PRE-IMO 2009 ALGEBRA POMERIGGIO

Titolo nota 25/05/2009

$\forall p, \quad \forall u, v$ t.c. $p \mid uv-1$

$$p \mid f(u)f(v)-1$$

Trovare gli f

Idea 1) $u \equiv v^{-1} \mod p$

$$\Downarrow$$

$$f(u) \equiv [f(v)]^{-1} \mod p$$

$u \cdot v$

$f(x)f(\underline{x^{-1}})x^n$ come polinomio $\mathbb{Z}[p]$

$n =$ grado di f

$$x^n \cdot f(x) \cdot f(x^{-1}) \equiv x^n \quad \mathbb{Z}[p]$$

?? COSA VUOL DIRE?

comunque scelgo $x \in \mathbb{F}_p$

$$X^n \cdot f(X) \cdot f(X^{-1}) - X^n = 0 \quad (\text{in } \mathbb{F}_p)$$

$X^p - X$ anche lui si annulla sempre!

Se scelgo $p >> n$

<u>dim</u>

Il polinomio

$$\underbrace{X^n \cdot f(X) \cdot f(X^{-1}) - X^n}_{\text{ha grado } 2n \text{ ("piccolo")}} = \underbrace{q(X) \cdot X \cdot (X-1)(X-2) \cdots (X-p+1)}_{\text{ha grado } \geq p \text{ se } q(X) \neq 0}$$

Dev'essere $q(X) \equiv 0 \pmod p$

<u>cioè</u>

Tutti i coefficienti di

$$X^n f(X) f(X^{-1}) - X^n$$

sono multipli di p

per ogni p abbastanza grosso

$$f(x) f(x^{-1}) = 1$$

idea <u>successiva</u>

f dev' essere omogeneo

se non è omogeneo,
ho termini di grado diverso in

$f(x)$ e $f(x^{-1})$ $\quad a > b$, $c > d$

$(x^a + x^b) \quad (x^c + x^d)$

$\Downarrow$

$x^{a+c} \qquad x^{b+d}$

chi sono i $p(x)$ omogenei?

Solo i monomi

+ VERIFICA $\square$

per quali a, n
$a x^n$ verifica l'ipotesi?

u, v inversi mod p

Tq $\iff$ $1 \equiv a\underline{u}^n \cdot a\underline{v}^n = a^2$

$a^2 \equiv 1 \mod p$ per tutti i p

OK se e solo se $a = \pm 1$

cioè vanno bene tutti e soli $f(x) = \pm x^n$

$$\underbrace{f(x)}_{deg = n} \underbrace{f(x^{-1})\, x^n}_{deg \le n} = x^n$$

(6) Trovare $u : \mathbb{R} \to \mathbb{R}$ t.c. $\exists f : \mathbb{R} \to \mathbb{R}$ strettamente monotona

$$f(x+y) = f(x)\,u(y) + f(y)$$

$$\cancel{f(0)} = f(0)\,u(0) + \cancel{f(0)}$$

$f(0) \neq 0 \Rightarrow u(0) = 0$ $\qquad f(x) = f(x)\cdot 0 + f(0)$

$\Rightarrow f(x) = f(0) \quad \forall x$ ✱

$$\boxed{f(0) = 0}$$

$0 = f(0) = f(x)\,u(-x) + f(-x)$ $\qquad u(-x) = \dfrac{-f(-x)}{f(x)}$

$$\boxed{u(x) = \frac{-f(x)}{f(-x)}}$$

$$f(x+y) = -f(x)\frac{f(y)}{f(-y)} + f(y)$$

$$f(x)\left\{1 - \frac{f(y)}{f(-y)}\right\} = f(y)\left\{1 - \frac{f(x)}{f(-x)}\right\}$$

$$\frac{f(x)}{f(y)} - \frac{f(x)}{f(-x)} = \frac{f(x)}{f(x)} - \frac{f(x)}{f(-y)} \qquad \forall x,y$$

