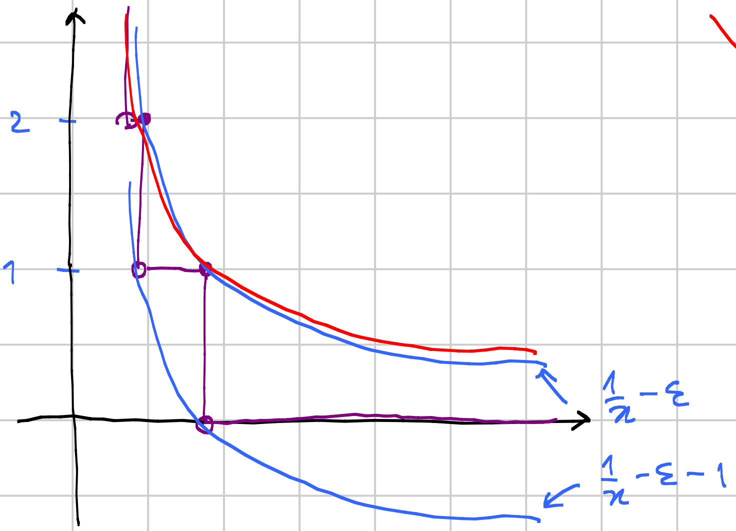
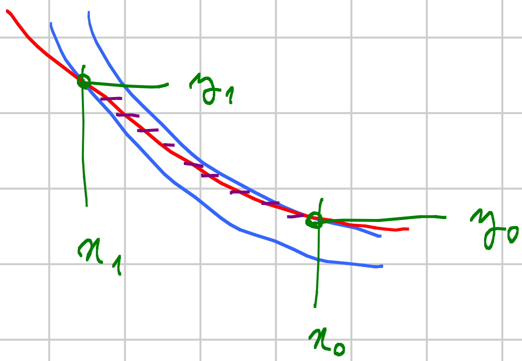
ALGEBRA

Titolo nota — 28/05/2013

$\boxed{1}$ $\quad \dfrac{1}{x+\varepsilon} = \left\lfloor \dfrac{1}{x} - \varepsilon \right\rfloor \qquad \varepsilon := n^{-2}$





$a \in \{0,1\}$

$$\frac{1}{x+\varepsilon} = \frac{1}{x} - \varepsilon - a \qquad \cancel{x} = \cancel{x} + \varepsilon - (\varepsilon + a)\, x\,(x+\varepsilon)$$

$$(\varepsilon+a)x^2 + (\varepsilon+a)\varepsilon x - \varepsilon = 0$$

$$x^2 + \varepsilon x - \frac{\varepsilon}{a+\varepsilon} = 0$$

$$x_{0,1} = \frac{\sqrt{\varepsilon^2 + \frac{4\varepsilon}{a+\varepsilon}} - \varepsilon}{2}$$

$$y_{0,1} = \frac{1}{x_{0,1}+\varepsilon} = \frac{2}{\sqrt{\varepsilon^2 + \frac{4\varepsilon}{a+\varepsilon}} + \varepsilon} = \frac{1}{\sqrt{\frac{\varepsilon}{a+\varepsilon} + \frac{\varepsilon^2}{4}} + \frac{\varepsilon}{2}}$$

$$\frac{1}{\sqrt{1+\frac{\varepsilon^2}{4}} + \frac{\varepsilon}{2}} < 1$$

$$\approx \frac{1}{\sqrt{\varepsilon}} = n$$

$\sqrt{1+x} \approx 1 + \dfrac{x}{2}$ per x piccolo

$\sqrt{1+x} \leq 1 + \dfrac{x}{2}$

$(1+\varepsilon)(1-\varepsilon) = 1 - \varepsilon^2 < 1$

$\dfrac{1}{1+\varepsilon} > 1 - \varepsilon$

$$y_1 = \frac{1}{\sqrt{\varepsilon}} \cdot \left(\sqrt{\frac{1}{1+\varepsilon} + \frac{\varepsilon}{4}} + \frac{\sqrt{\varepsilon}}{2} \right)^{-1} \approx \frac{1}{\sqrt{\varepsilon}} \left(\underbrace{\sqrt{1-\varepsilon+\frac{\varepsilon}{4}}}_{\sqrt{1-\frac{3}{4}\varepsilon} \,\approx\, 1-\frac{3}{8}\varepsilon} + \frac{\sqrt{\varepsilon}}{2} \right)^{-1}$$

