

TEOREMA DEI NUMERI - POMERIGGIO

PRELIM 2009

Titolo nota

28/05/2009

⑤ $m \geq 2$

$$A_m = \{m+1, 3m+2, 5m+3, 7m+4, \dots\}$$

$$A_m = \{x \in \mathbb{N} \mid x \equiv m+1 \pmod{2m+1}\}$$

Is. $\exists a \quad 1 \leq a \leq m$ t.c

$$2^a \in A_m \quad 2^{a+1} \in A_m$$

$$2^a \in A_m \Leftrightarrow 2^a \equiv m+1 \pmod{2m+1}$$

$$\Leftrightarrow 2^{a+1} \equiv 2m+2 \equiv 1 \pmod{2m+1}$$

$$\Leftrightarrow : \quad 2x \equiv 2y \pmod{2m+1} \Rightarrow x \equiv y \pmod{2m+1}$$

$$2m+1 \mid 2x-2y = 2(x-y) \quad (2m+1, 2)=1$$

$$\Rightarrow 2m+1 \mid x-y.$$

$$2^a + 1 \in A_m \quad 2^a + 1 \equiv m+1 \pmod{2m+1}$$

$$2^a \equiv m \pmod{2m+1}$$

$$2^{a+1} \equiv 2m \equiv -1 \pmod{2m+1}$$

$$\exists a \quad 1 \leq a \leq m \quad 2^a \in A_m \quad \vee \quad 2^a + 1 \in A_m$$

$$\exists a \quad 2 \leq a+1 \leq m \quad 2^{a+1} \equiv 1 \pmod{2m+1} \quad 2^{a+1} \equiv -1 \pmod{2m+1}$$

$$\text{ord}_{2m+1}(2) \leq m$$

OK

\rightarrow 2 genera tutto il gruppo $\mathbb{Z}/(2m+1)^\times$.

Quindi -1 appartiene al sottogruppo generato dalle potenze di 2

$$\text{ord}_{2m+1}(2) = h > m$$

$$h \leq 2m \quad (h \leq \phi(2m+1))$$

$$\exists k \nmid h \text{ s.t. } 2^k \equiv -1 \pmod{2m+1}$$

$$2^{2k} \equiv 1 \pmod{2m+1}$$

$$h \mid 2k$$

$$\text{ord}_{2m+1}(2) = \phi(2m+1) \text{ par} \quad (m \geq 2)$$

$$h = 2h_1$$

$$2h_1 \mid 2k$$

$$h_1 \mid k$$

$$S1: k_0 = \min\{k : 2^k \equiv -1\}$$

$$k_0 < h$$

Donner partie

$$h \mid 2k_0$$

$$\Rightarrow k_0 = h/2$$

$$h \leq 2m$$

$$\Rightarrow k_0 \leq m$$

(résulte la partie (1))

Case contr: $a+1=1$
 $a=0$

$$2 \leq a+1 \leq m$$

$$2 \equiv 1 \pmod{2m+1} \Rightarrow m=0 \quad (\text{imp.})$$

$$2 \equiv -1 \pmod{2m+1} \Rightarrow m=1$$

IMPOSIBLE

$$2^a \in A_m$$

$$2^{a+1} \equiv 1 \pmod{2m+1}$$

$$2^b \notin 1 \in A_m$$

$$2^{b+1} \equiv -1 \pmod{2m+1}$$

a_0, b_0 minimum

$$2(b_0+1) = a_0+1$$

6

$$r = k + \frac{1}{a_1} + \dots + \frac{1}{a_n}$$

n tale che tutti i razionali si ottengono
in questa forma per opportuni $k, a_1, \dots, a_n \in \mathbb{Z}$
 $a_i \neq 0$

NESUN n.

Per ogni n esistono interi intervalli di razionali
non rappresentabili.

Induciamo su n.

$$n=0$$

$$\alpha_0 = 1$$

$$\beta_0 = 2$$

$$n \Rightarrow n+1$$

α_n, β_n tale che nessun x

in $\alpha_n < x < \beta_n$ sia rappresentabile

Idea: Se prendo α'_n, β'_n

$$\alpha_n < \alpha'_n < \beta'_n < \beta_n$$

allora i numeri rappresentabili nell'intervallo
 $[\alpha'_n, \beta'_n]$ sono finiti.

