ALGEBRA MATTUTINA (PREIMO)

Titolo nota 28/05/2007

$$\frac{x(y-z)}{y+z}+\frac{y(z-x)}{z+x}+\frac{z(x-y)}{x+y}=0$$

$$\sum_{cyc} x(y-z)(x+z)(x+y)=0$$

$$\sum_{cyc}(xy-xz)(x^2+xz+xy+yz)=0$$

$$\sum_{cyc} x^3y-x^3z+\cancel{x^2yz}-x^2z^2+x^2y^2-\cancel{x^2yz}+xy^2z-xyz^2=0$$

$$\sum_{cyc} x^3(y-z)=0$$

$$\underline{x^3(y-z)+y^3(z-x)+z^3(x-y)}=0$$

$$(x-y)(y-z)(x-z)(x+y+z)=0$$

$P(x)$ $\qquad P(a)=0 \iff x-a \mid P(x)$

coeff. monomi in x e y
variabile z

$P(z)$ $\qquad z \to y$ $\qquad$ divisibile per $y-z$

$x-y \qquad x-z$

$$\mathbb{Z}[x,y,z] = \left(\mathbb{Z}[x,y]\right)[z]$$

$P(x,y,z,u,v)$ $\qquad P(x,x,z,u,v)=0 \quad \forall z,u,v$

$=$

$(y-x)\cdot$ qualcosa

$$\sum_{cyc} x^2y^2 = \sum_{cyc} x^2z^2$$

$$\sum_{cyc} (x^2y^2 - x^2z^2) = 0$$

$\boxed{\text{SI}}$ in 3 variabili

$$\sum_{cyc} x^2y = \sum_{cyc} x^2z$$

$\boxed{\text{NO}}$

$$(x-y)(y-z)(z-x) = -x^2y - y^2z - z^2x + xy^2 + yz^2 + zx^2$$

$$= \sum_{cyc} xy^2 - x^2y = \sum_{cyc} xy(y-x)$$

a) $a_1 + \ldots + a_n \geq n^2$

b) $a_1^2 + \ldots + a_n^2 \geq n^3 + 2$

i) $a_1 + \ldots + a_n \geq n^2 + 1$

$$\sqrt{\frac{a_1^2 + \ldots + a_n^2}{n}} \geq \frac{a_1 + \ldots + a_n}{n} \geq n + \frac{1}{n}$$

$$a_1^2 + \ldots + a_n^2 \geq n\left(n^2 + 2 + \frac{1}{n^2}\right) > n^3 + 2$$

$$(n + x_1, \dots, n + x_n) \qquad x_i \in \mathbb{Z} \qquad \sum x_i = 0$$

$$(n+x_1)^2 + \dots + (n+x_n)^2 \le n^3+2$$

$$\sum (n+x_i)^2 \le n^3+2$$

$$\sum \left(n^2 + 2n x_i + x_i^2 \right) \le n^3+2$$

$$\cancel{\sum n^2} + \cancel{2n \sum x_i} + \sum x_i^2 \le \cancel{n^3}+2$$

$$\sum x_i^2 \le 2$$

$$\left(n-1,\ n,\ \ldots,\ n,\ n+1 \right)$$

$$(n, \ldots, n)$$

$$a_i = n + x_i$$

$$n^2 \leq \sum a_i = \sum (n + x_i) = \sum n + \sum x_i = n^2 + \sum x_i$$

$$\Rightarrow \sum x_i \geq 0$$

$$\sum_{cyc} \frac{a^2}{b} \geq 3(a^2+b^2+c^2)$$

$$(a+b+c)\sum_{cyc} a^3 c \geq 3\sum_{cyc} a^3 bc$$

$$\sum_{cyc} a^4 c + \cancel{\sum_{cyc} a^3 bc} + \sum_{cyc} a^3 c^2 \geq \overset{2}{\cancel{3}} \sum_{cyc} a^3 bc$$