$$\approx \frac{1}{\sqrt{\varepsilon}} \left(1 + \frac{\sqrt{\varepsilon}}{2}\right)^{-1} \approx \frac{1}{\sqrt{\varepsilon}} \left(1 - \frac{\sqrt{\varepsilon}}{2}\right)$$

$$\approx n\left(1 - \frac{1}{2n}\right) \approx n - \frac{1}{2}$$

1 (2° approccio)

$$\frac{1}{x+\frac{1}{n^2}} = \left\lfloor \frac{1}{x} - \frac{1}{n^2} \right\rfloor = m$$

x positivo

$\Downarrow$

$\boxed{m < n^2}$

$$\frac{1}{x+\frac{1}{n^2}} = m \qquad x = \frac{1}{m} - \frac{1}{n^2}$$

$$m \leq \frac{1}{x} - \frac{1}{n^2} < m+1$$

$$0 \leq \frac{n^2 m}{n^2 - m} - \frac{1}{n^2} - m < 1$$

$$0 \leq \frac{m^2}{n^2 - m} - \frac{1}{n^2} < 1$$

$m \geq 1$ $\qquad$ $m^2 < n^2$

$m = 1, 2, \ldots, n-1$

$\boxed{4}$ **Soluzione furba**: $p(x) = a_n x^n + \dots + a_0$

$$q(x) = p(x+1) - p(x) = a_n \underbrace{\left((x+1)^n - x^n\right)} + a_{n-1}\left((x+1)^{n-1} - x^{n-1}\right) \dots \dots + (a_0 - a_0)$$

$$= a_n \cdot n x^{n-1} + \text{termini di grado minore}$$

Se ho un polinomio di grado n, se applico $n+1$ volte le diff. finite ottengo il polinomio nullo. Al passo prima ho un polinomio costante $= a_n \cdot n!$