P.es. $\beta_n - \alpha_n = d_n$

$$\alpha'_n = \alpha_n + \frac{d_n}{3}$$

$$\beta'_n = \beta_n - \frac{d_n}{3}$$

$$\alpha'_n \leq x \leq \beta'_n \quad x = k + \sum_{i=1}^{n+1} \frac{1}{a_i'}$$

$$1 \leq j \leq n+1$$

$$a_j' = k + \sum_{\substack{i=1 \\ i \neq j}}^{n+1} \frac{1}{a_i'}$$

Si hanno due possibilità.

$$\alpha' \leq \alpha_n$$

$$\alpha' \geq \beta_n$$

$$x \geq \alpha_n' = \alpha_n + \frac{d_n}{2}$$

$$x \leq \beta_n' = \beta_n - \frac{d_n}{2}$$

$$\frac{1}{\alpha_j} \geq \frac{d_n}{2}$$

$$\frac{1}{\alpha_j} \leq -\frac{d_n}{2}$$

$$|\alpha_j| \leq \frac{3}{d_n}$$

→ n° finite de possibilités.

In $[\alpha_{n+1}', \beta_{n+1}']$ as same in n° finite de rationnels représentables →

UN INTERVALLO $[\alpha_{n+1}, \beta_{n+1}]$ NON RAPPRESENTABILE.

$$r_i \rightarrow \sqrt{2}$$

$$i \in \mathbb{N}$$

$$r_i = k_i + \frac{1}{a_{i,2}} + \dots + \frac{1}{a_{i,n}}$$

$\underbrace{\hspace{1.5cm}}_{-2 \leq \cdot \leq 1}$

$$r_i \approx \frac{1}{\dots} + \frac{1}{a_{i,1}} + \dots + \frac{1}{\dot{a}_{i,n}}$$

⑦

$$\begin{aligned}
 \mathcal{L}^{(1)} &= 0, 0, 0, 0, 0, 0, - \\
 \mathcal{L}^{(2)} &= 1, 0, 0, 0, 0, 0, 0 \\
 \mathcal{L}^{(3)} &= 0, 2, 0, 0, 0, 0, - \\
 \mathcal{L}^{(4)} &= 1, 2, 0, 0, 0, 0, 0 \\
 \mathcal{L}^{(5)} &= 0, 1, 3, 0, 0, 0, 0 \\
 \mathcal{L}^{(6)} &= 1, 1, 3, 0, 0, 0, 0 \\
 \mathcal{L}^{(7)} &= 0, 0, 2, 4, 0, 0, 0 \\
 \mathcal{L}^{(8)} &= 1, 0, 2, 4, 0, 0, 0 \\
 \mathcal{L}^{(9)} &= 0, 2, 2, 4, 0, 0, 0 \\
 \mathcal{L}^{(10)} &= 1, 2, 2, 4, 0, 0, 0
 \end{aligned}$$

$S_{n,k}$ = somma dei primi k termini di $2^{(k)}$

Congettura: $\boxed{S_{n,k} \equiv n-1 \pmod{k+1}}$

Induzione su n .

$$n=1 \quad S_{n,k} \equiv 0 \pmod{k} \\ = 0$$

$$n \Rightarrow n+1$$

$$f(2^{(n)}) \begin{cases} > k \\ = i \leq k \end{cases} \quad \textcircled{A}$$

$$\textcircled{A} \quad S_{n+1,k} = S_{n,k} - k \equiv n-1-k \pmod{k+1} \quad = i \leq k \quad \textcircled{B} \\ \equiv n \pmod{k+1}$$

$$\textcircled{B} \quad S_{n+1,k} = S_{n,k} + i - (i-1) \equiv n-1+1 \equiv n \pmod{k+1}$$

$$2^{(n!)} = (x_1, x_2, x_3, \dots)$$

$$x_1 = S_{n!,1} \equiv n!-1 \pmod{2}$$

$$\equiv -1 \pmod{2}$$

$$\boxed{S_{n,k} \equiv n-1 \pmod{k+1}}$$

$n \geq 1$

$$\Rightarrow x_1 = 1$$

$$x_1 + x_2 = S_{n!,2} \equiv n!-1 \pmod{3}$$

$$x_i \leq i$$

$n \geq 2$

$$x_1 + x_2 \equiv 2 \pmod{3}$$

$$x_2 \equiv 1 \pmod{3}$$

$$\Rightarrow x_2 = 1$$

\dots

$$x_{n-1} \equiv 1 \pmod{n} \Rightarrow x_{n-1} = 1$$

$$x_1 + \dots + x_n = S_{n!,n} \equiv n!-1 \pmod{n+1}$$

$$x_1 = x_2 = \dots = x_{n-1} = 1$$

$$x_n \equiv n!-1 - (n-1) \pmod{n+1}$$

$$\equiv n!-n \equiv n!+1 \pmod{n+1}$$

Riduzione: $X_n = 0 \Leftrightarrow n! + 1 \equiv 0 \pmod{n+1}$

$n+1 = p$ primo (Wilson $\Rightarrow (p-1)! \equiv -1 \pmod{p}$)

