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Pre IMO 2012 ALGEBRA pomeriggio

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A5. Determinare tutte le funzioni f dai reali in sé tali che

$$f(f(x+y)) = f(x+y) + f(x)f(y) - xy;$$

per ogni x, y reali.

$y=0$ $\quad f(f(x)) = f(x) + f(x)f(0)$

Sull'immagine di f $\quad y = f(x)$: $f(y) = \alpha y$

$\alpha^2(x+y) = \alpha(x+y) + \alpha^2 xy - xy \qquad \forall x, y \in Im(f) = \mathbb{R}$

$\alpha^2 = 1 \qquad \alpha^2 = \alpha \qquad \alpha = 1 \qquad f(x) = x$

$$f(x\cancel{+}y) + f(0)f(x+y) = f(f(x+y)) = f(x\cancel{+}y) + f(x)f(y) - xy$$

(1) $\quad f(0)f(x+y) = f(x)f(y) - xy$

$y=1$ in (1)
$y=-1$ in (1)
$x \leftarrow x+1$ in (2)

$$\begin{cases} f(0)f(x+1) = f(x)f(1) - x \\ f(0)f(x-1) = f(x)f(-1) + x \\ f(0)f(x) = f(x+1)f(-1) + x + 1 \end{cases} \qquad (2)$$

$$f(x)f(1)f(-1) - xf(-1) = f(0)^2 f(x) - (x+1)f(0)$$

$$f(x)[f(0)^2 - f(1)f(-1)] = x[f(0) - f(-1)] + f(0)$$

$x=1 \quad y=-1$ in (1) $\qquad f(0)^2 = f(1)f(-1) + 1$

$$f(x) = x[f(0) - f(-1)] + f(0)$$

$\boxed{6}$ A6. Sia $n \geq \cancel{2}$ 3 e $a_1, a_2, \ldots, a_n$ reali positivi tali che $a_1 + a_2 + \cdots + a_n = 1$. Dimostrare che

$$\frac{a_2 \cdot a_3 \cdot \cdots \cdot a_n}{a_1 + n - 2} + \frac{a_1 \cdot a_3 \cdot \cdots \cdot a_n}{a_2 + n - 2} + \cdots + \frac{a_1 \cdot a_2 \cdot \cdots \cdot a_{n-1}}{a_n + n - 2} \leq \frac{1}{(n-1)^2}.$$

$$a_k = \frac{1}{n} \qquad \text{LHS} = n \cdot \frac{\left(\frac{1}{n}\right)^{n-1}}{\frac{1}{n} + n - 2} = \frac{n^2 \cdot n^{-n+1}}{1 + n^2 - 2n} = \frac{n^{-n+3}}{(n-1)^2} \leq \frac{1}{(n-1)^2}$$

$$\text{LHS} = \sum_{k=1}^{n} \frac{\prod_{i \neq k} a_i}{a_k + n - 2} \leq \sum_k \frac{\left(\sum_{i \neq k} a_i / n-1\right)^{n-1}}{a_k + n - 2} = \sum_k \frac{\left(\frac{1 - a_k}{n-1}\right)^{n-1}}{a_k + n - 2} = \sum f(a_k)$$

$$= n \cdot \frac{1}{n} \sum f(a_k) \overset{\text{Jensen}}{\geq} n\, f\left(\frac{1}{n} \sum a_k\right) = n f\left(\frac{1}{n}\right) = \frac{n^{-n+3}}{(n-1)^2}$$

$$\text{LHS} \leq \left(\frac{1}{n-1}\right)^{n-1} \sum_k \frac{1}{a_k + n - 2} \leq \left(\frac{1}{n-1}\right)^{n-1} \cdot n \cdot \frac{1}{n-2} = \frac{1}{(n-1)^2} \cdot \frac{n}{(n-2)(n-1)^{n-3}}$$

$$n \geq 4 \qquad \frac{n}{(n-2)(n-1)^{n-3}} < 1$$

Finito $n \geq 4$

$$\boxed{n=3} \qquad \sum_{cyc} \frac{ab}{c+1} \overset{?}{\leq} \frac{1}{4} \qquad \qquad \sum_c a = 1$$

