Pre IMO 2011 Algebra Pomeriggio

Titolo nota 25/05/2011

$\boxed{5}$ $f(x+y)f(x-y) = (f(x)+f(y))^2 - 4x^2 f(y)$

$x=y=0 \qquad f(0)^2 = 4f(0)^2 \qquad f(0)=0$

$x=y \qquad 0 = 4f(x)^2 - 4x^2 f(x) = 4f(x)\left[f(x)-x^2\right]$

$\forall x \begin{cases} \bullet\ f(x)=0 \\ \bullet\ f(x)=x^2 \end{cases}$

R.P.A. $y \neq 0 : f(y)=0 \qquad x\neq 0 : f(x)=x^2>0$

$\underset{\neq 0}{f(x+y)}\,\underset{\neq 0}{f(x-y)} = f(x)^2 > 0$

$\cancel{x^4} - 2x^2y^2 + y^4 = \cancel{x^4} \iff \cancel{y^2}\left(y^2-2x^2\right)=0$

$|y| = \sqrt{2}\,|x| \implies \forall z : |z| \neq \frac{|y|}{\sqrt{2}} \quad f(z)=0$

Prendo $|z|\neq |y|$ e x e arrivo $\times$.

$\boxed{6}$ trovare $\max\limits_{x_i} \prod\limits_{i=1}^{n} x_i \qquad n\geq 3$

vincolo : $\sum\limits_{i=1}^{n} \frac{x_i}{1+x_i} = 1$

$\frac{x_1}{1+x_1} + \frac{x_2}{1+x_2} + \dots + \frac{x_n}{1+x_n} = 1$

$\prod\limits_i (1+x_i) = \underbrace{1}_{\sigma_0} + \underbrace{\sum_i x_i}_{\sigma_1} + \underbrace{\sum_{i<j} x_i x_j}_{\sigma_2} + \dots + x_1x_2\cdots x_n =: \sigma_0 + \sigma_1 + \dots \sigma_n$

$$GM = \sqrt[n]{\sigma_n} \le \sqrt[n-1]{\frac{\sigma_{n-1}}{\binom{n}{n-1}}} \le \dots \le \sqrt{\frac{\sigma_2}{\binom{n}{2}}} \le \frac{\sigma_1}{\binom{n}{1}} = AM$$

$$\forall k \quad \sqrt[k]{\frac{\sigma_k}{\binom{n}{k}}} \ge \sqrt[n]{\sigma_n} = \theta$$

$$\boxed{\binom{n}{k}\theta^k \le \sigma_k} \qquad k = 1, 2, \dots, n$$

$$\frac{x_1}{1+x_1} + \frac{x_2}{1+x_2} + \dots + \frac{x_n}{1+x_n} = 1$$

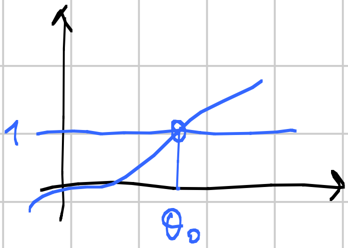
$$LHS = \prod_i (1+x_i)$$

$$LHS = x_1(1+x_2)\dots(1+x_n) + (1+x_1)x_2(1+x_3)\dots(1+x_n) + \dots$$
$$= \sigma_1 + 2\sigma_2 + 3\sigma_3 + \dots + n\sigma_n$$

$$RHS = \sigma_0 + \sigma_1 + \dots \sigma_n$$

$$\boxed{\sum_{k=1}^{n} (k-1)\sigma_k = 1}$$

$$\underbrace{\sum_{k=1}^{n} \binom{n}{k}(k-1)\theta^k}_{\text{crescente in } \theta} \le \sum (k-1)\sigma_k = 1$$



