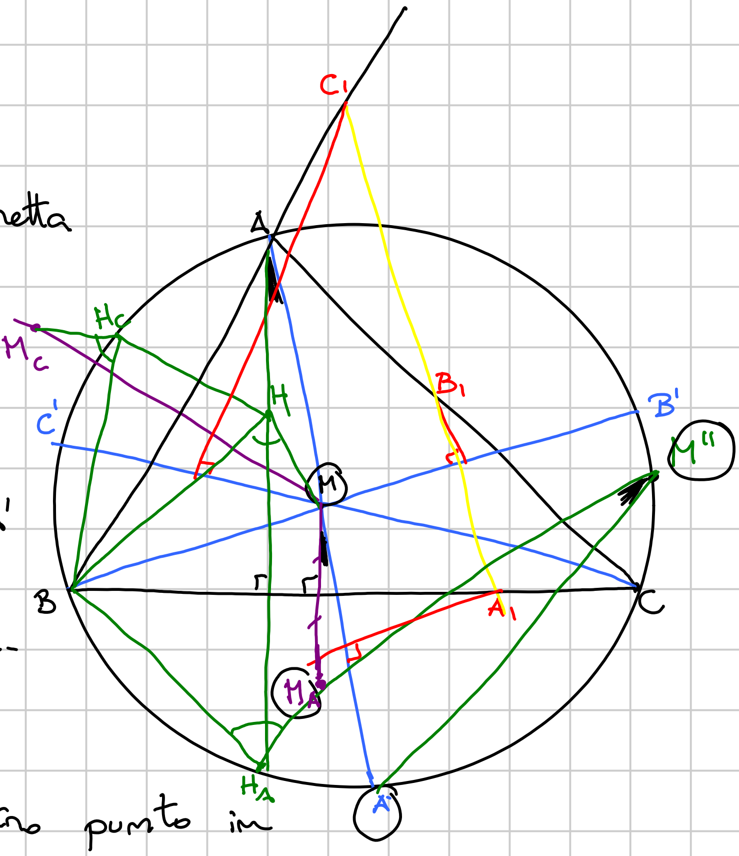
PREIMO 2014 - GP

Titolo nota 26/05/2014

Problema 7



Tesi: A_1, B_1, C_1 sono allineati.
Il simmetrico di M rispetto alla retta $A_1B_1C_1$ sta su Γ_{ABC}.

M_A simmetrico di M rispetto a BC.
Oss: A_1 è circocentro di MM_AA'

Tesi $\iff$ le circonf $\Gamma_{MM_AA'}, \Gamma_{MM_BB'}, \dots$ sono coassiali $\iff$ le tre circonf hanno un altro punto in comune (oltre a M).

M sta su tutte

Oss: il simmetrico di H rispetto a BC sta sulla circonf circoscr.
Dalla figura pare che H_A, M_A, M' siano allineati.

