negativi -3

$2X = 3$ ⇝ razionali $3/2$

$X^2 = 2$ ⇝ reali $\sqrt{2}$

$X^2 = -1$ ⇝ complessi $i^2 = -1$

$i + 1, \quad 2i, \quad \pi i - \sqrt{3}, \quad \dots$

Def. L'insieme dei numeri complessi è

parte immaginaria

$$\mathbb{C} = \{ a + b i \mid a, b \in \mathbb{R} \}$$

parte reale

Oss

- $(2+3i) + (4-i) = (2+4) + i(3-1) = 6+2i$
- $(2+3i)(4-i) = 2 \cdot 4 + 2(-i) + 3i \cdot 4 + 3i \cdot (-i)$

$$= 8 + 10i + \underbrace{(-3)\, i^2}_{3}$$

$$= 11 + 10i$$

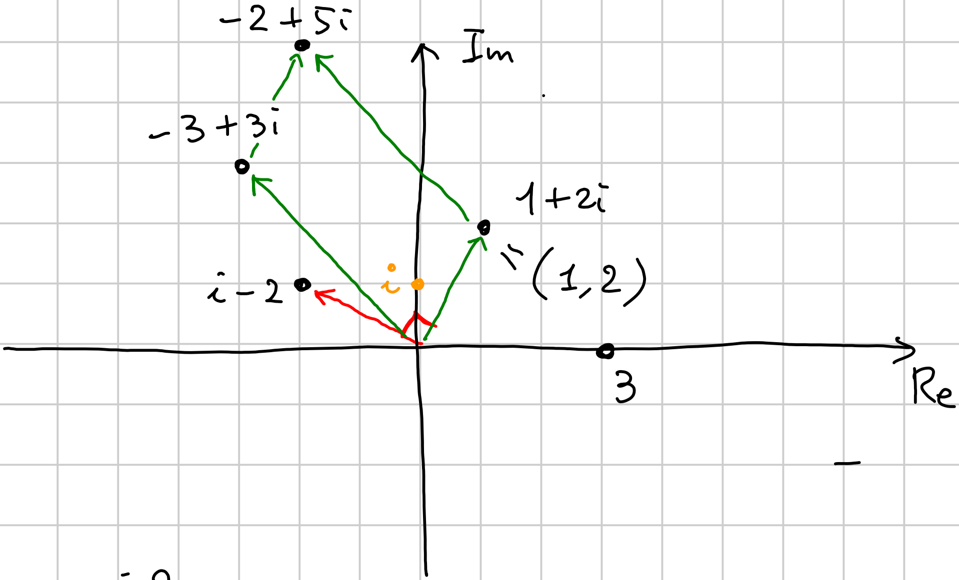
- $$\frac{a+bi}{c+di} = \frac{a+bi}{c+d\sqrt{-1}} = \frac{(a+bi)(c-di)}{(c+di)(c-di)} =$$

$$= \frac{(ac+bd) + i(bc-ad)}{c^2+d^2}$$

$$= \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}\, i$$

Piano complesso

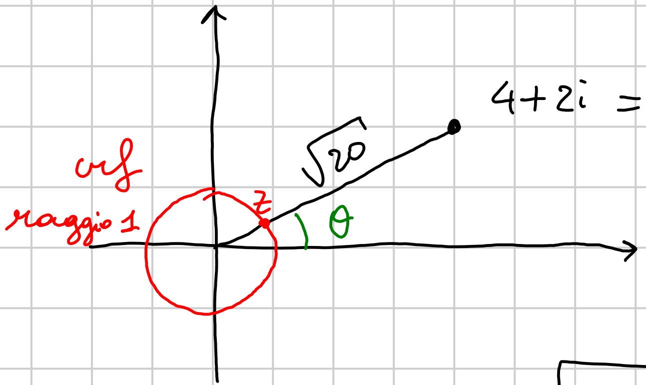
$(1+2i)\cdot i = i-2$



Formula magica (forma polare)

$$e^{i\vartheta} = \cos\vartheta + i\sin\vartheta$$

$e^a\, e^b = e^{a+b}$ anche se a, b sono numeri $\in \mathbb{C}$



$$4+2i = \sqrt{20}\cdot\underbrace{\left(\frac{4}{\sqrt{20}} + \frac{2}{\sqrt{20}}\,i\right)}_{z} = \sqrt{20}\cdot e^{i\vartheta}$$