$$x \neq 0 \qquad \frac{1}{f(y)} + \frac{1}{f(-y)} = \boxed{\frac{1}{f(x)} + \frac{1}{f(-x)} = k}$$

$$\frac{-f(y)}{f(y)} f(x) + f(x)\, u(y) = \frac{-f(y)}{f(-y)} f(x) + \frac{-f(y)}{f(y)} f(x) =$$

$$= -k f(x) f(y)$$

$$f(x+y) - f(y) - f(x) =$$

$$\boxed{f(x+y) = f(x) + f(y) - k f(x) f(y)}$$

$$1 - k f(x+y) = 1 - k f(x) - k f(y) + k^2 f(x) f(y)$$

$$= \big(1 - k f(x)\big)\big(1 - k f(y)\big)$$

$$\boxed{g(x) := 1 - k f(x)} \qquad g(x+y) = g(x)\, g(y)$$

$$\boxed{h(x) := \log g(x)} \qquad \boxed{h(x+y) = h(x) + h(y)}$$

$$h(x) = x\, h(1) = \lambda x$$ tutte e sole soluz in $\mathbb{Z}$ e $\mathbb{Q}$

Per chiudere: 1) a mano

2) h continua in un punto
3) h limitata su un intervallo
4) h monotona

☆ Per fare il log ci vuole $g > 0$ $\quad g(x) = g\left(\frac{x}{2} + \frac{x}{2}\right) = \left[g\left(\frac{x}{2}\right)\right]^2 \geq 0$

$$f(x+y+z)=f(x+(y+z))=f(x)u(y+z)+f(y+z)$$

$$=f((x+y)+z)=f(x+y)u(z)+f(z)$$

$$\cancel{f(x)}u(y+z)+\cancel{f(y)u(z)}+\cancel{f(z)}=\cancel{f(x)}u(y)u(z)+\cancel{f(y)u(z)}+\cancel{f(z)}$$

$$\boxed{u(y+z)=u(y)u(z)}$$

7 $$(ab)^{1/3}+(cd)^{1/3}\leq(a+c+b)^{1/3}(a+c+d)^{1/3}$$

$a,b,c,d\geq 0$

- Con Hölder

$$\sum a_ib_i\leq\sqrt{\sum a_i^2}\sqrt{\sum b_i^2}$$

$$\sum_{i=1}^{2}a_ib_ic_i\leq\sqrt[3]{\sum a_i^3}\sqrt[3]{\sum b_i^3}\sqrt[3]{\sum c_i^3}$$

HW: dimostrare con Hölder

$$\frac{a^{\frac{1}{3}}}{x}\cdot\frac{b^{\frac{1}{3}}}{y}\cdot xy+\frac{c}{z}^{\frac{1}{3}}\frac{d^{\frac{1}{3}}}{t}\cdot zt$$

$$\leq\sqrt[3]{\frac{a}{x^3}+\frac{d}{t^3}}\sqrt[3]{\frac{b}{y^3}+\frac{c}{z^3}}\sqrt[3]{x^3y^3+z^3t^3}$$

$$\sqrt[3]{a+c+d}\sqrt[3]{a+c+b}$$

$$\sqrt[3]{a\frac{a+c}{a}+\frac{d}{1}}\sqrt[3]{c\frac{a+c}{c}+\frac{b}{1}}\overbrace{\sqrt{\frac{a}{a+c}1+\frac{c}{a+c}}}^{1}$$

- Con AM-GM al contrario

$$3\sqrt[3]{xAyB\frac{1}{xy}} = xA+yB+\frac{1}{xy} = x(a+c+b)+y(a+c+d)+\frac{1}{xy}$$