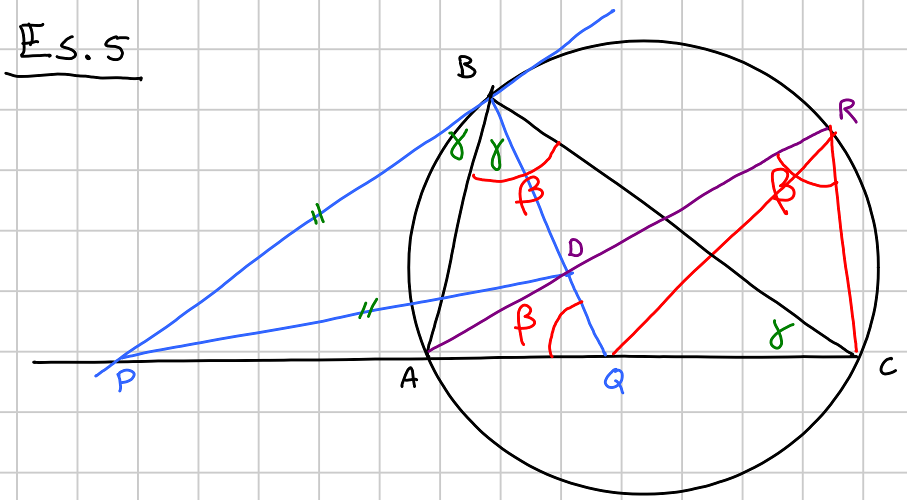
Definiamo $M' = H_AM_A \cap \Gamma$

$M''H_AB = BHM = BH_CM_C$

Segue che $M''H_CM_C$ sono allineati

Per concludere, basta far vedere che $M'' \in \Gamma_{MM_AA'}$

Es. 5



Th : $BQ = QR$

Sol 1

$\rightarrow$ dim che $BQ = QR$

$A\overset{\Delta}{Q}B \sim A\overset{\Delta}{B}C \rightarrow \frac{BQ}{AB} = \frac{BC}{AC} \rightarrow BQ = \frac{a \cdot c}{b}$

calcoliamo QR

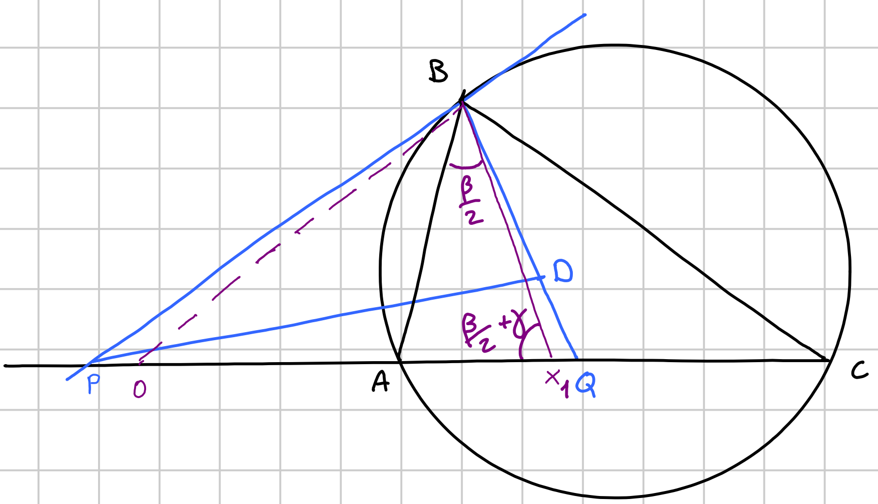
Oss $\rightarrow$ DQCR ciclico ($A\hat{R}C = A\hat{B}C = A\hat{Q}B$)

$A\overset{\Delta}{D}C \sim A\overset{\Delta}{Q}R \longrightarrow \frac{QR}{AQ} = \frac{CD}{AD} \rightarrow QR = AQ \cdot \frac{CD}{AD}$

$\frac{AQ}{AB} = \frac{AB}{AC} \rightarrow AQ = \frac{c^2}{b}$

Th $\Leftrightarrow \frac{a \cdot c}{b} = \frac{c^2}{b} \cdot \frac{CD}{AD} \Leftrightarrow \frac{CD}{AD} = \frac{a}{c}$

Th $\Leftrightarrow$ D sta sulla cfr di Apollonio che passa per B, x_1, x_2
(x_1, x_2 . piedi delle bisettrici da B)



O centro della circonferenza

OBx_1 isoscele

$x_1\hat{B}O = \frac{\beta}{2} + \gamma$

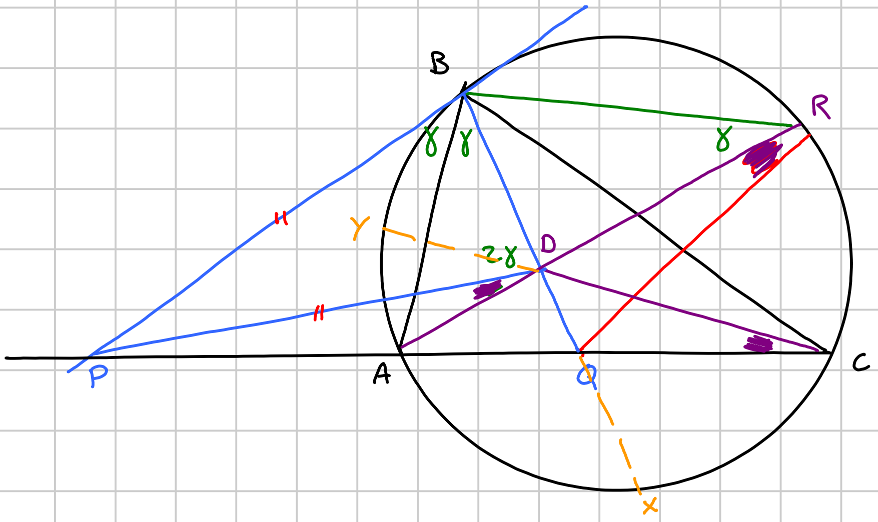
$A\hat{B}O = \gamma$

$A\hat{B}P = \gamma$ (ipotesi)

$O \equiv P$

$PB = PD$ per ipotesi

C.V.D.



