ALGEBRA

Pre-IMO 2013 pomeriggio

Titolo nota 28/05/2013

ES 5

$N \in \mathbb{N}_0 \qquad a \in (0,1)$

Determinare $C(a,N)$ t.c.

per ogni scelta di $k_1, \dots, k_n \in \mathbb{Z}$ distinti

$$(1) \qquad \left(\sum_{i=1}^{n} a^{k_i}\right)^N \le C(a,N) \sum_{i=1}^{n} a^{k_i \cdot N}$$

WLOG $k_1 < \dots, < k_n$

WLOG $k_1 = 0$

$$(2) \qquad a^{-Nk_1}\left(\sum_{i=1}^{n} a^{k_i}\right)^N \le Ca(N)\left(\sum_{i=1}^{n} a^{k_i \cdot N}\right) a^{-Nk_1}$$

la (1) è vera $\Leftrightarrow$ la (2) è vera

Per induzione

"togliere il primo elemento".

IDEA

$n=1$ vera

$$(a^0)^N \le \frac{1-a^N}{(1-a)^N}\, a^{0\cdot N}$$

$$1 \qquad\qquad 1$$

Claim

$$C(a,N) = \frac{1-a^N}{(1-a)^N}$$

$$(1-a)^N \overset{?}{\le} 1-a^N$$

$$a^N + (1-a)^N \overset{?}{\le} 1$$

$$\| $$

$$(a + (1-a))^N$$

$\boxed{n-1 \Rightarrow n}$

$$\sum_{i=1}^{n} a^{k_i} = 1 + a^{k_2} \sum_{i=2}^{n} a^{k_i - k_2}$$

$$\text{LHS} = \Big(1 + \overbrace{a^{k_2} \sum_{i=2}^{n} a^{k_i - k_2}}^{S}\Big)^N =$$

$$= \underbrace{1 + NS + \binom{N}{2} S^2 + \dots + N S^{N-1}} + S^N$$

$$S^N = \Big(a^{k_2} \underbrace{\sum_{i=2}^{n} a^{k_i - k_2}}_{n-1 \text{ termini}}\Big)^N \leq C(a,N) \sum_{i=2}^{n} a^{N k_i}$$

$$\| \quad \sum_{i=2}^{n} a^{k_i}$$

$$\sum_{i=2}^{n} a^{k_i - k_2} \leq \underbrace{\sum_{i=0}^{k_n - k_2} a^i}_{+ \text{ termini}} \leq \underbrace{\sum_{i=0}^{k_n} a^i}_{\text{somma "geometrica"}} =$$

$$\overset{(a \neq 1)}{=} \frac{a^{k_n+1} - 1}{a - 1} = \frac{1 - a^{k_n+1}}{1-a} \leq \frac{1}{1-a}$$

$$S = a^{k_2} \sum_{i=2}^{n} a^{k_i - k_2} \leq a \frac{1}{1-a}$$

$$a^{k_2} \leq a \ ; \quad k_2 \geq 1 \quad 0 < a < 1$$

aggiungo e tolgo

$$\leq \underbrace{1 + N\frac{a}{1-a} + \binom{N}{2}\Big(\frac{a}{1-a}\Big)^2 + \dots + N\Big(\frac{a}{1-a}\Big)^{N-1} + \Big(\frac{a}{1-a}\Big)^N}_{\text{termine a termine}} - \Big(\frac{a}{1-a}\Big)^N =$$

$$= \Big(\underbrace{1 + \frac{a}{1-a}}_{\frac{1}{1-a}}\Big)^N - \frac{a^N}{(1-a)^N} = \frac{1 - a^N}{(1-a)^N} = C(a,N)$$

Manca l'ottimalità.