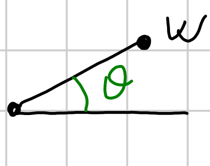
$$\text{dist}(z,0) = \sqrt{\left(\frac{4}{\sqrt{20}}\right)^2 + \left(\frac{2}{\sqrt{20}}\right)^2} = \sqrt{\frac{16+4}{20}} = 1$$

$$z = \cos\theta + i\sin\theta = e^{i\vartheta}$$

In generale: riesco a scrivere ogni $w \in \mathbb{C}$ come

$$w = |w|\cdot e^{i\vartheta}$$

dove, se $w = a+ib$, $|w| = \sqrt{a^2+b^2}$ e ϑ è l'angolo fra la congiungente



e l'asse x

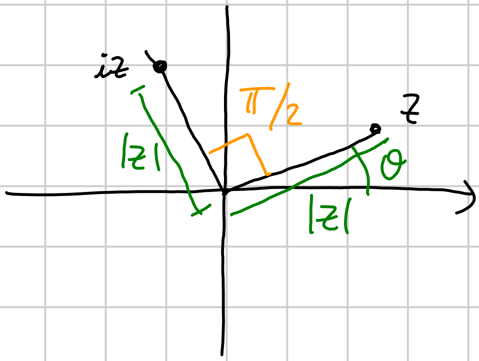
Esempio $i = |i|\cdot e^{i\vartheta} = 1\cdot e^{i\pi/2}$

$$z = |z| \cdot e^{i\vartheta}$$

$$i \cdot z = |z| \cdot e^{i\vartheta} e^{i\pi/2} = |z| e^{i(\vartheta + \pi/2)}$$

$= n^\circ$ complesso con lo stesso modulo di z

e angolo $\vartheta + \pi/2$



Formule di addizione di seno/coseno

$$\cos(\alpha + \beta) = ?$$

Oss Sui complessi c'è il CONIUGIO: se $z = a + bi$,

scrivo $\bar{z} = a - bi$. Buone proprietà:

$$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2, \qquad \overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$$

$$a + bi = w$$
$$a - bi = \bar{w}$$
$$a = \frac{w + \bar{w}}{2}$$

$$e^{i\vartheta} = \cos\vartheta + i \sin\vartheta$$

$$e^{i(2\pi - \theta)} = \cos\vartheta - i \sin\vartheta$$
$$\parallel$$
$$e^{-i\theta}$$

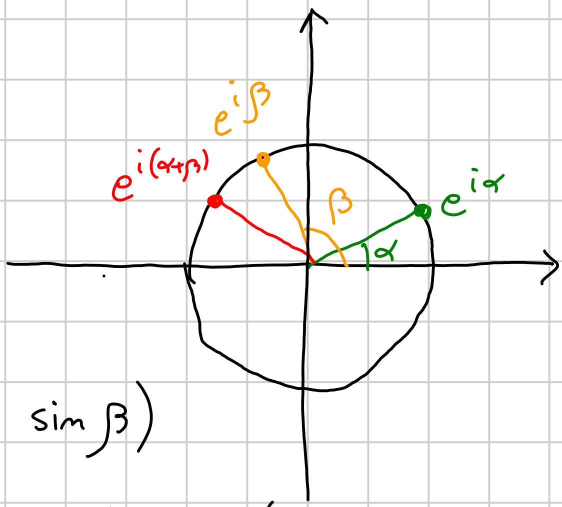
$$e^{2\pi i} = \cos 2\pi + i \sin 2\pi$$
$$= 1$$

