Pre IMO 2011 Algebra M

Titolo nota 25/05/2011

$\boxed{1}$ $$\sum_{cyc} \frac{1}{4a+3b+c} \geq 2 \qquad a,b,c,d>0 \quad \sum_{cyc} a = 1$$

HM^{-1}

$$\left(\frac{1}{4}\sum_{cyc}\frac{1}{4a+3b+c}\right)^{-1} \leq GM \leq AM \leq QM$$

$$\left(\frac{1}{4}\sum_{cyc}\frac{1}{4a+3b+c}\right)^{-1} \leq \frac{1}{4}\sum_{cyc}(4a+3b+c) = \frac{1}{4}(4+3+1) = 2$$

$$\left(\sum_{cyc}\frac{1}{4a+3b+c}\right)^{-1} \leq \frac{1}{4}\cdot 2 = \frac{1}{2}$$

CS, anzi, Lemma di Titu

$$\sum_{cyc}\left(\frac{1}{\sqrt{x}}\right)^2 \sum_{cyc}\left(\sqrt{x}\right)^2 \geq \left(\sum_{cyc} 1\right)^2 = 4^2$$

$$\sum_{cyc}\frac{1}{x} \geq \frac{4^2}{\sum_{cyc} x}$$

$$\boxed{\sum_{i=1}^{n}\frac{a_i^2}{b_i} \geq \frac{\left(\sum a_i\right)^2}{\sum b_i}}$$

$$\sum_{cyc}\frac{1}{4a+3b+c} \geq \frac{16}{\sum_{cyc}(4a+3b+c)} = \frac{16}{8} = 2$$

$\boxed{2}$ $x > 0$ trovare n tale che

$\exists\ y_1, y_2, \ldots, y_n \in (-1, 1)$ t.c.

(i) $y_1 + y_2 + \cdots + y_n = 0$ e

(ii) $y_1^2 + y_2^2 + \cdots + y_n^2 = x$

$y_1, y_2, \ldots, y_k$ positivi (>0)

$y_{k+1}, \ldots, y_n$ non positivi (≤ 0)

$z_i = -y_{k+i}$

(i) $\longrightarrow$ $y_1 + y_2 + \cdots + y_k = z_1 + z_2 + \cdots + z_{n-k}$

(ii) $\longrightarrow$ $y_1^2 + y_2^2 + \cdots + y_k^2 = x - (z_1^2 + z_2^2 + \cdots + z_{n-k}^2)$

Prendiamo $k \leq n-k$ $\quad 2k \leq n$ $\quad k \leq \left\lfloor \frac{n}{2} \right\rfloor$

$z_1^2 + z_2^2 + \cdots + z_{n-k}^2 = x - (y_1^2 + \cdots + y_k^2) > x - k$

$z_1^2 + z_2^2 + \cdots + z_{n-k}^2 \leq z_1 + z_2 + \cdots + z_{n-k} =$

$= y_1 + y_2 + \cdots + y_k < k$

$k > x - k \Rightarrow 2k > x$

$$2\lfloor \tfrac{n}{2} \rfloor > x \qquad (\lfloor \tfrac{x}{2} \rfloor + 1)$$

$$\left\lfloor \frac{n}{2} \right\rfloor \geq k > \frac{x}{2} \qquad \left\lfloor \frac{n}{2} \right\rfloor > \frac{x}{2} \rightsquigarrow \left\lfloor \frac{n}{2} \right\rfloor \geq \left\lceil \frac{x}{2} \right\rceil$$

$$n \geq 2\left\lceil \frac{x}{2} \right\rceil \longleftarrow \frac{n}{2} \geq \left\lfloor \frac{n}{2} \right\rfloor \geq \left\lceil \frac{x}{2} \right\rceil$$

$$n = 2\left\lceil \frac{x}{2} \right\rceil \qquad \left\lceil \frac{x}{2} \right\rceil = k \qquad n = 2k$$

$$y_1 = y_2 = y_3 = \dots = y_k = z_1 = z_2 \dots = z_k = \lambda$$

Con questa scelta (i) è assicurata

$$y_1 + y_2 + \dots + y_k = k\lambda = z_1 + z_2 + \dots + z_k$$

$$y_1^2 + y_2^2 + \dots y_k^2 + z_1^2 + \dots + z_k^2 = \lambda^2 \cdot 2k = \lambda^2 n$$

posso scegliere $\lambda \in (0,1)$ t.c. $\lambda^2 \cdot n = x$?