Consideriamo $k_1 = 0, k_2 = 1, \dots, k_n = n-1$

$$\underbrace{\left(\sum_{i=0}^{n-1} a^i\right)^N} \le C \underbrace{\sum_{i=0}^{n-1} a^{Ni}}$$

la costante C deve verificare: (per ogni n)

$$\left(\frac{1-a^n}{1-a}\right)^N \le C \, \frac{1-a^{nN}}{(1-a^N)}$$

$$C \ge \frac{(1-a^n)^N}{(1-a)^N} \, \frac{(1-a^N)}{1-a^{nN}} \ge C(a,N) \cdot \underbrace{\frac{(1-a^n)^N}{(1-\underbrace{a^{nN}})}}$$

se n "tende all'infinito" $\Rightarrow$ tendono a 0

questo termine diventa arbitrariamente vicino ad 1

Dunque $C \ge k \cdot C(a,N)$ per ogni $k < 1$

$$C \ge C(a,N)$$

Es. n° 6

$P(x,y)$: $$f(x+yf(x)) = f(f(x)) + xf(y)$$

$P(0,y)$: $$f(yf(0)) = f(f(0))$$

$$\boxed{y = \frac{f(x)-x}{f(x)}}$$

2 casi : $f(0)=0$ ok. ho guadagnato $f(0)=0$

altrimenti $yf(0)$ può essere qualunque cosa

$$f(\text{qualunque cosa}) = f(f(0)) = c$$

Sostituisco e ottengo $c=0$.

D'ora in poi $f(0)=0$ e $f \not\equiv 0$

$P(x,0)$: $$f(x) = f(f(x))$$

Quindi $f(z) = z$ se $z \in \operatorname{Im} f$.

Prendiamo c tale che $f(c)=0$

$P(c,y)$ $$\underbrace{f(c+yf(c))}_{f(c)=0} = \underbrace{f(f(c))}_{0} + cf(y)$$

$$0 = cf(y)$$

abbiamo già escluso $f \equiv 0$, ma allora $c=0$.

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iniettività : supponiamo che f non sia iniettiva, dunque esistono $x \neq x'$ t.c. $f(x) = f(x')$

$$f(x + yf(x)) = f(x) + xf(y)$$

pongo $x + yf(x) = x'$ $\qquad \dfrac{x'-x}{f(x)} = y$ $\qquad$ se $f(x) \neq 0$ ok. ma l'abbiamo detto prima.

$$\cancel{f(x')} = \cancel{f(x)} + xf(y)$$

$$0 = xf(y) \qquad y = 0 \Rightarrow x' = x.$$

o

$$\cancel{f}(f(x)) = \cancel{f}(x) \qquad f(x) = x$$

$$f(x + yf(x)) = f(x) + xf(y)$$

$$f(x + f(y)f(x)) = f(x) + xf(f(y)) = f(x) + xf(y)$$

$$\cancel{x} + y\cancel{f(x)} = \cancel{x} + f(y)\cancel{f(x)}.$$

A7, marzo 0/3 trovare il più piccolo $M \in \mathbb{R}$ t.c.

$$\left| ab(a^2-b^2) + bc(b^2-c^2) + ca(c^2-a^2) \right| \leq M(a^2+b^2+c^2)^2$$

$\forall$ $a,b,c \in \mathbb{R}$.

- $ab(a^2-b^2) + bc(b^2-c^2) + ca(c^2-a^2) = (a-b)(b-c)(c-a)(a+b+c)$

$$ab(a-b) + bc(b-c) + ca(c-a) = (a-b)(b-c)(c-a)$$

$$ab(a^3-b^3) + bc(b^3-c^3) + ca(c^3-a^3) =$$

$$\left|(a-b)(b-c)(c-a)(a+b+c)\right| \leq M(a^2+b^2+c^2)^2$$

$$\underset{x}{(a-b)^2} + \underset{y}{(b-c)^2} + \underset{z}{(c-a)^2} + \underset{t}{(a+b+c)^2} = 3(a^2+b^2+c^2)$$