$$q_m(x) = \sum_{i=0}^{m} \binom{m}{i} (-1)^i \, p(x+i)$$

$p(x+1) - p(x)$

$p(x+2) - 2p(x+1) + p(x)$

$p(x+3) - 3p(x+2) + 3p(x+1) - p(x)$

$p(x) = x^2$

p	0	1	4	9	16	25	..
$m=1$	1	3	5	7	9		
$m=2$	2	2	2	2			
$m=3$	0	0	0				

$p(k) = 2^k$ se $k = 0, \dots, n$ $\quad p$ di grado n

$p(n+1)$

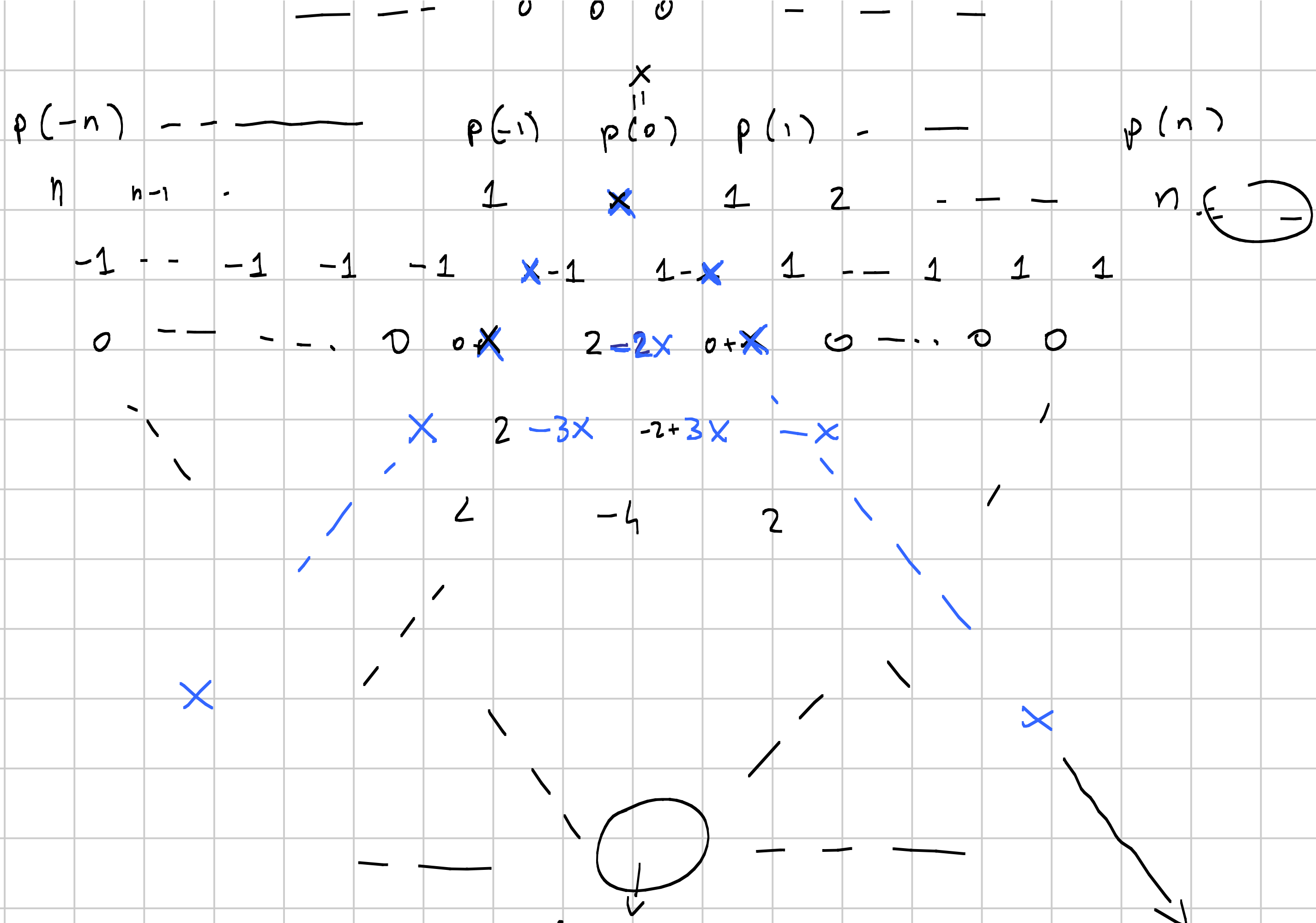
$1 \quad 2 \quad 4 \quad \dots \quad 2^{n-1} \quad 2^n \quad 2^{n+1}-1$

$1 \quad 2 \quad \dots \quad 2^{n-2} \quad 2^{n-1}$

$\cdot 4 \quad 7$

$1 \quad 2 \quad 3$

$1 \quad 1$



$$2\binom{2n-2}{n-1} - \binom{2n}{n} x \qquad x = \frac{n}{2n-1}$$

Polinomi di Lagrange

$p(x_j) = y_j \quad j = 1, 2, \dots, m \qquad \deg p = m-1$

$\exists!$ soluzione se $x_i \neq x_j \quad \forall i \neq j$ esiste ($\exists$) ed è unica ($!$)

un modo: pol di Lagrange

$j = 1, \dots, m \qquad \ell_j(x) := \prod_{\substack{i=1 \\ i \neq j}}^{m} \frac{x - x_i}{x_j - x_i} = \frac{x-x_1}{x_j - x_1} \cdot \ldots \widehat{\frac{x - x_j}{x_j - x_j}} \cdot \ldots \cdot \frac{x - x_m}{x_j - x_m}$

$\ell_j(x_i) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} = \delta_{ij}$ delta di Kronecker