Posso farlo solo se $x < n$

$$n = 2\left(\left\lfloor \frac{x}{2} \right\rfloor + 1\right) > x \Longleftarrow \lfloor \alpha \rfloor > \alpha - 1$$

———— o ————

Idea: faccio il problema "al contrario"

Trovare il minimo n t.c. $\exists \begin{cases} x_1^2 + \dots + x_n^2 = x \\ x_1 + \dots + x_n = 0 \\ x_i \in (-1,1) \end{cases}$

(Fissato n) Trovare il massimo di $x_1^2 + \dots + x_n^2$ $\begin{cases} x_1 + \dots + x_n = 0 \\ x_i \in (-1,1) \end{cases}$

lo chiamo $m(n)$

e poi trovo il minimo n t.c. $m(n) \geq \kappa$

$\boxed{n=2}$ $\quad x_1 + x_2 = 0 \qquad x_1, x_2 \in (-1,1)$

$x_1^2 + x_2^2$ massimo possibile

$x_1 = -x_2 \qquad 2x_1^2 < 2 \qquad x_1 = 1$

$x_2 = -1$

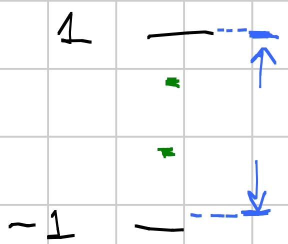
$x_1 + x_2 = c \qquad x_1 > x_2 \qquad x_1, x_2 \in (-1,1)$

$x_1^2 + x_2^2$

$(x_1 + \varepsilon) + (x_2 - \varepsilon) = c$

$$(x_1+\varepsilon)^2 + (x_2-\varepsilon)^2 = x_1^2 + x_2^2 + \underset{\geq 0}{2\varepsilon^2} + \underbrace{2\varepsilon(x_1 - x_2)}_{\geq 0}$$

$\geq x_1^2 + x_2^2$



n qualsiasi. Supponiamo ci sia una coppia x_1, x_2 t.c. $-1 < x_1 < x_2 < 1$

Applico lo "spiattellamento" a questi due e quindi uno dei 2 andrà a 1 oppure -1

Permettiamo agli x_i di andare a 1 o -1

$\boxed{n \text{ pari}}$

Problema nuovo ($x_i \in [-1, 1]$)

il massimo è proprio n e lo ottengo quando $x_i = (-1)^i$.

Problema vecchio ($x_i \in (-1, 1)$)

il sup è $\leq n$

$$x_1^2 < 1 \qquad x_2^2 < 1 \quad -$$

$$\underbrace{x_1^2 + x_2^2 + \dots + x_n^2}_{\substack{\text{II} \\ n\lambda^2}} < n$$

$$\sup(x_1^2 + x_2^2 + \dots + x_n^2) = n$$

ma NON è un massimo

$\boxed{n \text{ dispari}}$

Problema nuovo ($x_i \in [-1, 1]$)

$$\sum_1^n x_i = 0$$

$$x_1 = -\sum_{i=2}^{n} x_i \qquad \boxed{x_1 = 0}$$

massimo è $(n-1) = \overset{0}{\overset{\shortparallel}{x_1^2}} + \overset{1}{\overset{\shortparallel}{x_2^2}} + \dots + \overset{1}{\overset{\shortparallel}{x_n^2}}$

Problema vecchio ($x_i \in (-1, 1)$)

$$\sup(x_1^2 + x_2^2 + \dots + x_n^2) = (n-1)$$

Anche qui non ho massimo.

$$m(n) = \begin{cases} n & \text{se } n \text{ è pari} \\ n-1 & \text{se } n \text{ è dispari} \end{cases} = 2\left\lfloor \frac{n}{2} \right\rfloor$$

x non è intero

$m(n) > x \qquad 2\left\lfloor \frac{n}{2} \right\rfloor > x \qquad n = 2\left(\left\lfloor \frac{x}{2} \right\rfloor + 1\right)$

$\boxed{3}$ $P(n) \qquad n = 0, 1, 2, \ldots, d$

$\mathrm{DEG}\, P(n) = d \qquad P(n) = \dfrac{1}{\binom{d+1}{n}}$

$P(d+1) = ?$

$$\sum_{k=0}^{d} \frac{x(x-1)(x-2)\cdots(x-d)}{(x-k)\, a_k} = P(x)$$

$$h \rightarrow \quad \frac{1}{a_h} = P(h) \frac{1}{\prod_{k \neq h} (h-k)}$$

$h = 2 \qquad 2 \cdot 1 \cdot (-1)(-2) \cdots -(d-2)$

$$h!\,(d-h)!\,(-1)^{d-h}$$

$$P(h) = \frac{h!\,(d+1-h)!}{(d+1)!}$$

$$\frac{1}{a_h} = \frac{\cancel{h!}\,(d+1-h)!}{(d+1)!} \; \frac{1}{\cancel{h!}\,\cancel{(d-h)!}\,(-1)^{d-h}}$$