$$|xyzt| \leq \frac{M}{9}(x^2+y^2+z^2+t^2)^2 \qquad (x+y+z=0)$$

$|x|, |y|, |z|$ $\qquad |x|+|y| = |z|$

$$|xy| \leq \frac{z^2}{4} \qquad \sqrt{|xy|} \leq \frac{|x|+|y|}{2} = \frac{|z|}{2} \leq \sqrt{\frac{|x|^2+|y|^2}{2}}$$

$$|x|^2+|y|^2 \geq \frac{z^2}{2}$$

$x = -\frac{z}{2}$ $\quad y = -\frac{z}{2}$

$$|xyzt| \leq \frac{|z^3 t|}{4} \leq \frac{M}{9}\left(\frac{3}{2}z^2+t^2\right)^2 \leq \frac{M}{9}(x^2+y^2+z^2+t^2)^2$$

$$z^3 t \le \frac{4}{9} M \left(\frac{3}{2}z^2 + t^2\right)^2$$

$$\frac{\frac{3}{2}z^2 + \frac{1}{2}\cdot 2t^2}{\frac{3}{2}+\frac{1}{2}} = \frac{\frac{3}{2}z^2 + \frac{1}{2}2t^2}{\frac{3}{2}+\frac{1}{2}} \ge \left((z^2)^{\frac{3}{2}} \cdot (2t^2)^{\frac{1}{2}}\right)^{\frac{1}{2}}$$

$$\left(\frac{\frac{3}{2}z^2+t^2}{2}\right)^2 \ge \sqrt{2}\, t z^3$$

$$\frac{4M}{9} = \frac{1}{4\sqrt{2}} \qquad \boxed{M = \frac{9}{16\sqrt{2}}} \qquad \frac{1}{4\sqrt{2}} \ge \frac{t z^3}{\left(\frac{3}{2}z^2+t^2\right)^2}$$

UNSMOOTHING $\quad f(a,b,c) = LHS$

$a \le b \le c$

a^2+c^2 fissato $\quad$ LHS è più grande se $a=0$

$$f(a,b,c) \le f(0, b, \sqrt{a^2+c^2})$$

SMOOTHING $\quad f(a,b,c) \le f\left(\frac{a+b}{2}, \frac{a+b}{2}, c\right)$

N° 8

Trovare i polinomi $p(x)$ tali che $p(a)+p(b)$ è un quadrato perfetto ogni volta che $a, b \in \mathbb{N}_0$, $a+b=\square$.

Lemma. Se $q(x)$ è un polinomio tale che $q(n)$ è un quadrato di un intero $\forall\, n \geq n_0$, allora $q(x)=(r(x))^2$.

$$p(x^2-a)+p(a)=q_a(x)$$

$\forall\, n>\sqrt{a}$ so che $q_a(n)=\square$

$$=(r_a(x))^2$$

$x^2-a=y$

$$p(y)+p(a)=(r_a(\sqrt{a+y}))^2 \qquad a+y\geq 0$$

$\qquad\qquad\qquad\qquad\qquad\qquad x$

$$p(y)+p(a)=(r_1(a+y)+\sqrt{a+y}\; r_2(a+y))^2$$

$$r_a(x)=a_n x^n+a_{n-1}x^{n-1}+\dots+a_0$$

$$r_a(\sqrt{a+y})=a_n(\sqrt{a+y})^n+\dots$$

$$a_{2i}(\sqrt{a+y})^{2i}=a_{2i}(a+y)^i \qquad a_{2i+1}(\sqrt{a+y})^{2i+1}=a_{2i+1}(a+y)^i\sqrt{a+y}$$

$$p(y) + p(a) = r_1^2(a+y) + (a+y)\, r_2^2(a+y) + \underbrace{2\sqrt{a+y}\; r_1(a+y) r_2(a+y)}_{=0 \; *}$$

$$p(y) + p(a) - r_1^2(a+y) - (a+y)\, r_2^2(a+y) \quad \text{e' un polinomio}$$

$$\left\{ a+y \neq \left(\frac{p_1(a,y)}{p_2(a,y)}\right)^2 \qquad V_a\left(\frac{a_n y^n + \dots}{a_m y^m + \dots}\right) = n-m \right\}$$