$$\boxed{\cos\vartheta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \sin\vartheta = \frac{e^{i\theta} - e^{-i\theta}}{2i}}$$

$$\cos(\alpha+\beta) = \frac{e^{i(\alpha+\beta)} + e^{-i(\alpha+\beta)}}{2}$$

$$\cos\alpha = \frac{e^{i\alpha}+e^{-i\alpha}}{2} \qquad \sin\alpha = \frac{e^{i\alpha}-e^{-i\alpha}}{2i}$$

$$\cos\beta = \frac{e^{i\beta}+e^{-i\beta}}{2}$$



$$e^{i(\alpha+\beta)} = \cos(\alpha+\beta) + i\sin(\alpha+\beta)$$

$$\parallel$$

$$e^{i\alpha}e^{i\beta} = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$$

$$= (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\cos\alpha\sin\beta + \sin\alpha\cos\beta)$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\sin(\alpha+\beta) = \cos\alpha\sin\beta + \sin\alpha\cos\beta$$

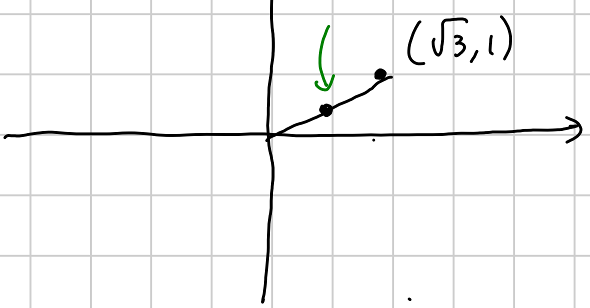
Esercizio Calcolare $(i+\sqrt{3})^{2020}$

$$z = i+\sqrt{3} \qquad z = |z|\cdot e^{i\vartheta}$$

$$|z| = \sqrt{\sqrt{3}^2+1^2} = 2$$

$$z = 2\cdot\left(\frac{i+\sqrt{3}}{2}\right) = 2\cdot e^{i\theta} \qquad \vartheta = \frac{\pi}{6}$$

$$\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right) = (\cos\theta, \sin\theta)$$



$$z = 2 \cdot e^{i\pi/6} \Rightarrow z^{2020} = 2^{2020} \cdot e^{i \cdot \left(\frac{2020\pi}{6}\right)} = 2^{2020} e^{i \cdot \frac{2}{3}\pi}.$$

$$\frac{2020\pi}{6} = 2k\pi + r$$

$$\parallel$$

$$\frac{2016\pi + 4\pi}{6} = 168 \cdot 2\pi + 2/3\,\pi$$

$$z = 2^{2020}\left(\cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right)\right).$$

Oss $\quad |z_1 \cdot z_2| = |z_1| \cdot |z_2|$

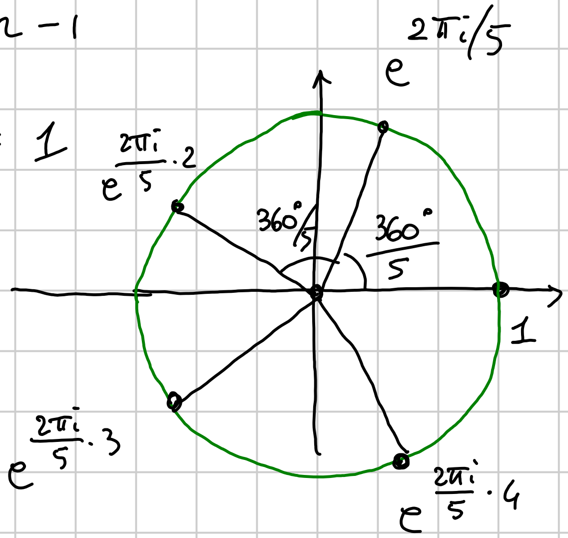
$$z_1 = |z_1| \cdot e^{i\theta_1} \qquad z_2 = |z_2| \cdot e^{i\theta_2} \qquad z_1 \cdot z_2 = |z_1| \cdot |z_2| \cdot e^{i(\theta_1+\theta_2)}$$