$$\text{LHS} = \sum_{cyc} \frac{ab}{(a+b+2c)(a+b+c)} \overset{?}{\leq} \frac{1}{4}$$

$$\underbrace{(a+b+c)(a+b+2c)}_{a^2+b^2+2c^2+2ab+3bc+3ac}(a+2b+c)(2a+b+c) \overset{?}{\geq} 4 \sum_c ab\, \underbrace{(a+2b+c)(2a+b+c)}_{2a^2+2b^2+c^2+5ab+3bc+3ac}$$

$$\text{LHS} = \sum_c \left(2a^4 + a^2b^2(2+2+10) + a^3b(4+5) + ab^3 \cdot 9 + a^2bc(6+15+6+3)\right)$$

$$= \sum_s \left(a^4 + 7a^2b^2 + 9a^3b + 15a^2bc\right)$$

$$RHS = \sum_{S}\left(8a^3b + 14a^2bc + 10a^2b^2\right)$$

$$LHS - RHS = \sum_{S}\left(a^4 + a^3b + a^2bc\right) - \sum_{S} 3a^2b^2$$

$$\sum_{S}\left(a^4 + a^3b + a^2bc\right) \geq \sum_{S} 3a^2b^2$$

$$\sum_{S}\left(a^3 + abc\right) \geq \sum_{S} 2a^2b$$

$$\sum_{S}\left(a^4 + a^2bc\right) \geq \sum_{S} 2a^3b$$

Schur

$$\sum_{S}\left(a^4 + a^3b + a^2bc\right) \geq \sum_{S} 3a^3b \geq \sum_{S} 3a^2b^2$$

$$\sum_{cyc} \frac{ab}{(a+b+2c)(a+b+c)} \overset{?}{\leq} \frac{1}{4}$$

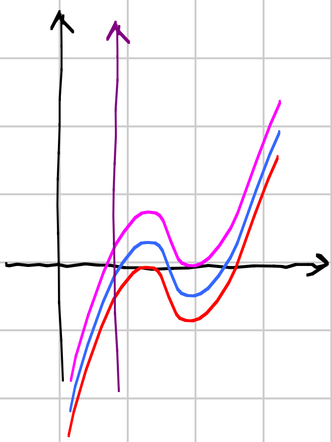
Metodo ABC o SPQ(R)

$$(*) = \underbrace{(a+b+c)(\ldots)(\ldots)(\ldots) - \sum_{c} ab(\ldots)(\ldots)}_{\text{polinomio in 3 var simmetrico di IV grado}} \overset{?}{\geq} 0$$

$$S = a+b+c \qquad Q = ab+bc+ca \qquad P = abc$$

$$(*) = R(P, Q, S) = \alpha \cdot S \cdot P + R_1(Q, S)$$

$$p(x) = (x-a)(x-b)(x-c) = x^3 - Sx^2 + Qx - P$$



i casi limite, se pongo $a \leq b \leq c$

sono $a = 0$, $b = c$, $a = c$

In pratica $a = 0$, $a = b$

Basta verificare la disuguaglianza originale $a=0$, $a=b$ (fatelo)

$$\sum \frac{1}{a} \geq \frac{n^2}{\sum a}$$

Viene anche di smoothing

$$\sum_{cyc} \frac{4ab}{1+c} \leq 1 \qquad ab \frac{4}{a+b+2c} \leq ab\left(\frac{1}{a+c} + \frac{1}{b+c}\right)$$

$$LHS \leq \sum_{cyc}\left(\frac{ab}{a+c} + \frac{ab}{b+c}\right) = \sum_{cyc} \frac{ab+cb}{a+c} = \sum_{cyc} a = 1$$

$\boxed{7}$

A7. Determinare tutte le soluzioni di

$$\begin{cases} a+b+c+d+e=0 \\ a^3+b^3+c^3+d^3+e^3=0 \\ a^5+b^5+c^5+d^5+e^5=10 \end{cases}$$

con $a,b,c,d,e \in [-2,2]$.