$* \Rightarrow \quad r_1 \equiv 0 \quad$ opp. $\quad r_2 \equiv 0$

1. $p(y) + p(a) = r_1^a(y+a)^2$

2. $p(y) + p(a) = (y+a)\, r_2(y+a)^2$

$p(x) = 2k^2$

$(p \not\equiv c)$

(1)

$$p(y) + p(a) = r_1^a(y+a)^2$$

$$p(y) + p(b) = r_1^b(y+b)^2$$

$$p(b) - p(a) = r_1^b(y+b)^2 - r_1^a(y+a)^2 =$$

$$= \underbrace{\left(r_1^b(y+b) - r_1^a(y+a)\right)}\underbrace{\left(\quad + c\right)}$$

$\Rightarrow \quad \text{↯}$

(2) $$p(x)+p(a) = \boxed{(a+x)\,r_a(x+a)^2}$$
$$p(x)+p(b) = \boxed{(b+x)\,r_b(x+b)^2}$$

CLAIM: $\exists$ a,b t.c. $\big(p(x)+p(a),\ p(x)+p(b)\big)=1$;
ogni volta che $a \neq b$.

$\deg = n$: per Bezout (polinomi) esistono
due polinomi Q_1 e Q_2 di $\deg \leq n-1$
tali che

$$Q_1 P_1 + Q_2 P_2 = \text{cost.}$$

$$Q_1(x)\big(p(x)+p(a)\big) + Q_2(x)\big(p(x)+p(b)\big) = \text{cost.}$$

$$\underbrace{\big(Q_1(x)+\cancel{Q_2(x)}\big)p(x)}_{\geq n} + \underbrace{p(a)\,Q_1(x)+p(b)\,Q_2(x)}_{\deg \leq n-1} = \text{cost.}$$

$$Q_1 = -Q_2$$

$$p(a)\,Q_1 - p(b)\,Q_2 = \text{cost.}$$

$$Q_1 = -Q_2 = 1$$

$$\big(p(x)+p(a)\big) - \big(p(x)+p(b)\big) = \text{cost.}$$

———— 0 ————

A domani vi faccio la fine. –

$\cancel{p(x^2) = (q(x))^2}$

$$P(x) = \square \quad \text{per} \quad n \geq n_0.$$

p di grado pari $\quad \partial p = 2n$

1° $\exists\, A$, Q monico tale $\partial\left(P(x) - A\,(Q(x))^2\right) \leq n-1$

$$A = a_{2n} \qquad 2\, q_{n-3} \cdot a_{2n} = \boxed{}$$

2° $$\sqrt{P(x)} - \sqrt{A}\, Q(x) = \frac{\overbrace{P(x) - A Q^2(x)}^{\partial \leq n-1}}{\underbrace{\sqrt{P(x)} + \sqrt{A}\, Q(x)}_{\partial = n}} \sim \frac{1}{x} \xrightarrow[x \to \infty]{} 0$$

$$a_i = \sqrt{P(m+i)}$$

$$\sum_{i=1}^{n} a_i \binom{n}{i} (-1)^i = C \qquad \text{per } m \text{ abbastanza grande}$$

$$\sum_{i=1}^{n} q(i) \binom{n}{i} (-1)^i = n! \qquad \text{sempre}$$

$$\sum_{i=1}^{n} \underbrace{\left(a_i - \sqrt{A}\, q_i\right)}_{\downarrow\, 0} \binom{n}{i} (-1)^i \longrightarrow 0$$

se m è abbastanza grande

$$\underbrace{\sum_{i=1}^{n} a_{m+i} \binom{n}{i} (-1)^i}_{\in \mathbb{N}} \longrightarrow n!\sqrt{A} \quad \text{per } m \to \infty$$