$\exists x,y$ t.c. $=$

$$\frac{1}{(x+y)}\cdot\frac{x+y}{xy}$$

$$= x(a+c+b)+\frac{1}{x(x+y)} + y(a+c+d)+\frac{1}{y(x+y)}$$

$$= (x+y)a + xb + \frac{1}{x(x+y)} + (x+y)c + yd + \frac{1}{y(x+y)}$$

$$\geq 3\sqrt[3]{ab} + 3\sqrt[3]{cd}$$

$$\sqrt{\sum a_i^2}\sqrt{\sum b_i^2} = \frac{x\left(\sum a_i^2\right)+\frac{1}{x}\left(\sum b_i^2\right)}{2} =$$

$$= \sum \frac{x a_i^2 + \frac{1}{x} b_i^2}{2} \geq \sum a_i b_i$$

- $\sqrt{A}+\sqrt{B} \leq \sqrt{C}$ $\qquad \sqrt{\frac{A}{C}} + \sqrt{\frac{B}{C}} \leq 1$

$$\sqrt[3]{\frac{ab\ ?}{(a+c+b)(a+c+d)\ ?}} + \sqrt[3]{\frac{cd}{(a+c+b)(a+c+d)}} \leq 1$$

$$\sqrt[3]{\frac{ab(a+c)}{(a+c+b)(a+c+d)(a+c)}} + \sqrt[3]{\frac{cd(a+c)}{(a+c+b)(a+c+d)(a+c)}} \overset{!}{\leq} 1$$

$$\leq \frac{\frac{b}{a+c+b} + \frac{a+c}{a+c+d} + \frac{a}{a+c} + \frac{a+c}{a+c+b} + \frac{d}{a+c+d} + \frac{c}{a+c}}{3} = 1$$

$$a^{\frac{1}{3}} b^{\frac{1}{3}} + c^{\frac{1}{3}} d^{\frac{1}{3}} \leq (\underbrace{a+c}_{S}+b)^{\frac{1}{3}} (\underbrace{a+c}_{S}+d)^{\frac{1}{3}}$$

Fissati b, d, S, quando è max il LHS

$$\mathrm{Max} \left\{ a^{\frac{1}{3}} A + c^{\frac{1}{3}} B \ : \ a+c = S \right\}$$

Se uno è disperato, si usa l'analisi.
In alternativa: Hölder

$$\sum a_i b_i \leq \left(\sum a_i^p\right)^{\frac{1}{p}} \left(\sum b_i^q\right)^{\frac{1}{q}} \qquad \frac{1}{p}+\frac{1}{q} = 1$$

$$\frac{1}{3} + \frac{2}{3} = 1 \qquad p = 3 \qquad q = \frac{3}{2}$$

$$a^{\frac{1}{3}} A + c^{\frac{1}{3}} B \leq (a+c)^{\frac{1}{3}} \left(A^{\frac{3}{2}} + B^{\frac{3}{2}} \right)^{\frac{2}{3}}$$
$$= S^{\frac{1}{3}} \left(A^{\frac{3}{2}} + B^{\frac{3}{2}} \right)^{\frac{2}{3}}$$

$$\mathrm{LHS} = a^{\frac{1}{3}} b^{\frac{1}{3}} + c^{\frac{1}{3}} d^{\frac{1}{3}} \leq S^{\frac{1}{3}} \left(\sqrt{b}+\sqrt{d}\right)^{\frac{2}{3}} \overset{\text{Hope}}{\leq} (S+b)^{\frac{1}{3}} (S+d)^{\frac{1}{3}}$$

$$S \left(\sqrt{b}+\sqrt{d}\right)^2 \overset{?}{\leq} S^2 + \cancel{Sb} + \cancel{Sd} + bd$$
$$\cancel{Sb} + \cancel{Sd} + 2S\sqrt{bd} \qquad \text{AM-GM.}$$

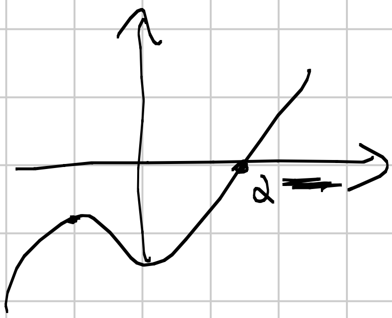
Quando si ha $=$?