$$p(x) = \prod_c (x-a) = \ldots = x^5 + cx^3 + dx - 2$$

$a = 2\cos\alpha$ e cicliche

$$\sum_c \cos\alpha = 0 \qquad \sum_c \cos^3\alpha \qquad \sum_c \cos^5\alpha = \frac{5}{16}$$

$$\cos 2\alpha = 2\cos^2\alpha - 1$$

$$\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha = P_3(\cos\alpha)$$

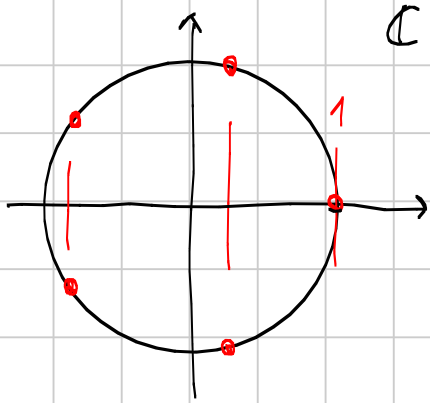
$$\cos n\alpha = 2\cos\alpha\cos((n-1)\alpha) - \cos((n-2)\alpha)$$

$$P_n(\cos\alpha) = 2\cos\alpha\, P_{n-1}(\cos\alpha) - P_{n-2}(\cos\alpha)$$

$$\boxed{P_n(x) = 2x\, P_{n-1}(x) - P_{n-2}(x)}$$ polinomi di Chebyshev

$$\boxed{\cos 5\alpha = \ldots = 16\cos^5\alpha - 20\cos^3\alpha + 5\cos\alpha}$$

$$\sum_c \cos 5\alpha = 16\sum_c \cos^5\alpha = 5 \iff \cos 5\alpha = 1 \qquad \alpha, \beta, \gamma, \ldots$$



$\sum \cos\alpha = 0$

$(a, b, c, d, e) = \left(2, \frac{-1+\sqrt{5}}{2}, id, \frac{-1-\sqrt{5}}{2}, id\right)$

$\frac{-1\pm\sqrt{5}}{4}$

che verificano

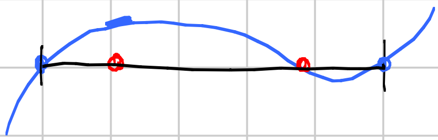
$\boxed{8}$

A8. Sia n un intero positivo. Dimostrare che esistono n reali $a_1, a_2, \ldots, a_n$ nell'intervallo $(0,1)$ tali che la seguente condizione è soddisfatta per ogni $i = 1, 2, \ldots, n$: se un polinomio $p_i(x)$ di grado $n+1$ si annulla in $0, 1, a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n$, allora $\max_{x\in[0,1]} |p_i(x)| = |p_i(a_i)|$.

$n = 1$ $\quad p_1$ di grado 2 $\quad p_1(x) = x(1-x)a$ è max in $\frac{1}{2}$

$a_1 = \frac{1}{2}$

$n = 2$



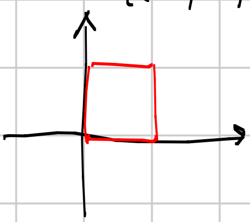
$a_0 = 0 \qquad a_{n+1} = 1$

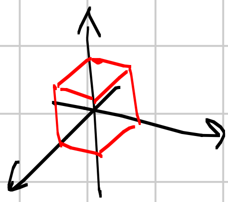
$p_i(a_i) = (a_i - a_0)(a_1 - a_1) \cdot \ldots \cdot \widehat{(a_i - a_i)} \cdot \ldots \cdot (a_i - a_{n+1})$

$f(a_1, a_2, \ldots, a_n) = \prod_{0 \le i < j \le n+1} |a_i - a_j|$

$f : [0,1]^n \longrightarrow \mathbb{R}$

$\{(a_1, a_2, \ldots, a_n) \in \mathbb{R}^n : a_i \in [0,1]\ \forall i\}$ chiuso e limitato





$\exists\ (a_1, a_2, \ldots, a_n)$ tale che f è max in quel punto

$f(a_1, a_2, \ldots, \underset{\uparrow a_i}{x}, \ldots, a_n) = |p_i(x)| \cdot |q(a_1, \ldots, a_n)|$

Osservazione, nessuno degli a_i è sul bordo, perché $f = 0$ sul bordo