Radici dell'unità

Radice n-esima dell'unità: numero z t.c. $z^n = 1$

$$z = e^{i \cdot \frac{2\pi}{n} \cdot k} \qquad k = 0, 1, \ldots, n-1$$

$$z^n = \left(e^{\frac{2\pi i k}{n}}\right)^n = \left(e^{2\pi i}\right)^k = 1$$



Polinomi

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$$

con $a_i \in \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \dots$

$$x^2 - 3x + 2 = (x-1)(x-2) \qquad [\text{Radici } 1, 2]$$

Def Le RADICI di un polinomio $p(x)$ sono quei numeri α t.c. $p(\alpha) = 0$

Ruffini (regola) $\quad x^3 + 3x + 4 = p(x), \quad p(-1) = 0$

$$x^3 + 3x + 4 = (x+1) \cdot (x^2 - x + 4)$$

coeff. x^2

$$\begin{array}{c|ccc|c} & 1 & 0 & 3 & 4 \\ -1 & \downarrow & -1 & 1 & -4 \\ \hline & 1 & -1 & 4 & / \end{array}$$

Divisione con resto: $\quad p(x) = q(x)\,s(x) + r(x)$

Dati $p(x)$ e $q(x)$ esistono $s(x)$ e $r(x)$ come sopra

con $\text{grado}(r(x)) < \text{grado}(q(x))$

(Esiste se i coeff. sono $\mathbb{Q}, \mathbb{R}, \mathbb{C}$, ma in generale non $\mathbb{Z}$)

$x^2 + 1$ diviso $2x$ → non e' in generale a coeff. interi

$$x^2 + 1 = 2x \left(\tfrac{1}{2} x \right) + 1$$

Teo (Ruffini) $\quad p(x)$ diviso $x - a$

$$p(x) = q(x) \cdot (x-a) + r \quad \text{numero}$$
$$\Downarrow$$
$$p(a) = q(a) \cdot \cancel{(a-a)} + r$$

$\Rightarrow$ resto della divisione è $p(a)$

Teo (fondam. dell'algebra)

Sia $p(x)$ un polinomio a coeff in $\mathbb{C}$.

Allora esistono $\alpha_1, \dots, \alpha_m \in \mathbb{C}$ e $c \in \mathbb{C}$ t.c.

$$p(x) = c\,(x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_m)$$

Es. $x^n - 1$ in $\mathbb{R}$ ha ≤ 2 radici

$$x^n - 1 = (x-1)\left(x - e^{\frac{2\pi i}{n}}\right)\left(x - e^{4\pi i/n}\right)\dots\left(x - e^{\frac{2\pi i}{n}(n-1)}\right)$$

Es $x^2 - 1 = (x+1)(x-1)$

$2x^2 - 2 = 2(x+1)(x-1)$

Oss ① Se $p(x)$ ha coeff. reali e $z \in \mathbb{C}$ è radice,

allora $\bar{z}$ è ancora radice

$$p(\bar{z}) = a_n \bar{z}^n + a_{n-1}\bar{z}^{n-1} + \dots + a_0$$
$$= \overline{a_n}\, \bar{z}^n + \dots + \overline{a_0}$$
$$= \overline{a_n z^n + a_{n-1} z^{n-1} + \dots + a_0}$$
$$= \overline{p(z)} = 0$$

② Se $p(x) = q(x)$ per infiniti valori di x

$\iff$ sono uguali coefficiente per coefficiente

In effetti: se $x_1, x_2, \dots, x_n$ sono tali che

$$p(x_1) = q(x_1), \dots, p(x_n) = q(x_n),$$

allora $p(x) - q(x)$ ha $x_1, x_2, \dots, x_n$ come radici

$$p(x) - q(x) = \underbrace{(x-x_1)(x-x_2)\cdots(x-x_n)\, r(x)}_{\text{grado } \geq n \ (\text{oppure } r(x)=0)}$$