$$\frac{1}{a_h} = \frac{d+1-h}{(d+1)!}(-1)^{d-h}$$

$$p(d+1) = \sum_{k=0}^{d} \frac{(\cancel{d+1})!}{(d+\cancel{1-k})} \; \frac{d\cancel{+1-k}}{\cancel{(d+1)!}}(-1)^{d-k}$$

$$p(d+1) = \begin{cases} 0 & d \text{ DISP.} \\ 1 & d \text{ PARI} \end{cases}$$

Sol. 2:

$$p(x) = \sum_{i=0}^{d} a_i \binom{x}{i} \overset{Hp:}{=} \frac{1}{\binom{d+1}{x}} \qquad \forall x = 0, 1, 2, \dots d$$

$$\to \sum_{i=0}^{d} a_i \binom{x}{i}\binom{d+1}{x} = 1$$

$$\binom{x}{i}\binom{d+1}{x} = \binom{d+1}{i} \cdot \binom{d+1-i}{x-i}$$

$$\frac{\cancel{x!}}{i!\,(x-i)!} \cdot \frac{(d+1)!}{\cancel{x!}\,(d+1-x)!} \cdot \frac{(d+1-i)!}{(d+1-i)!} \qquad \text{si vede.}$$

$$p(x) = \sum_{i=0}^{d} \underline{a_i \binom{d+1}{i}} \binom{d+1-i}{x-i} = 1 \quad \text{per } x = 0, \dots, d$$

$$p(0) - p(1) + p(2) - \dots + (-1)^d p(d)$$

$$p(d+1) = \sum a_i \binom{d+1}{i}$$

$$\sum_{i=0}^{d} \sum_{x=0}^{d} \boxed{(-1)^x} \underbrace{a_i \binom{d+1}{i}} \boxed{\binom{d+1-i}{x-i}}$$

$$① = \sum_{x=0}^{d} (-1)^x = \begin{cases} 0 & \text{se } d \text{ disp} \\ 1 & \text{se } d \text{ pari} \end{cases}$$

$$\|$$

$$② \quad \sum_{i=0}^{d} a_i \binom{d+1}{i} (-1)^d = (-1)^d p(d+1)$$

$P(1) = P(d) \qquad P(2) = P(d-1) \quad \dots$

Simmetria in $\frac{d+1}{2}$

$Q(x) = \underline{P(x) - P(d+1-x)}$ $\qquad \boxed{1, 2, \dots, d}$

$\deg(Q) \quad \checkmark + \checkmark = 2\checkmark$

d è pari $\deg Q < d$

$Q(x) = 0 \ \forall x \in \mathbb{R}$

$P(x) = P(d+1-x) \qquad P(d+1) = P(0) = 1$

d dispari $\qquad \deg Q = d$

$Q(x) = \alpha \cdot (x-1)(x-2) \dots (x-d)$

$\overset{\|}{2[x^d]P(x)}$

$$P(x) = \frac{\begin{vmatrix} & \binom{d+1}{0}^{-1} \\ & \binom{d+1}{1}^{-1} \\ & \vdots \\ & \binom{d+1}{d}^{-1} \end{vmatrix}}{\begin{vmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 2^d \\ \vdots & & & \\ 1 & d & \cdots & d^d \end{vmatrix}} =$$

$$= (-1)^{d+k} \frac{d! \cdot (d-1)! \cdots 2! \cdot 1!}{k!\,(d-k)!}$$

$$P(x) = \begin{vmatrix} & \binom{d+1}{0}^{-1} \\ & \binom{d+1}{1}^{-1} \\ & \vdots \\ & \binom{d+1}{d}^{-1} \end{vmatrix} =$$

$$\sum_{i=0}^{d} \binom{d+1}{i}^{-1} (-1)^{d+1+i} \qquad \frac{\checkmark}{i!\,(d-i)!} (-1)^{d-i}$$

$$\frac{1}{2\,d!}$$

$\circledast$ $\quad a_k = P(k) \qquad k = 0, 1, \ldots, d$

$a_3 - 2a_2 + a_1$

$$P(x) = a_0 + x\left(a_1 - a_0 + \frac{(x-1)}{2}\left(\boxed{a_2 - 2a_1 + a_0} + \frac{x-2}{3}\left(\qquad \right)\right)\right)$$

$$= a_0 + x(a_1 - a_0) + \frac{x(x-1)}{2}(a_2 - 2a_1 + a_0) + \binom{x}{3}\left(\qquad \right) + \ldots$$

$$\Delta P(x) := P(x) - P(x-1) \qquad \Delta^k P(x) := \Delta^{k-1}P(x) - \Delta^{k-1}P(x-1)$$

$$\Delta^2 P(x) = P(x) - P(x-1) - \big(P(x-1) - P(x-2)\big) = P(x) - 2P(x-1) + P(x-2)$$