$$(a,c) = \lambda (A,B) \qquad = \text{ in Hölder}$$
$$S = \sqrt{bd} \qquad = \text{ in AM-GM.}$$

$$\left(\sqrt[3]{ab}+\sqrt[3]{cd}\right)^3 = ab+cd+3\sqrt[3]{abcd}\left(\sqrt[3]{ab}+\sqrt[3]{cd}\right)$$

α

$$x^3-3\sqrt[3]{abcd}\cdot x-(ab+cd)$$



$$3x^2-3\sqrt[3]{abcd} \qquad x=-\sqrt[6]{abcd}$$

HOPE

$$P(RHS) \geq 0$$

$$(a+c+b)(a+c+d)-3\sqrt[3]{abcd(a+c+b)(a+c+d)}-ab-cd \geq 0$$

$$\frac{a^2+c^2+2ac+ad+bc+bd}{3} \geq \sqrt[3]{abcd(a+c+b)(a+c+d)}$$

$$\frac{(a+c+b)c+(a+c+d)a+bd}{3} \geq RHS$$

AM - GM

$$x^3-3\sqrt[3]{abcd}\,x-(ab+cd)$$

$$x=-\sqrt[6]{abcd}$$

$$P(\varepsilon)=2\sqrt{abcd}-(ab+cd)\leq 0 \qquad \text{AM - GM}$$

$ab=cd$ SI AGGIUSTA.

MI DICONO $P(0)<0$ E SI CAPISCE CHE QUESTO FACENDO IL DISEGNO!

(8) $a,b,c,d > 0 \qquad abcd = 1 \qquad a+b+c+d > \underbrace{\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}}_{x}$

$$a+b+c+d < \overbrace{\frac{b}{a}+\frac{c}{b}+\frac{d}{c}+\frac{a}{d}}^{y}$$

$$a = \sqrt[4]{a^4} = \sqrt{\frac{a^4}{abcd}} = \sqrt{\frac{a}{b}\cdot\frac{b}{c}\cdot\frac{a}{b}\cdot\frac{a}{d}}$$

$$\leq \left(\frac{a}{b}+\frac{a}{b}+\frac{b}{c}+\frac{a}{d}\right)\frac{1}{4}$$

$$\sum_{cyc} a \leq \frac{1}{4}\sum_{cyc}\left(3\frac{a}{b}+\frac{b}{a}\right) = \frac{3}{4}x + \frac{1}{4}y$$

$$\frac{3}{4}(a+b+c+d) + \frac{1}{4}(a+b+c+d) \leq \frac{3}{4}x + \frac{1}{4}y$$

$$\vee$$

$$\frac{3}{4}x + \frac{1}{4}(a+b+c+d)$$

$$a+b+c+d < y$$

$abc = 1$

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq \sqrt{a}+\sqrt{b}+\sqrt{c}$$

$$3\frac{a}{b}+2\frac{b}{c}+\frac{c}{a} \geq 6\sqrt[6]{\frac{a^3}{b^3}\cdot\frac{b^2}{c^2}\cdot\frac{c}{a}} = 6\sqrt[6]{a^3} = 6\sqrt{a}$$

$$2\frac{a}{b}+\frac{b}{c} \geq 3\sqrt[3]{a^3} = 3a$$

$$3\frac{a}{b} + 2\frac{b}{c} + \frac{c}{d} \geq 6\sqrt[6]{\frac{a^3}{bcd}} = 6\sqrt[6]{a^4} =$$

$$= 6a^{2/3}$$

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} \geq \sum_{cyc} a^{2/3}$$

$$2\frac{a}{b} + \frac{b}{c} + \frac{a}{d} \geq 4\sqrt[4]{\frac{a^3}{bcd}} = 4a$$