$$2\sum_{cyc} a^4 c + 2\sum a^3 c^2 \geq 4\sum a^3 bc$$

$$\sum a^4 c \geq \sum a^3 bc \qquad w_1 = \frac{9}{13} \qquad w_2 = \frac{3}{13}$$

$$\sum a^4 c + 2\sum a^3 c^2 \geq 3\sum a^3 bc \qquad w_3 = \frac{1}{13}$$

$$LHS = \sum_{cyc}(a^4 c + a^3 c^2 + b^3 a^2) \geq 3\sum a^3 bc \quad \text{AM-GM}$$

$$\underbrace{13\,a^4c}_{\omega_1} + \underbrace{13\,b^4a}_{\omega_2} + \underbrace{13\,c^4b}_{\omega_3} \geq 13\,a^3bc + 13\,b^3ca + 13\,c^3ab$$

$$\frac{9a^4c + 3b^4a + 1\,c^4b}{13} \geq \sqrt[13]{a^{4\cdot 9+3}\cdot b^{3\cdot 4+1}\cdot c^{9+4}}$$

$$= \sqrt[13]{a^{39}\,b^{13}\,c^{13}} = 13\,a^3bc$$

$$\underset{A}{a^4c} + \underset{B}{b^4a} + \underset{C}{c^4b} \rightsquigarrow a^3bc$$

$4A + B = 3(A+B+C)$ ← esponente di a

$4B + C = 1(A+B+C)$ ← " " b

$4C + A = 1(A+B+C)$ ← " " c

(A, B, C) sol. $\Rightarrow$ $(\lambda A, \lambda B, \lambda C)$

$$\sum_{cyc} a^4 b \geq \sum_{cyc} a^3 b c$$

$$a^4 b + b^4 c + c^4 a =$$

$$a^3 \cdot ab + b^3 \cdot bc + c^3 \cdot ca \geq a^3 \cdot bc + b^3 \cdot ac + c^3 \cdot ab$$

$a \geq b \geq c$ $a \geq c \geq b$

$ab \geq ac \geq bc$

$$\sum_{cyc} a^3 b^2 \geq \sum_{cyc} a^3 b c$$

$$f(x+f(y)) = f(x)+y+7$$

$$f: \mathbb{Q} \to \mathbb{Q}$$

$$g(x) = f(x) - 7$$

$$g(x+g(y)+7) = g(x)+y+7$$

$$x = -7$$

$$g(g(y)) = g(-7)+y+7$$

g è iniettiva

$$y = g(t)$$

$$g(x + g(y(t)) + 7) = g(x) + g(t) + 7$$

$$g(x + t + g(-7) + 14) = g(x) + g(t) + 7$$

$$14 + g(-7) = k$$

$$7 = h$$

$$g(x + t + k) = g(x) + g(t) + h$$

$$t + k = z \quad \Rightarrow g(x + z) = g(x) + g(z - k) + h$$

$t = c$

$g(x+c) = g(x) + g(c-k) + h$

$s + g(x+z) = g(x) + g(z) + s + s$

$s + g(x) = h(x)$

$h(x+y) = h(x) + h(y)$

$h : Q \to Q$

$$h(x) = \lambda x$$

$$\rightarrow \quad f(x) = \lambda x + \kappa$$

$$f(x) = -x - 7$$

$$f(x) = x + 7$$

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq 3\left(a^2+b^2+c^2\right)$$

$$\underbrace{\frac{a^2}{a^2+b^2+c^2}}_{\lambda_1} \cdot \frac{1}{b} + \underbrace{\frac{b^2}{a^2+b^2+c^2}}_{\lambda_2} \cdot \frac{1}{c} + \underbrace{\frac{c^2}{a^2+b^2+c^2}}_{\lambda_3} \cdot \frac{1}{a} \geq 3$$

