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# Pre IMO 2012 ALGEBRA mattino

Titolo nota 21/05/2012

$\boxed{1}$ $x, y, z \in \mathbb{R}$

$$\begin{cases} \sum x = 0 \\ \sum x^3 = 18 \\ \sum x^7 = 2058 = 6 \cdot 343 = 6 \cdot 7^3 \end{cases}$$

$x + y + z = 0$ $\boxed{z = -x-y}$

$z^3 = -(x+y)^3 = -x^3 - y^3 - 3xy(x+y)$

$18 = \sum x^3 = -3xy(x+y) = 3xyz$ $\boxed{xyz = 6}$

$z^7 = z^3 \cdot z^3 \cdot z \quad \ldots$

$= -(x+y)^7 = -x^7 - y^7 - 7xy\left(x^5 + 3x^4y + 5x^3y^2 + 5x^2y^3 + \ldots\right)$

$6 \cdot 7^3 = -7xy\left( \ldots \right)$

$\cancel{6} \cdot 7^2 = x\cancel{y}z\left(x^4 + 2x^3y + 3x^2y^2 + 2xy^3 + y^4\right)$

$7^2 = \left(x^2 + xy + y^2\right)^2$

$\boxed{x^2 + xy + y^2 = \pm 7}$

$\boxed{x^2y + xy^2 = -6}$

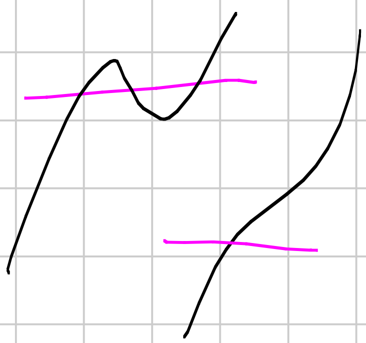
| | | | | | |
|---|---|---|---|---|---|
| 1 | 3 | 5 | 5 | 3 | 1 |
| | 1 | 2 | 3 | 2 | 1 |
| 1 | 2 | 3 | 2 | 1 | |
| 1 | 1 | 1 | | | |
| | 1 | 1 | 1 | | |
| | | 1 | 1 | 1 | |

$x^3 + x^2y + xy^2 = \pm 7x$ $x^3 \mp 7x - 6 = 0$

$x^3 - 7x - 6 = (x+1)(x+2)(x-3)$

$x^3 + 7x - 6$ ha una sola radice reale

$3x^2 + 7$ non ha radici reali



$\alpha, \beta, \gamma$ i numeri cercati

$$p(x) = (x-\alpha)(x-\beta)(x-\gamma) = x^3 - \cancel{ax^2} + bx - c$$

$$a = \alpha + \beta + \gamma = 0 \qquad p(\alpha) = 0 \quad \alpha^3 = c - b\alpha$$

$$b = \alpha\beta + \beta\gamma + \gamma\alpha \qquad \beta^3 = c - b\beta$$

$$c = \alpha\beta\gamma \qquad \gamma^3 = c - b\gamma$$

$$18 = \sum \alpha^3 = 3c \qquad c = 6 \qquad p(x) = x^3 + bx - 6$$

$$\sum \alpha^7 \qquad \alpha^7 = (6 - b\alpha)^2 \alpha = \alpha 36 - 12b\alpha^2 + b^2(6 - b\alpha)$$

$$6 \cdot 7^3 = \sum \alpha^7 = 18b^2 - 12b\sum\alpha^2 = 18b^2 + 24b^2 = 42b^2$$

$$\sum \alpha^2 = \left(\sum \alpha\right)^2 - 2\sum \alpha\beta = 0 - 2b$$

$$b^2 = 7^2 \qquad b = \pm 7$$

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$$p(x) = x^3 + bx - 6 \qquad \alpha^3 = -b\alpha + 6 \qquad \alpha^7 = -b\alpha^5 + 6\alpha^4$$

$$\sum \alpha^7 = -b\sum\alpha^5 + 6\sum\alpha^4 = -b\left(-b\sum\alpha^3 + 6\sum\alpha^2\right) + 6\left(-b\sum\alpha^2 + 6\sum\alpha\right)$$