Scegliendo $n >$ grado $p(x)$, grado $q(x)$ trovo che $r(x) = 0$. $\Rightarrow p(x) - q(x) = 0 \Rightarrow p(x) = q(x)$

Esercizio $x^{20200} - x^{2019}$ diviso x^2+x+1: che resto dà?

$$x^{20200} - x^{2019} = (x^2+x+1)\, q(x) + r(x)$$

con $r(x)$ di grado ≤ 1 $\qquad r(x) = ax + b$

Cerco $x^2+x+1 = 0$. $\iff x = \dfrac{-1 \pm \sqrt{-3}}{2}$

$$= \frac{-1 \pm i\sqrt{3}}{2} \qquad (x^3 = 1)$$

$$x^3 - 1 = (x-1)(x^2+x+1)$$

$$X^{20200} - X^{2019} = (x^2+x+1)\, q(x) + r(x) \qquad r(x) = ax+b$$

Sostituisco $x = \dfrac{-1+\sqrt{-3}}{2}$.

$$X^{3K} \cdot X - 1 = X - 1 = \frac{-3+\sqrt{-3}}{2} \qquad (\text{lato sx})$$

$$20200 = 3K + 1$$

$$a \cdot x + b = a\left(\frac{-1+\sqrt{-3}}{2}\right) + b \qquad (\text{lato dx})$$

Quindi voglio $\boxed{-\frac{3}{2} + \frac{\sqrt{3}}{2}\, i = \left(-\frac{a}{2} + b\right) + \frac{a}{2}\sqrt{3} \cdot i}$

$$\Leftrightarrow \begin{cases} -3/2 = -a/2 + b & \Rightarrow \quad b = -1 \\ a/2\, \sqrt{3} = 1/2\, \sqrt{3} & \Rightarrow \quad a = 1 \end{cases}$$

FORMULE DI VIÈTE / ESPRESSIONI SIMMETRICHE

$$(x-x_1)(x-x_2) = x^2 - (x_1+x_2)x + x_1x_2$$

$$(x-x_1)(x-x_2)(x-x_3) = x^3 - (x_1+x_2+x_3)x^2 + (x_2x_3 + x_1x_3 + x_1x_2)x - x_1x_2x_3$$

$$(x-x_1)(x-x_2)(x-x_3)(x-x_4) = x^4 + (x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4)x^2$$
$$-(x_1x_2x_3 + x_1x_2x_4 + x_2x_3x_4 + x_1x_3x_4)x + x_1x_2x_3x_4$$
$$- x^3(x_1+x_2+x_3+x_4)$$

In generale:

$$(x-x_1)\cdots(x-x_n) = x^n - (\text{somma } x_i)\, x^{n-1} +$$
$$+ (\text{somma prodotti a 2 a 2})\, x^{n-2} - (\text{somma prodotti a 3 a 3})\, x^{n-3}$$
$$+ \dots + (-1)^n x_1 x_2 \dots x_n$$

Esempio $p(x) = x^3 + 3x + 5$, $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$ radici

Quanto vale $\alpha_1^2 + \alpha_2^2 + \alpha_3^2$

$\alpha_3^2 + \alpha_2^2 + \alpha_1^2$

$a^2b + b^2c + c^2a$ non è simmetrica: $a \leftrightarrow b$

$b^2a + a^2c + c^2b$

ma è "ciclica": $a \to b \to c \to a$

$b^2c + c^2a + a^2b$

$a^2b + b^2c + c^2a + b^2a + c^2b + a^2c$

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = \underbrace{(\alpha_1+\alpha_2+\alpha_3)}_{-\text{coeff } x^2}{}^2 - 2\underbrace{(\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1)}_{\text{coeff. di } x}$$