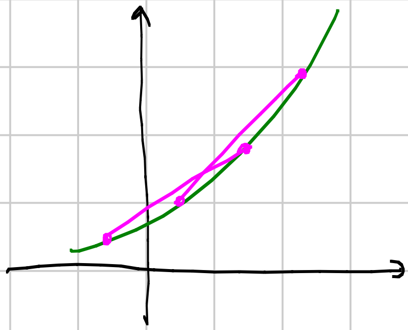
$$\frac{x_i}{1+x_i} = \frac{1}{n} \Rightarrow x_i = \frac{1}{n-1} \qquad \theta_0 = \sqrt[n]{\prod_i \left(\frac{1}{n-1}\right)} = \frac{1}{n-1}$$

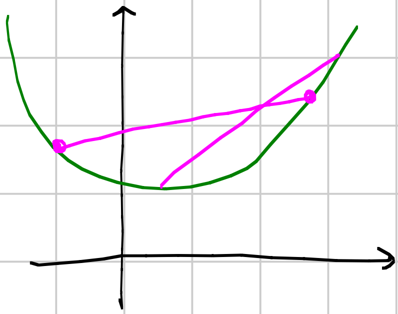
$$\sqrt[n]{\prod x_i} \le \theta_0 = \frac{1}{n-1}$$

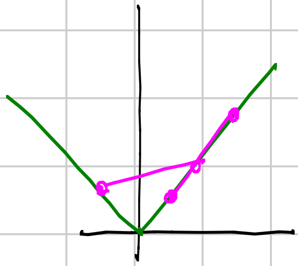
Allenamento 1

$$\min\{f(y_1)+\dots+f(y_n) \;:\; y_1+\dots+y_n = \underset{\uparrow\,\text{dato}}{k}\}$$

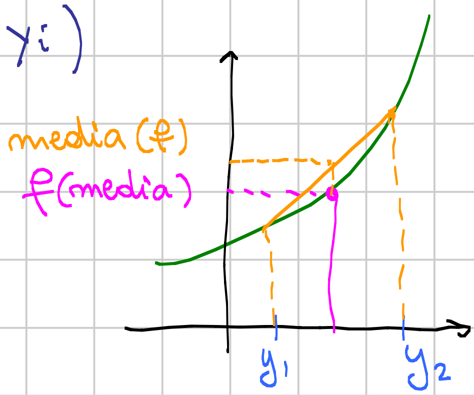
Se $f(x)$ è convessa allora il min è quando sono tutti uguali
Se inoltre $f(x)$ è strett. convessa, allora è l'unico caso di minimo.







Dim. Jensen $\qquad \frac{1}{n}\sum f(y_i) \geq f\left(\frac{1}{n}\sum y_i\right)$



Verifica che $f(x)$ è convessa:

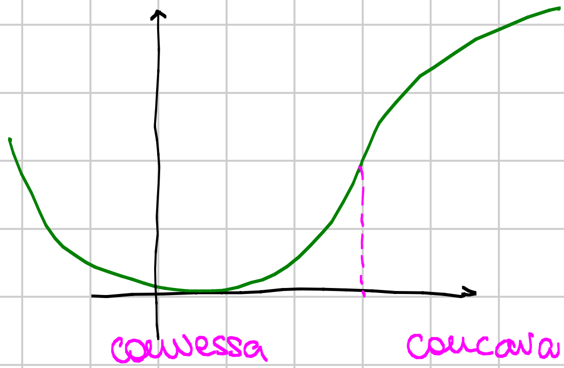
- se esiste $f''(x)$, allora f convessa $\Leftrightarrow$ $f''(x) \geq 0$.