$\boxed{2}$ $$\sum_{cyc} \frac{a}{b+2c+d} \geq k \qquad a, b, c, d > 0$$

omogenea, ciclica ma non simmetrica

tutti $= 1$    LHS $= 1$    $k \leq 1$

$a = 0$   altri $= 1$    LHS $= 7/6$

$a = b = 0$   $c = d = 1$    LHS $= 2$

$a = c = 0$   $b = d = 1$    LHS $= 1$

Dimostro che $\displaystyle\sum_{cyc} \frac{a}{b+2c+d} \geq 1$

HOPELESS

$$\sum_{cyc} \frac{\sqrt{a}^2}{b+2c+d} \overset{!}{\geq} \frac{16}{\sum_{cyc}\left(\frac{b}{a}+2\frac{c}{a}+\frac{d}{a}\right)} \geq 1$$

$$\sum \frac{1}{x} \geq \frac{4^2}{\sum x}$$

$$\boxed{\sum \frac{a^2}{x} \geq \frac{\left(\sum a\right)^2}{\sum x}}$$

Lemma di Titu

Disug. di BERGSTROM

$$\sum \sqrt{x}^2 \sum \frac{a^2}{\sqrt{x}^2} \geq \sum \sqrt{x}\,\frac{a^2}{\sqrt{x}} = \sum a^2$$

HOPELESS

$$LHS \geq \frac{\left(\sum \sqrt{a}\right)^2}{\sum (b+2c+d)} \geq 1$$

HOPE

$$\sum_{cyc} \frac{a^2}{ab+2ac+ad} \geq \frac{\left(\sum a\right)^2}{\sum(ab+2ac+ad)} \geq 1$$

$$\left(\sum_{cyc} a\right)^2 \overset{?}{\geq} \sum_{cyc}(ab+2ac+ad) = \frac{1}{2}\sum_{sym} ab + 2ac + 2bd$$

$$\left(\sum_{cyc} a\right)^2 = \sum_{cyc} a^2 + \frac{1}{2}\sum_{sym} ab$$

$$4ab + 4ac + 4ad + 4bc + 4bd + 4cd$$

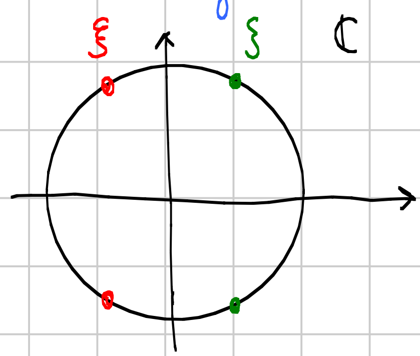
$$a^2+b^2+c^2+d^2 \geq 2ac+2bd$$

smoothing

$$\sum_{cyc} \frac{a}{b+2c+d} \overset{!}{\geq} \left( \frac{\overbrace{\frac{a+c}{2}}^{x}}{b+2\frac{a+c}{2}+d} + \frac{\overbrace{\frac{b+d}{2}}^{y}}{c+2\frac{b+d}{2}+a} \right) \cdot 2 = 1$$

