pre IMO 2014

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$\boxed{1}$ $f(x+y-z)+f(2\sqrt{xz})+f(2\sqrt{yz})=f(x+y+z)$

$\forall x,y,z\in\mathbb{R}^+ : z\leq x+y$ $\qquad f:\mathbb{R}^+\to\mathbb{R}^+$

$x=y=z=0 \qquad 3f(0)=f(0) \Rightarrow \boxed{f(0)=0}$

$x=y \quad z=2x \qquad 0+2f\left(2\sqrt{x\cdot 2x}\right)=f(4x)$

$2f(2\sqrt{2}x)=f(4x) \qquad x=\frac{1}{2\sqrt{2}}y$

$\boxed{2f(y)=f(\sqrt{2}y)}$

$f(2x)=4f(x)$

$f(x)\approx x^2 \qquad g(x):=\sqrt{f(x)}$ non è utile in questo caso

$\boxed{h(x):=f(\sqrt{x})} \qquad h:\mathbb{R}^+\to\mathbb{R}^+$

$y=0 \qquad\qquad 0\leq z\leq x$

$h(\underbrace{x^2+z^2-2xz}_{a})=f(x-z)=f(x+z)-f(2\sqrt{xz})=h(\underbrace{x^2+z^2+2xz}_{a+b})-h(\underbrace{4xz}_{b})$

Cauchy? $\quad \forall a,b\in\mathbb{R}^+ \quad h(a)=h(a+b)-h(b)$

domanda: fissati a,b, è vero che $\exists\ x,z$:

$$\left.\begin{array}{l} a=x^2+z^2-2xz \\ b=4xz \end{array}\right\} a+b=x^2+z^2+2xz$$

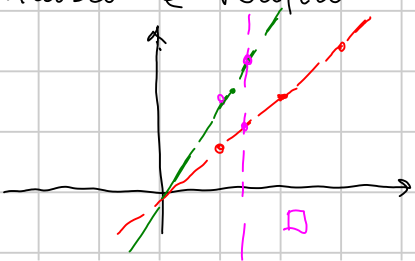
$\sqrt{a}=x-z \qquad \sqrt{a+b}=x+z \qquad x=\frac{\sqrt{a+b}+\sqrt{a}}{2} \qquad z=\frac{\sqrt{a+b}-\sqrt{a}}{2}$

Cauchy ok. Ma la condizione?

Siccome $h:\mathbb{R}^+\to\mathbb{R}^+$ non ci sono punti del grafico nel IV quadr.

$h(x)=\lambda x \qquad \lambda\in\mathbb{R}$

Sostituisco e verifico e trovo $f(x)=\lambda x^2$ con $\lambda\geq 0$



$\boxed{2}$ $p(x)$ di grado n $\rightarrow$ $n+1$ coeff.

$\alpha_0, \alpha_1, \dots, \alpha_n$

$y_0, y_1, \dots, y_n$ $\qquad p(\alpha_i) = y_i \quad \forall i = 0, 1, \dots, n \longrightarrow n+1$ condizioni

(sse $\alpha_i \neq \alpha_j \quad \forall i \neq j$)

Come trovo p ?

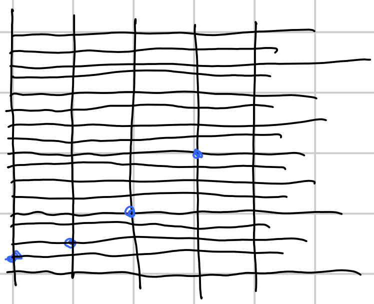
1) Inverto la matrice di Vandermonde
2) Formula di Lagrange

$$p(x) = \sum_{i=0}^{n} \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{(x-\alpha_j)}{(\alpha_i - \alpha_j)} y_i$$

3) Metodi con le mani

$$p(x) = y_0 + (x-\alpha_0)\left(\frac{y_1 - y_0}{\alpha_1 - \alpha_0} + (x-\alpha_1)\left(\frac{y_2 - y_1}{\alpha_2 - \alpha_1} - \frac{y_1 - y_0}{\alpha_1 - \alpha_0} + (x-\alpha_2)\left(\dots \right)\right)\right)$$