— o — o —

Allenamento 2

Come sopra, ma con $f(x)$ convex – concave

Se la giocano 2 configurazioni



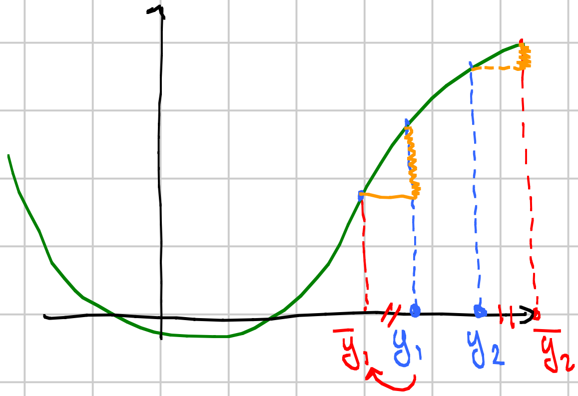
- $y_1 = \dots = y_n$ e nella zona convessa
- $y_1 = \dots = y_{n-1}$ nella zona convessa e y_n nella zona concava.

Per il secondo caso pongo $y_n = x$ e mi riduco a studiare

$$f(x) + (n-1)\, f\left(\frac{k-x}{n-1}\right)$$ e siamo in 1 variabile.

Idea: smoothing

Idea: sposto tutti meno uno in zona convessa e poi li prendo uguali nella zona convessa.

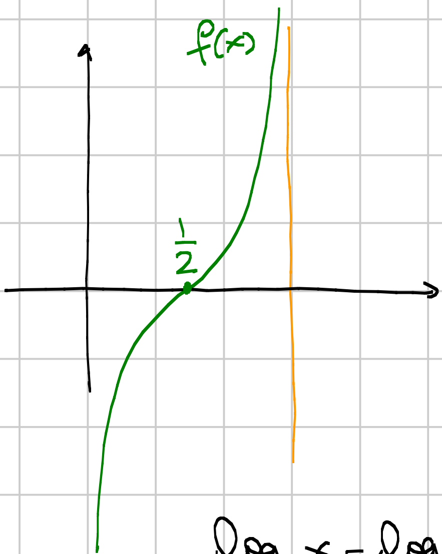


— o — o —

$y_i = \frac{x_i}{1+x_i}$ $\qquad$ $\sum y_i = 1$ $\qquad$ $x_i = \frac{y_i}{1-y_i}$

Devo massimizzare $\prod \frac{y_i}{1-y_i}$, il che equivale a max

$$\sum \log\left(\frac{y_i}{1-y_i}\right) = \sum f(y_i) \quad \text{con} \quad f(x) = \log\frac{x}{1-x}$$
$$= \log x - \log(1-x)$$



Questa è concave - convex, MA io devo massimizzare, quindi ho 2 config:

1. tutti uguali nella concava
2. tutti meno 1 nella concava ed il restante nella convessa.

$$\log x - \log(1-x) + (n-1)\log\frac{1-x}{n-1} - \log \cdots$$

La derivata si annulla in $x = 1-\frac{1}{n}$ ... o simili.

$$\sum \frac{X_i}{1+X_i} = 1 \qquad \boxed{\frac{X_1}{1+X_1} + \frac{X_2}{1+X_2} = a}$$

$X_1 X_2$ si massimizza se $X_1 = X_2$

$x_1, x_2, - - - -, x_n$

se x_i costante

$x_1 \neq x_2$ $\quad \frac{c}{1+c} + \frac{d}{1+d} = \frac{x_1}{1+x_1} + \frac{x_2}{1+x_2}$ $\quad c = d$