$n+1$ non è primo $n+1 = a \cdot b$ $a, b < n+1$
 $n! \equiv 0 \pmod{n+1}$ $n! + 1 \equiv 1 \pmod{n+1}$
 (Vedere $n=4$ separatamente).

$$\textcircled{8} \quad \binom{2^{n+1}}{2^n} - \binom{2^n}{2^{n-1}}$$



Massima potenza di 2 che divide

$$\binom{2^n}{2^{n-1}} = \frac{2^n (2^n - 1) \dots (2^n - 2^{n-1} + 1)}{\underbrace{2^{n-1} (2^{n-1} - 1) \dots 2^1}_2 \cdot 1} \quad \begin{matrix} 2^n - a \\ a \end{matrix}$$

$$\begin{aligned} \binom{2^{n+1}}{2^n} - \binom{2^n}{2^{n-1}} &= 2D - 2D' \\ &= 2(D - D') \end{aligned}$$

$$\binom{2^n}{2^{n-1}} = \frac{(2^n)!}{[(2^{n-1})!]^2} \quad 2^n!!$$

$$(2^n)! = \{1 \cdot 3 \cdot 5 \cdots (2^n-1)\} \{2 \cdot 4 \cdots 2^n\}$$

$$= A_n \cdot 2^{2^{n-1}} \cdot (2^{n-1})!$$

↓
dispon!

$$(2^n)! = A_n \cdot 2^{2^{n-1}} \cdot A_{n-1} \cdot 2^{2^{n-2}} \cdot (2^{n-2})! = \dots$$

$$= A_n \cdot A_{n-1} \cdots A_1 \cdot 2^{2^n - 1}$$

$$\binom{2^n}{2^{n-1}} = \frac{2^{2^{n-1}} A_1 \cdots A_n}{(2^{2^{n-1}-1})^2 (A_1 \cdots A_{n-1})^2} = \frac{2 A_n}{A_1 \cdots A_{n-1}}$$

$$\binom{2^{n+1}}{2^n} - \binom{2^n}{2^{n-1}} = \frac{2 A_{n+1}}{A_1 \cdots A_n} - \frac{2 A_n}{A_1 \cdots A_{n-1}} =$$

$$\frac{2 (A_{n+1} - A_n^2)}{A_1 \cdots A_n}$$

$$A_{n+1} = \underbrace{1 \cdot 3 \cdot 5 \cdots (2^n-1)}_{2^n} \underbrace{(2^n)(2^{n+1}) \cdots (2^{n+1}-1)}_{2^{2^n}}$$

$$= \prod_{k=1}^{2^n} (2^{2^n} - k^2)$$

k dispon!

Congetura:
la potencia gausiana
 $e^{-\frac{1}{2} 2^{2^n}}$

$$A_{n+1} \equiv ? \pmod{2^{4^n}}$$

$$A_{n+1} \equiv \prod_{\substack{k=1 \\ k \neq i}}^{2^n} k^2 - \left(\sum_i \prod_{k \neq i} k^2 \right) 2^{2^n}$$

$n \geq 1$

$$\equiv A_n^2 - \sum \frac{A_n^2}{i^2} \cdot 2^{2n}$$

Denn also

$$\sum_{\substack{i=1 \\ i \text{ ungerade}}}^{2^n} \frac{1}{i^2} \pmod{2^n}$$

$$\sum_{i \text{ ungerade}} \frac{1}{i^2} \equiv \sum i^2 \pmod{2^n}$$

$$\sum_{i=1}^{2^n} i^2 - \sum_{i \text{ ger.}} i^2 = \frac{2^n (2^n + 1) (2^{n+1} + 1)}{6}$$

$$= \frac{2^{n-1}}{3} (4^n - 1) - \frac{2^{n-1} 2^{n-1} (2^{n-1} + 1) (2^n + 1)}{6}$$

divisibel es ist immer
für 2^{n-1} .

→ H. $2n \neq (n-1) + 1 = 3n$.

$$2n!! = 1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdots (2n-1)$$

$$= \frac{2n!}{2^n \cdot n!}$$

$$\begin{aligned}
 &= \frac{2 \cdot 4 \cdot 6 \cdot 8 \cdots 2n}{2 \cdot 1 \cdot 2 \cdot 2 \cdot 3 \cdots 2 \cdot n} \\
 &= 2^n n!
 \end{aligned}$$