$\sum_{j=1}^{m} y_j \ell_j(x) = p(x)$

$r(x) := p(-x)$ passa per gli stessi punti + unicità $\Rightarrow r(x) = p(x)$

$p(x)$ è pari $\quad p(-x) = p(x) = a_0 + a_2 x^2 + a_4 x^4 + \ldots + a_{2n-2} x^{2n-2} =: q(x^2)$

$q(x)$ è un pol. di grado $n-1$ tale che $q(k^2) = k \qquad k = 1, 2, \dots, n$

$x_k \qquad y_k$

$$p(0) = q(0) = \sum_{j=1}^{n} j\,\ell_j(0) = \sum_{1}^{n} j \prod_{\substack{i\neq j\\ i=1}}^{n} \frac{0-i^2}{j^2-i^2} = \sum j\,(-1)^{n-1}\frac{(n!)^2}{j^2}\prod_{i\neq j}\frac{1}{(j+i)(j-i)}$$

$$= \sum (-1)^{n-1}\frac{(n!)^2}{j}\,\frac{j!\cdot 2j}{(j+n)!}\underbrace{\prod_{i=1}^{j-1}\frac{1}{j-i}}_{\frac{1}{(j-1)!}}\ \underbrace{\prod_{i=j+1}^{n}\frac{1}{j-i}}_{(-1)^{n-j}\frac{1}{(n-j)!}} = \sum_{j=1}^{n}(-1)^{j-1}(n!)^2\frac{\cancel{j!}^{\,j}\,2}{(j+n)!\,(\cancel{j-1})!\,(n-j)!}$$

$$2\frac{(n!)^2}{(2n)!}\sum_{j=1}^{n}\binom{2n}{n-j}(-1)^{j-1}(j-n) + n\,2\frac{(n!)^2}{(2n)!}\sum_{j=1}^{n}\binom{2n}{n-j}(-1)^{j-1}$$

$$(\qquad)\sum_{j=1}^{n-1}2n\binom{2n-1}{n-j-1}(-1)^{j-1} + (\qquad)\sum_{j=1}^{n}\underbrace{\binom{2n}{n-j}}_{\binom{2n-1}{n-j-1}+\binom{2n-1}{n-j}}(-1)^{j-1}$$

$\boxed{3}$ $a,b,c \geq 0 \qquad S:=a+b+c=3$

$$3\sum_{cyc}a^4 + 33 \overset{?}{\geq} 14\sum_{cyc}a^2 \qquad\qquad \sum_{cyc}a^2b = a^2b+b^2c+c^2a$$

$$3\sum_{cyc}a^4 + 33\frac{S^4}{3^4} \overset{?}{\geq} 14\frac{S^2}{3^2}\sum_{cyc}a^2$$

$S = a+b+c$
$Q = ab+bc+ca$
$P = abc$

$$S^2 = \sum_{cyc}a^2 + 2Q \qquad\qquad \sum_{cyc}a^2 = S^2-2Q$$

$$\Big(\sum_{cyc}a^2\Big)^2 = \sum_{cyc}a^4 + 2\sum_{cyc}a^2b^2 \qquad\qquad Q^2 = \sum_{cyc}a^2b^2 + \overbrace{2\sum_{cyc}abc^2}^{2PS}$$

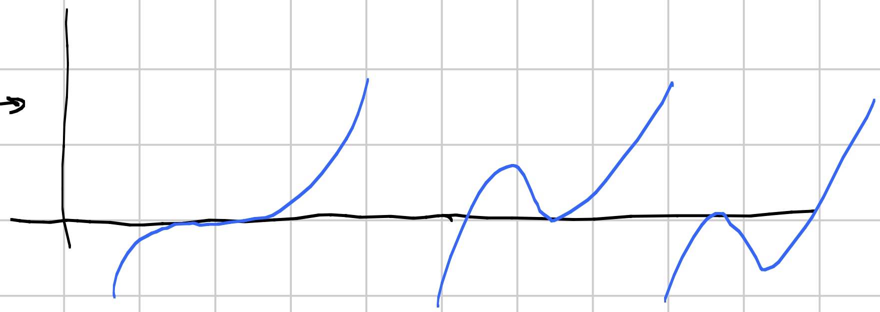
$$\underbrace{(S^2-2Q)^2-2Q^2+4PS} \qquad\qquad Q^2-2PS$$