$$= 0^2 - 2 \cdot 3 = -6$$

$$\alpha_1^3 + \alpha_2^3 + \alpha_3^3 = (\alpha_1 + \alpha_2 + \alpha_3)^3 - 6\alpha_1\alpha_2\alpha_3 - 3(\alpha_1^2\alpha_2 + \alpha_1^2\alpha_3 + \alpha_2^2\alpha_1 + \alpha_2^2\alpha_3 + \alpha_3^2\alpha_1 + \alpha_3^2\alpha_2)$$

$$(\underline{\alpha_1} + \alpha_2 + \alpha_3)(\alpha_1 + \alpha_2 + \alpha_3)(\alpha_1 + \alpha_2 + \alpha_3)$$

$\sum_{sym} \alpha_1^2 \alpha_2$ = somma di $\alpha_1^2\alpha_2$ e tutte le espr. che ottengo rinominando gli indici

$$\sum_{sym} a^3 b = a^3b + a^3c + b^3a + c^3b + b^3c + c^3a$$

$$\sum_{sym} a^3 = 2(a^3 + b^3 + c^3)$$

$$\alpha_1^3 + \alpha_2^3 + \alpha_3^3 = (\alpha_1 + \alpha_2 + \alpha_3)^3 - 6\alpha_1\alpha_2\alpha_3 - 3\sum_{sym} \alpha_1^2\alpha_2$$

$$\sum_{sym} \alpha_1^2\alpha_2 = (\alpha_1 + \alpha_2 + \alpha_3)(\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1) - 3\alpha_1\alpha_2\alpha_3$$

$\alpha_1 + \alpha_2 + \alpha_3$ $\qquad$ $\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1$ $\qquad$ $\alpha_1\alpha_2\alpha_3$

(Una) disuguaglianza

$a_1, a_2, \dots, a_n$ reali ≥ 0

Media (aritmetica) : $AM = \frac{a_1 + \dots + a_n}{n}$

Media (geometrica) : $GM = \sqrt[n]{a_1 a_2 \dots a_n}$

Teo $AM \geq GM$ (sempre)

$$\frac{a_1 + a_2}{2} \geqslant \sqrt{a_1 a_2} \iff \frac{a_1^2 + 2a_1 a_2 + a_2^2}{4} \geqslant a_1 a_2$$

$$\iff a_1^2 + 2a_1 a_2 + a_2^2 \geqslant 4 a_1 a_2$$

$$\iff a_1^2 - 2a_1 a_2 + a_2^2 \geqslant 0 \iff (a_1 - a_2)^2 \geqslant 0$$

PROBLEMI CONSIGLIATI

- A2, A3, A4, A6
- Sia $p(x)$ un polinomio di grado 99 t.c.

$$p(k) = \frac{1}{k} \quad \text{per} \quad k = 1, \dots, 100. \text{ Calcolare } p(101)$$

$\boxed{A2}$ a, b, c, d interi $\Rightarrow$ esistono x, y interi t.c.

$$(a^2+b^2)(c^2+d^2) = x^2+y^2$$

$z = a+bi$ $\qquad |z| = \sqrt{a^2+b^2}$ $\qquad u = x+iy$

$w = c+di$ $\qquad |w| = \sqrt{c^2+d^2}$

$$|z|^2 \cdot |w|^2 = |u|^2 \iff \underbrace{|z| \cdot |w|}_{= |z \cdot w|} = |u|$$