$$\lambda_1 + \lambda_2 + \lambda_3 = 1$$

$$\lambda_1 f(b) + \lambda_2 f(c) + \lambda_3 f(a) \geq f\left(\lambda_1 b + \lambda_2 c + \lambda_3 a\right) \qquad f(x) = \frac{1}{x} \leftarrow \text{convesse}$$

$$= \frac{1}{\frac{a^2 b}{a^2+b^2+c^2} + \frac{b^2 c}{\cdots} + \frac{c^2 a}{\cdots}} = \frac{a^2+b^2+c^2}{a^2b+b^2c+c^2a} \underset{\substack{\uparrow \\ \text{HOPE}}}{\geq} 3$$

$$\left(a^2+b^2+c^2\right) \overset{?}{\geq} 3\left(a^2b+b^2c+c^2a\right)$$

$$(a+b+c)$$

$$a^3+b^3+c^3 + \cancel{\sum_{cyc} a^2 b} + \sum_{cyc} ab^2 \overset{?}{\geq} \overset{2}{\cancel{3}} \sum_{cyc} a^2 b$$

$$\sum_{cyc} (a^3+ab^2) \geq \sum_{cyc} 2\sqrt{a^4b^2} = 2\sum_{cyc} a^2b.$$

— o — o —

$$f(x+f(y)) = f(x)+y+7 \qquad f:\mathbb{Q}\to\mathbb{Q}$$

$\boxed{\text{CLASSICA}}$ $\quad f(x+f(y)) = f(x)+y$

$\boxed{x=0}$ $\quad f(f(y)) = y+f(0) \implies f$ iniett. + surg.

$\boxed{y=0}$ $\quad \cancel{f}(f(0)) = 0+f(0) = \cancel{f}(0) \implies f(0)=0$

$\cancel{f}(a) = \cancel{f}(b) \implies a=b$

DUNQUE $\quad f(f(y)) = y$

$y=f(z)$ $\quad f(x+f(f(z))) = f(x)+f(z)$

$f(x+z) = f(x)+f(z) \qquad \forall x\in\mathbb{Q},\ \forall z\in\mathbb{Q}$

CAUCHY

$\Rightarrow \quad f(x) = \lambda x \qquad \forall x \in \mathbb{Q}$ [No $\mathbb{R}$]

$\lambda = f(1)$

Sostituisco nell' eq. iniziale e trovo i λ per cui va bene

— o — o —

$$f(x+f(y)) = f(x) + y + 7$$

$\boxed{x=0}$ $\quad f(f(y)) = y + f(0) + 7 \quad \Rightarrow$ iniett. + surg.

$\boxed{y=0}$ $\quad f(x+f(0)) = f(x) + 7$

$\boxed{y=f(z)}$ $\quad f(x+f(f(z))) = \boxed{f(x) + f(z) + \cancel{7}}$

$$= f(\boxed{x+z+7} + f(0))$$

$$= \boxed{f(x+z+7) + \cancel{7}}$$

$$f(x+y+k) = f(x) + f(y)$$

$$f(x) = \lambda(x+k)$$

Pongo $f(x) = g(x+k)$

$$g(x+k+y+k) = g(\underbrace{x+k}_{u}) + g(\underbrace{y+k}_{v})$$

$$g(u+v) = g(u) + g(v)$$

g risolve la CAUCHY

$g(x) = \lambda x \quad \forall x \in \mathbb{Q}$

$\boxed{\text{ANALOGA}}$

$$f(x+y)^{+k} = f(x) + f(y)^{+k} + k$$

$$\underbrace{f(x+y)+k}_{g(x+y)} = \underbrace{f(x)+k}_{g(x)} + \underbrace{f(y)+k}_{g(y)}$$

$\Rightarrow g(x) = \lambda x$

$f(x) + k = \lambda x$

ANCORA

$$f(x+y+k) = f(x) + f(y) + k$$

$$f(x+y+k) + k = f(x) + k + f(y) + k$$

$g(x) = f(x) + k$

$$g(x+y+k) = g(x) + g(y)$$