$\alpha_{i+1} - \alpha_i \equiv \text{cost}$



$$1 + x + \frac{x(x-1)}{2} + \frac{x(x-1)(x-2)}{6} +$$

$1 + 3 + 3 + 1$

$1 + 4 + 6 + 4 + 1$

$$p_n(x) = \sum_{k=0}^{n} \binom{x}{k} = \sum_{k=0}^{n} \frac{x(x-1)\cdot \dots \cdot (x-k+1)}{k!} \qquad \text{ha grado } n$$

$$p_n(n) = \sum_{k=0}^{n} \binom{n}{k} = 2^n \qquad p_n(m) = 2^m \qquad m \leq n$$

$\exists!$ un polinomio $p_n(x)$ tale che $p_n(i) = 2^i \quad i = 0, 1, \dots, n$

$$p_n(n+1) = \binom{n+1}{0} + \binom{n+1}{1} + \dots + \binom{n+1}{n} = 2^{n+1} - 1$$

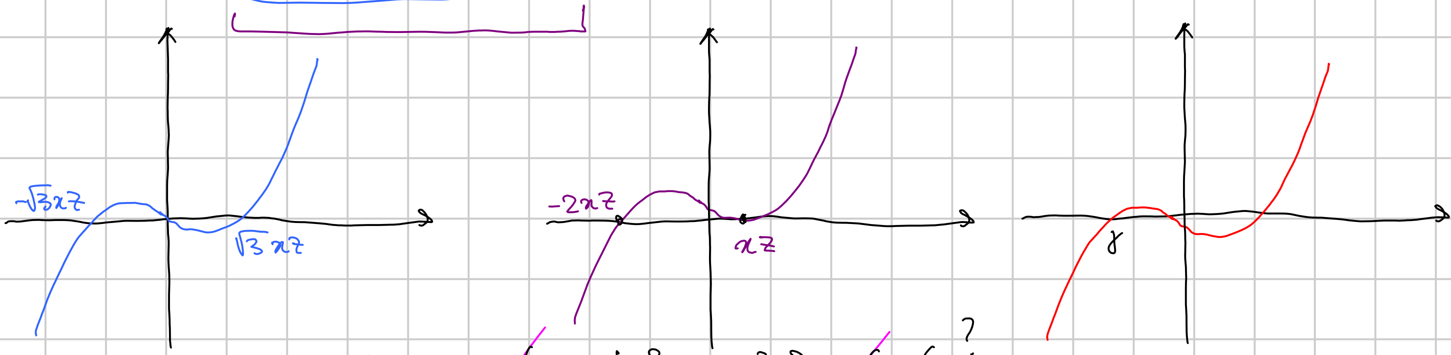
$$p_n(n+2) = 2^{n+2} - 1 - (n+2) = 2^{n+2} - n - 3$$

$\boxed{3}$ $a,b,c \geq 0$ $\qquad \dfrac{a+b+c}{3} - \sqrt[3]{abc} \overset{?}{\leq} \dfrac{2}{3} \max\left\{ (\sqrt{a}-\sqrt{b})^2 ; (\sqrt{b}-\sqrt{c})^2 ; (\sqrt{c}-\sqrt{a})^2 \right\}$

wlog $a \leq b \leq c$ $\qquad \dfrac{a+b+c}{3} - \sqrt[3]{abc} \overset{?}{\leq} \dfrac{2}{3}\left(c+a-2\sqrt{ac}\right)$

$a = x^6$ e cicliche $\qquad x^6 + y^6 + z^6 - 3x^2y^2z^2 \overset{?}{\leq} 2z^6 + 2x^6 - 4x^3z^3$

$t = y^2$ $\qquad \underbrace{\underbrace{t^3 - 3x^2z^2t}_{} + 4x^3z^3}_{} - x^6 - z^6 \overset{?}{\leq} 0$



$t = x^2$ $\qquad \cancel{x^6} - 3x^4z^2 + 4x^3z^3 - \cancel{x^6} - z^6 \overset{?}{\leq} 0$ $\qquad$ fattorizzabile

$t = z^2$ $\qquad$ uguale

$a_0 = -1$

$\boxed{4}$ $f(x) := \displaystyle\sum_{i=1}^{5} \frac{a_i}{x+i} - \frac{1}{x} = \frac{\overbrace{\sum_{j=0}^{5} a_j \prod_{\substack{i=0 \\ i\neq j}}^{5} (x+i)}^{p(x)}}{\underbrace{\prod_{i=0}^{5}(x+i)}_{q(x)}}$ $\qquad f(k^2) = 0 \quad k = 1,2,\dots,5$