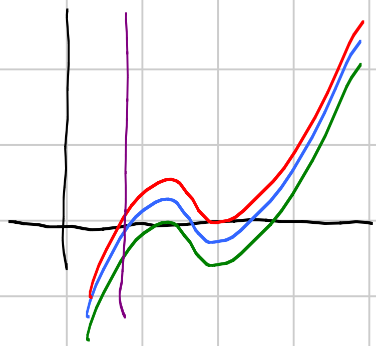
$4SP + \text{roba senza } P \overset{?}{\geq} 0$

$$x^3 - Sx^2 + Qx - P = (x-a)(x-b)(x-c)$$

4S è positivo, quindi basta dimostrare la disug con P minimo

intendo P minimo fissati Q e S



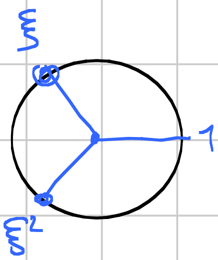


Casi estremali di P: una variabile 0 oppure due uguali
Basta (per questo problema) verificare la disug. con $a=b$

$\boxed{2}$ $$(\sqrt[3]{2}-1)^{n+3} = (a_n + \sqrt[3]{2}\, b_n + \sqrt[3]{4}\, c_n)(2 - 3\sqrt[3]{4} + 3\sqrt[3]{2} - 1)$$
$$= (a_{n+3} + \sqrt[3]{2}\, b_{n+3} + \sqrt[3]{4}\, c_{n+3})$$
$$c_{n+3} = c_n + 3b_n - 3a_n \equiv c_n \pmod 3 \qquad + \text{ base}$$

$$(\sqrt[3]{2}-1)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} (\sqrt[3]{2})^k$$

$$= \underbrace{\sum_{k\equiv 0} \binom{n}{k}(-1)^{n-k} 2^{k/3}}_{a_n} + \sqrt[3]{2}\underbrace{\sum_{k\equiv 1} \binom{n}{k}(-1)^{n-k} 2^{\frac{k-1}{3}}}_{b_n} + \sqrt[3]{4}\underbrace{\sum_{k\equiv 2} \binom{n}{k}(-1)^{n-k} 2^{\frac{k-2}{3}}}_{c_n}$$



$$\xi(\xi\sqrt[3]{2}-1)^n = \xi a_n + \xi\cdot\xi\sqrt[3]{2}\, b_n + \xi\cdot\xi^2\sqrt[3]{4}\, c_n$$
$$\xi^2(\xi^2\sqrt[3]{2}-1)^n = \xi^2 a_n + \xi^2\cdot\xi^2\sqrt[3]{2}\, b_n + \xi^2\cdot\xi\sqrt[3]{4}\, c_n$$
$$(\sqrt[3]{2}-1)^n = a_n + \sqrt[3]{2}\, b_n + \sqrt[3]{4}\, c_n$$
$$\xi\alpha^n + \xi^2\beta^n + \gamma^n = 0 + 0 + 3\sqrt[3]{4}\, c_n \qquad \text{sommo}$$

$$c_n = \frac{1}{3\sqrt[3]{4}}\left[\xi\alpha^n + \xi^2\beta^n + \gamma^n\right]$$

cerco il polinomio che ha per radici α, β, γ
$$(x-\alpha)(x-\beta)(x-\gamma) = \dots = x^3 + 3x^2 + 3x - 1$$
$$c_{n+3} = -3c_{n+2} - 3c_{n+1} + c_n$$

$$f_{n+2} = f_{n+1} + f_n$$
$$x^2 = x + 1$$

$$(\sqrt[3]{2}-1)^n(\sqrt[3]{2}-1)$$

$$x^2 - x - 1 \to \alpha, \beta$$
$$f_n = a\alpha^n + b\beta^n$$