$\boxed{3}$ $(x^2+x+1)f(x^2-x+1) = (x^2-x+1)\,g(x^2+x+1) \quad \forall x \in \mathbb{R}$

sono uguali come polinomi




$$0 = LHS(\xi) = RHS(\xi) = (\underbrace{\xi^2-\xi+1}_{-2\xi})\,g(0)$$

$$g(x) = x\,g_1(x)$$

analogamente

$$f(x) = x\,f_1(x)$$

$f_1(x^2-x+1) = g_1(x^2+x+1) \qquad \Rightarrow f_1(x) = g_1(x) = a$

1) LHS simmetrica risp a $\frac{1}{2}$, RHS simm risp a $-\frac{1}{2}$

$\Rightarrow$ periodico $\Rightarrow$ costante

2) $f_1(x) = \sum_0^n a_k x^k \qquad g_1(x) = \sum_0^n b_k x^k$

$$\sum_0^n a_k (x^2-x+1)^k = \sum_0^n b_k (x^2+x+1)^k$$

$$a_n x^{2n} - n a_n x^{2n-1} + \dots = b_n x^{2n} + n b_n x^{2n-1}$$

$n=0 \quad a_0 = b_0 = a.$

3) $f_1(x) = \prod_{k=1}^n (x-\alpha_k) \qquad g_1(x) = \prod_1^n (x-\beta_k)$

$$\prod_1^n (x^2-x+1-\alpha_k) = \prod (x^2+x+1-\beta_k)$$

$(x-?)\,(x-??)$

somma 2n radici è n $\qquad$ somma 2n radici è $-n$

$\boxed{4}$ hp $\begin{cases} f(1) = 1 \\ f(n) = n - f(f(n-1)) & n \geq 2 \end{cases}$

ts $\{ f(n+f(n)) = n \qquad n \geq 1$

$f(n) \stackrel{?}{=} \lfloor \alpha n + \beta \rfloor \qquad \alpha = \frac{\sqrt{5}-1}{2} \qquad \beta \stackrel{?}{=} \alpha$

| n | f(n) | f(f(n)) |
|---|---|---|
| 1 | 1 | 1 |
| 2 | 1 | 1 |
| 3 | 2 | 1 |
| 4 | 3 | 2 |
| 5 | 3 | 2 |
| 6 | 4 | 3 |
| 7 | 4 | 3 |
| 8 | 5 | 3 |
| 9 | 6 | 4 |
| 10 | 6 | 4 |
| 11 | 7 | 4 |

☆ $f(n+1) - f(n) = 1 - [f(f(n)) - f(f(n-1))]$

$f(n+1) - f(n) \in \{0,1\}$ per induzione

$f(n) \leq n$

$\rightarrow f(n) = f(n-1) \Rightarrow f(n+1) - f(n) = 1$

$* = f(n+1+f(n+1)) - f(n+f(n)) \stackrel{?}{=} n+1-n = 1$

$n+1+f(n+1) - (n+f(n)) \in \{1,2\}$

$* \in \{0,1,2\}$

Ora escludo che sia 0

$f(n+1+f(n+1)) = f(n+f(n)) \Rightarrow f(n+1) = f(n)$

$n-1 \stackrel{ind}{=} f(n-1+f(n-1)) = f(n-1+f(n)-1) = f(n+f(n)-2)$

$f(n+f(n)) \stackrel{ind}{=} n$

$n \quad n \quad ? \quad n-1$ $\qquad f(n+f(n)+1) = n$

$n-1$

$n-1 = f(n+f(n)-1) = n-1+f(n) - \overbrace{f(\underbrace{f(n+f(n)-2)}_{n-1})}^{f(n)-1} = n$ assurdo

Ora escludo che sia 2

$f(n+1) = f(n)+1$

$f(n+f(n)+2) = 1 + f(n+f(n)+1) = 2 + f(n+f(n)) = 2+n$

$f(f(n+f(n)+1)) - f(f(n+f(n))) = f(f(n+f(n)-1))$

$f(n+f(n)+1) = n$ $\qquad \overset{\shortparallel}{f(n)}$

$f(n+f(n)+1) = n+f(n)+1 - f(f(n+f(n))) = n+1$

$n = f(n+f(n))$

$f(n) \stackrel{?}{=} f(f(n)+f(f(n))) = f(f(n)+n+1-f(n+1))$ $\quad f(n)$ $\quad f(n+1) = f(n)$