<u>Lemma</u> : $\Delta^k P(x) = \sum_{i=0}^{k} \binom{k}{i} (-1)^i P(x-i)$

$$0 = \Delta^{d+1} P(x) = \sum_{i=0}^{d+1} \binom{d+1}{i} (-1)^i P(x-i)$$

$$x = d+1 \qquad P(x-i) = P(d+1-i) = \binom{d+1}{d+1-i}^{-1} = \binom{d+1}{i}^{-1}$$

$$0 = \sum_{i=1}^{d+1} (-1)^i + P(d+1)$$

$$\boxed{\Delta^d P(x) = d!\, a_d} \qquad P(x) = \sum_{k=0}^{d} a_k x^k$$

$$\Delta P(x) = \sum_{k=0}^{d} a_k \left(\cancel{x^k} - \left(\cancel{x^k} - k x^{k-1} + \binom{k}{2} x^{k-2} - \dots\right)\right)$$

$$= a_d \cdot d \cdot x^{d-1} + \dots$$

$\boxed{4}$
$$\begin{cases} x^2 - yz - zt - ty = a \\ y^2 - \dots \qquad b \\ \quad \dots \\ \quad \dots \end{cases}$$

$$\underbrace{\sum x^2}_{Q} - 2 \underbrace{\sum xy}_{P} = \boxed{\sum a}$$

$$a = x^2 - yz - zt - ty = x^2 - P + x\big(\overbrace{y+z+t}^{\overset{S}{\sum x - x}}\big) = x^2 - P + x(S - x)$$
$$= xS - P$$

$$xS - P = a \qquad x = \frac{a+P}{S}$$

$$yS - P = b \qquad y = \frac{b+P}{S}$$

$$\dots$$

$$\left(\begin{aligned} a &= x^2 - yz - zt - ty = x^2 - \frac{1}{2}\left((y+z+t)^2 - y^2 - z^2 - t^2\right) \\ &= x^2 - \frac{1}{2}\left((S-x)^2 - Q + x^2\right) = xS + \frac{Q}{2} - \frac{S^2}{2} \end{aligned} \right)$$

$$S = x+y+z+t = \frac{\overbrace{\Sigma a}^{u} + 4P}{S} \qquad \boxed{S^2 = u + 4P}$$

$$P = xy + yz + \dots = \frac{ab + P^2 + aP + bP}{S^2} + \dots \qquad v = \sum ab$$

$$\boxed{P = \frac{v + 6P^2 + 3uP}{S^2}}$$

$$uP + 4P^2 = PS^2 = v + 6P^2 + 3uP$$

$$2P^2 + 2uP + v = 0 \qquad P_{1,2} = \frac{-u \pm \sqrt{u^2 - 2v}}{2}$$

Noto che $u^2 \geq 2v$ sempre

$$u + 4P = -u \cancel{\pm} 2\sqrt{u^2 - 2v}$$

$$u^2 \leq 4 \sum a^2 = 4(u^2 - 2v)$$

$$u \leq 2\sqrt{u^2 - 2v}$$

$$0 \leq -u + 2\sqrt{u^2 - 2v}$$

$$P = \frac{-u + \sqrt{u^2 - 2v}}{2} \qquad S = \pm\sqrt{u + 4P}$$

$x = \frac{\alpha + P}{S} \quad y = \dots$ occhio a $S=0$

$$x^2 - yz - zt - ty = a$$
$$y^2 - xz - zt - tx = b$$

$(x-y)\underline{(x+y+z+t)} = a-b$ $\quad (x,a)\ (y,b)$

$(y-z)(x+y+z+t) = b-c$

$a = \alpha x + \beta$

$b = \alpha y + \beta$

$c = \alpha z + \beta$

$d = \alpha t + \beta$

$\alpha = 0$ può capitare solo se $a=b=c=d$

Se $a=b=c=d$ $<$ $\alpha \neq 0$, $\alpha = 0$

$x+y+z+t=0$

$x+y+z+t=0$

$(\sigma x, \sigma y, \sigma z, \sigma t)$

$x^2 - yz - zt - ty = \lambda$

$\sigma^2 \lambda = a$