$p(x) = f(x)q(x)$ $\qquad p(k^2) = 0 \qquad k = 1,2,\dots,5$

$\overset{!}{=} \underbrace{\sum_{j=0}^{5} a_j}_{d} x^5 + \dots \qquad = d(x-1)(x-4)(x-9)(x-16)(x-25)$

$p(0) = -d\,(5!)^2 = -d \cdot 120^2$

$p(x) = \displaystyle\sum_{j=1}^{5} a_j \prod_{\substack{i=0 \\ i\neq j}}^{5} (x+i) - \prod_{i=1}^{5}(x+i)$

$= \displaystyle\sum_{j=1}^{5} a_j x \prod_{\substack{i=1 \\ i\neq j}}^{5} (x+i) - \prod_{i=1}^{5}(x+i)$ $\qquad p(0) = -\displaystyle\prod_{i=1}^{5}(0+i) = -5! = -120$

$d = \dfrac{1}{120}$

$\displaystyle\sum_{1}^{5} \frac{a_i}{36+i} = f(36) + \frac{1}{36} = \frac{p(36)}{q(36)} + \frac{1}{36} = \frac{\frac{1}{120}(36-1)(36-4)\cdot\ \dots\ \cdot(36-25)}{36\cdot 37\cdot 38 \cdot \dots \cdot 41} + \frac{1}{36}$

$p(x)$ di grado n

$$\begin{array}{ccccccc} p(0) & & p(1) & & p(2) & & p(3) \; ---- \\ & q(0) & & q(1) & & q(2) \; -- & \\ & & r(0) & & r(1) \; --- & & \end{array}$$

dopo

$q(x) = p(x+1) - p(x)$

$q(x)$ ha grado $n-1$

$r(x) = q(x+1) - q(x)$

ha grado $n-2$

Finite difference array

$p(x)$ grado n
passa per $(0,1), \; -- \; (n, 2^n)$

$$\begin{array}{cccccc} 2^0 & 2^1 & 2^2 & ---- & & 2^n \\ 2^0 & 2^1 & 2^2 & ---- & & \\ & & & & & \\ & & 2^0 & 2^0 & 2^0 & 2^0 \; -- \end{array}$$

$\leftarrow p(x)$ grado n

$\leftarrow q(x)$ grado $n-1$

$\leftarrow r(x)$ grado $n-2$

$\leftarrow$ polinomio di grado 0

$p(x) = \binom{x}{\ell}$

Cosa succede se faccio questo lavoro invece partendo da

$$\begin{array}{ccccc} p(0) & & p(1) & & p(2) \quad \cdots \\ & \diagdown & & \diagup & \\ & & q(0) & & \cdots \\ & & \diagdown & \diagup & \end{array}$$

$$q(x) = \binom{x+1}{l} - \binom{x}{l} = \binom{x}{l-1}$$

$$r(x) = \binom{x}{l-2}$$

$$\vdots$$

Se faccio la tabella partendo da $p(x) = \binom{x}{l}$,
la prima diagonale vale, $(\underbrace{0,\ 0,\ 0,\ \ldots}_{l}\ 1,\ 0,\ 0,\ 0\ \ldots)$

$$p(x) = \sum_{i=0}^{d} c_i \binom{x}{l},$$ se partita da

la prima "colonna" vale $c_0\ c_1\ c_2\ c_3\ \ldots\ c_d\ 0\ 0\ 0\ \ldots$

Data la FDA, il polinomio è

$$p(x) = \sum_{l=0}^{\infty} c_l \binom{x}{l},$$

dove c_i sono i valori della prima "colonna storta"

$$p(x) = \sum_{l=0}^{\infty} \binom{x}{l}$$

Usate $\binom{x}{l}$ per il problema dei binomiali

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polinomio di grado n ("$n+1$ gradi di libertà")

$n+3$ condizioni

Possono esistere $q(x)$, $r(x)$ diversi che soddisfano $n+2$ delle condizioni ognuno?

In quanti punti coincidono questi polinomi? $n+1$!