$cd > x_1 x_2$

x_i $\quad 0 \leq x_i \leq$ qualcosa < 1

$\boxed{7}$ $\quad f(x + f(x) + 2f(y)) = f(2x) + f(2y)$

$\uparrow$ $P(x,y)$

<u>f e' suriettiva</u>

(i) $f(x_1) = f(x_2) \rightsquigarrow f$ è $(x_1 - x_2)$-periodica

$P(x, x_1)$ $\quad f(x + f(x) + 2f(x_1)) = f(2x) + f(2x_1)$

$P(x, x_2)$ $\quad f(x + f(x) + 2f(x_2)) = f(2x) + f(2x_2)$

$\rightsquigarrow f(2x_1) = f(2x_2)$

$P(x_1, y)$ $\quad f(x_1 + f(x_1) + 2f(y)) = f(2x_1) + f(2y)$

$P(x_2, y)$ $\quad f(x_2 + f(x_2) + 2f(y)) = f(2x_2) + f(2y)$

$f(x_1 + a(y)) = f(x_2 + a(y))$

suriettività $\Rightarrow$ a può assumere qualunque valore

(ii) c t.c. $f(c) = 0$

$P(c,c)$ $\quad f(c+0+2\cdot 0) = 2f(2c)$

$$\Downarrow$$

$f(2c) = 0$ $\quad \underset{\shortparallel}{} \; f(c) = 0$

$P(x,c)$ $\quad f(x+f(x)+2\underset{\overset{\shortparallel}{0}}{f(c)}) = f(2x) + \underset{\overset{\shortparallel}{0}}{f(2c)}$

$$f(\underbrace{x+f(x)}_{x_1}) = f(\underbrace{2x}_{x_2}) \qquad \forall x$$

f è $\quad 2x - (x+f(x)) = \underbrace{x - f(x)}_{T}$ - periodica

$$f(x-T) = f(x)$$

$$f(x - x + f(x)) = f(x) \qquad f(f(x)) = f(x)$$

$\forall y \quad \exists x \quad$ t.c. $\quad y = f(x)$

$$f(y) = y \qquad \forall y \in \mathbb{R}.$$

$\boxed{8}$

$$\frac{1}{x^2(y+1)} + \frac{1}{y^2(z+1)} + \frac{1}{z^2(x+1)} \geq \frac{3}{4(x+y+z)}$$

con $\quad xyz = 3(x+y+z) \qquad \frac{1}{3} = \frac{1}{xy} + \frac{1}{yz} + \frac{1}{zx}$

$$\sum_1 (y+1) \sum_1 \frac{\frac{1}{x^2}}{(y+1)} \overset{CS}{\geq} \left(\sum_1 \frac{1}{x}\right)^2 \geq 3\left(\frac{1}{xy} + \frac{1}{yz} + \frac{1}{zx}\right)$$

$$= 1$$

$$\sum_{cyc} \frac{1}{x^2(y+1)} \geq \frac{1}{S+3} \overset{H}{\geq} \frac{3}{4S}$$

$$4S \overset{H}{\geq} 3S + 9$$

$$S \overset{H}{\geq} 9$$

$$S \geq 3\,GM = 3\sqrt[3]{xyz} = 3\sqrt[3]{3S}$$

$$\frac{S}{3} \geq \sqrt[3]{3S}$$

$$S^2 \geq 81 \quad \Rightarrow \quad S \geq 9$$

$\square$.

$$\sum \frac{1}{x^2(y+1)} \geq \frac{9}{4xyz}$$

$$\sum \frac{yz}{x(y+1)} \geq \frac{9}{4}$$

$x \to \frac{1}{a}$ $y \to \frac{1}{b}$ $z \to \frac{1}{c}$ $ab+bc+ca = \frac{1}{3}$

$$\sum \frac{\frac{1}{bc}}{\frac{1}{a}\left(\frac{1}{c}+1\right)} \geq \frac{9}{4}$$

$$\sum \frac{a}{b(c+1)} \geq \frac{9}{4}$$

$$\sum \frac{a^2}{abc+ab} \geq \frac{9}{4}$$

$$RHS \geq \frac{(\sum a)^2}{\sum abc + \sum ab} = \frac{S^2}{3abc + \frac{1}{3}} \overset{?}{\geq} \frac{9}{4}$$

$$abc \leq \frac{1}{27}$$

$$\frac{S^2}{\frac{1}{9}+\frac{1}{3}} \overset{?}{\geq} \frac{9}{4}$$

$$\frac{9}{4} S^2 \overset{?}{\geq} \frac{9}{4}$$

$$S^2 \overset{?}{\geq} 1$$

$$\frac{S}{3} \geq \sqrt{\frac{ab+bc+ca}{3}} = \frac{1}{3}$$