$$a_{m+i} = \sqrt{p(m+i)} \in \mathbb{N}$$

$$a_{m+i} \in \mathbb{N} \Rightarrow \sum_{i=1}^{n} a_{m+i} \binom{n}{i} (-1)^i = S_m \in \mathbb{N}$$

$$S_m \longrightarrow l \quad \Rightarrow \quad l \in \mathbb{N} \qquad n!\sqrt{A} \in \mathbb{N}$$

$$|S_m - l| < \frac{1}{2} \quad \text{per } m \geq m_0 \quad \Rightarrow \quad \underset{m \geq m_0}{S_m = l}$$

$$a_m = f(m) \quad \forall \; m \geq m_0$$

$$\partial f \leq n$$

$$p(x) - (f(x))^2 = 0 \quad \text{per } x = m_0, \; m_0+1, \; m_0+2 \; \ldots$$

$$p(x) = (f(x))^2 \quad \text{come poli.}$$

$$p(x^2) = (q(x))^2 \qquad q(x)^2 = q(-x)^2$$

$$(q(x) - q(-x))(q(x) + q(-x)) = 0$$

$$p(x^2) = (q(x^2))^2 \qquad p(x^2) = x^2 (r(x^2))^2$$

$$p(x) = x\, r(x)^2$$

Parte II° $\quad p$ ha grado dispari

a) $p(x) + p(a) = (x+a)\, R(x+a)^2$

$p(x) + p(b) = (x+b)\, K(x+b)^2 \implies 2RR'(x+a) + R^2 = K'K(x+b) + K^2$

$$2R'(x+a) + R = K(x+b)$$

$$= \delta(x+c)$$

$$\text{|||}$$

cost.

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b) $p(2x^2) + p(2x^2) = (Q(x))^2$

$2p(2x^2) = (Q(x))^2 \rightsquigarrow a_n \cdot 2^{n+1} = b_n^2 \Rightarrow a_n = \square$ (n dispari)

$p(x^2) + p(3x^2) = (Q(x))^2 \rightsquigarrow (3^n + 1)\, a_n = c_n^2$

$\Rightarrow 3^n + 1 = \square \qquad n$ dispari

$3^n + 1 = k^2 \qquad k = 2$

$3^n = (k-1)(k+1) \qquad \boxed{n=1}$.

II^a parte   A8: p ha grado dispari.

$$p(x) + p(a) = (x+a)\, R(x)^2$$

$R(x)^2 (x+a) - S(x)^2 (x+b) = p(a) - p$

$$p(x) + p(b) = (x+b)\, S(x)^2$$

$$p(x) + p(c) = (x+c)\, T(x)^2$$

$$p'(x) = R(x)^2 + 2(x+a) R' \cdot R$$

$$p'(x) = S^2 + 2(x+b) S' \cdot S$$

$$R\left(R + 2(x+a)R'\right) = S\left(S + 2(x+b)S'\right)$$

$$R \mid \cancel{S}\left(S + 2(x+b)S'\right)$$

$$\Downarrow$$

$$R = S + 2(x+b)S'$$

$$T = S + 2(x+b)S'$$

$$R = T, \qquad (R, T) = 1$$

$$\Rightarrow \quad R = T \equiv c$$

$$\rightsquigarrow \quad p(x) + p(a) = c(x+a) \qquad p(x) = cx$$

$$p(x) = k^2 x$$

$$p(2x^2) + p(2x^2) = Q(x) = \left(\tilde{Q}(x)\right)^2$$

$$2\,p(2x^2) = \left(\tilde{Q}(x)\right)^2$$

$\downarrow$ Leading coeff.

$$2 \cdot 2^n \cdot a_n = r^2 \qquad 2^{n+1} a_n = r^2 \Rightarrow a_n = \square.$$

$$p(x^2) + p(3x^2) = \left(Q_2(x)\right)^2$$

$$(3^n + 1) = k^2 \qquad 3^n + 1 = k^2$$

$$3^n = (k-1)(k+1)$$

$$\boxed{n = 1}$$