Scelgo

$$u = z \cdot w = (a+bi)(c+di) = (ac-bd) + (ad+bc)i$$

Mi viene in mente che $(a^2+b^2)(c^2+d^2) = (ac-bd)^2 + (ad+bc)^2$

$\boxed{A3}$ x_1, x_2, x_3, x_4 radici di $x^4+ax+b=0$. Calcolare

$$x_1^4+x_2^4+x_3^4+x_4^4$$

$$x_1^4 + a x_1 + b = 0 \quad \Rightarrow \quad x_1^4 = -ax_1 - b$$

$$x_2^4 = -ax_2 - b$$

$$\vdots$$

$$x_1^4 + \dots + x_4^4 = -a \underbrace{(x_1+x_2+x_3+x_4)}_{0} - 4b$$

$$= -4b$$

$\boxed{A4}$ $P(p) = P(1/p) = 0$ per i primi 21 n^i primi dispari
$\deg P = 42$

$$P(x) = c \cdot \underbrace{(x-3)\left(x-\tfrac{1}{3}\right)}\ \underbrace{(x-5)\left(x-\tfrac{1}{5}\right)} \cdots \left(x - p_{21}\right)\left(x - \tfrac{1}{p_{21}}\right)$$

$$\frac{(x-q)\left(x-\tfrac{1}{q}\right)}{(x-q)\left(x-1/q\right)} \quad \begin{array}{l} \text{per } x = 2 \\ \text{per } x = 1/2 \end{array} \quad \begin{array}{l} (2-q)\left(\frac{2q-1}{q}\right) \\ \left(\frac{1-2q}{2}\right)\left(\frac{q-2}{2q}\right) \end{array}$$

$$\frac{\cancel{(2-q)}\,\cancel{(2q-1)}\,/\,\cancel{q}}{\cancel{(1-2q)}\,\cancel{(q-2)}\,/\,4\cancel{q}} = 4$$

$$\Rightarrow \quad P(2)/P(1/2) = 4^{21}$$

$\boxed{A6}$ $a, b, c \in \mathbb{R} \Rightarrow a^2+b^2+c^2 \geq ab+bc+ca$

$$\underbrace{a^2+b^2+c^2 - (ab+bc+ca)} \geq 0$$

Candidati quadrati:

- $a+b-2c$
- $a-b \rightsquigarrow$

$$\begin{aligned} (a-b)^2 &= a^2 - 2ab + b^2 \geq 0 \\ (b-c)^2 &= b^2 - 2bc + c^2 \geq 0 \\ (c-a)^2 &= c^2 - 2ca + a^2 \geq 0 \\ \hline & 2a^2+2b^2+2c^2 - 2(ab+bc+ca) \geq 0 \end{aligned}$$

Sia $P(x)$ di grado 99, $P(1) = 1/1$, $P(2) = 1/2$, ..., $P(100) = 1/100$

Calcolare $P(101)$

Se $P_1(x)$ e $P_2(x)$ hanno questa proprietà,

$Q(x) = P_1(x) - P_2(x)$ è di grado ≤ 99 e ha

≥ 100 radici $\Rightarrow Q(x)$ è $0 \Rightarrow P_1 = P_2$

$P(x) = 1/x \iff x P(x) = 1$ per $x = 1, \dots, 100$

$R(x) = x P(x) - 1$ è polinomio di grado 100 con radici $1, 2, \dots, 100$

$$x P(x) - 1 = R(x) = c \cdot (x-1)(x-2) \dots (x-100)$$

Metto $x = 0 \Rightarrow -1 = c \cdot (-1)(-2) \dots (-100)$

$$\Rightarrow c = \frac{-1}{100!}$$

$$P(x) = \frac{-\frac{1}{100!}(x-1)\dots(x-100) + 1}{x}$$

$$P(101) = \frac{-\frac{1}{100!}\,100! + 1}{101} = 0$$

$\boxed{\text{A3, reprise}}$ $\quad e_1 = x_1 + x_2 + x_3 + x_4 = 0 \qquad x^4 + ax + b = 0$

$e_2 = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_3x_4 = 0$

$e_3 = x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4$

$e_4 = x_1x_2x_3x_4$

$\cancel{e_1^4}, \ \cancel{e_1^2 \cdot e_2}, \ \cancel{e_1 \cdot e_3}, \ \cancel{e_2^2}, \ e_4$

Risposta: $\underset{-4}{k} \cdot b \qquad x^4 + 1 = 0 \qquad x^4 = -1$