Sol 2 $\rightarrow$ $Q\hat{B}R = Q\hat{R}B$

Oss 1 $\rightarrow$ DQCR ciclico

Oss 2 $\rightarrow$ $P\overset{\triangle}{A}D \sim P\hat{D}C$

[
- PB tangente
- $PB^2 = PA \cdot PC$
- $PD^2 = PA \cdot PC$
]

$Q\hat{B}R = A\hat{D}B - D\hat{R}B = 2\gamma + P\hat{D}A - \gamma = \gamma + P\hat{D}A$

$Q\hat{R}B = \gamma + Q\hat{R}D$

$Th \iff P\hat{D}A = Q\hat{R}D$

$\big(P\hat{D}A \underset{\text{oss 2}}{=} Q\hat{C}D \underset{\text{oss 1}}{=} Q\hat{R}D \big)$

CVD

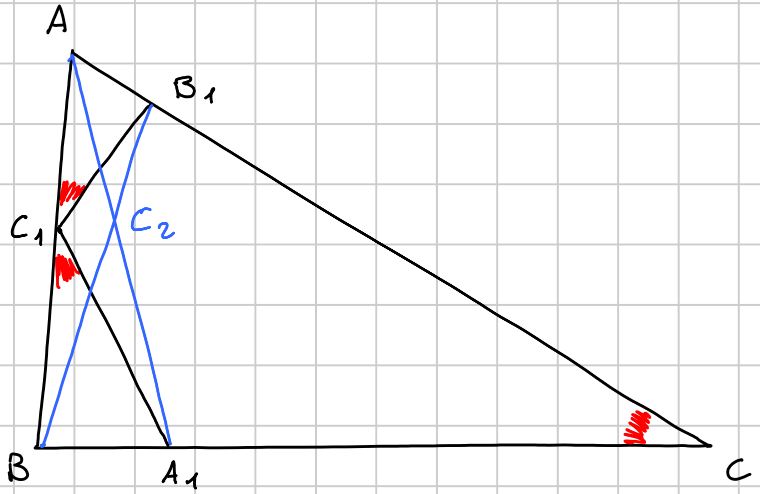
[
altra sol

$X = BD \cap \Gamma$

$Y = CD \cap \Gamma$

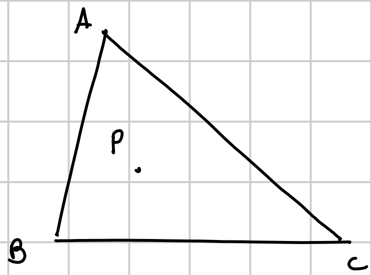
XYR isoscele e simile a BQR
]

Es 6



SOL 1 (CONTI)

COORDINATE BARICENTRICHE



$$P = \big[\lambda[BPC], \lambda[APC], \lambda[APB]\big] \qquad \lambda \in \mathbb{R},\ \lambda \neq 0$$

$$r: \quad \lambda_1 x + \lambda_2 y + \lambda_3 z = 0$$

$$P = [\lambda_1, \lambda_2, \lambda_3] \qquad Q = [\mu_1, \mu_2, \mu_3]$$

$$r_{PQ}: \quad (\lambda_2\mu_3 - \lambda_3\mu_2)x + (\lambda_3\mu_1 - \lambda_1\mu_3)y + (\lambda_1\mu_2 - \lambda_2\mu_1)z = 0$$

$$r_1: \quad \lambda_1 x + \lambda_2 y + \lambda_3 z = 0 \qquad r_2: \mu_1 x + \mu_2 y + \mu_3 z = 0$$

$$W = r_1 \cap r_2 = [\lambda_2\mu_3 - \lambda_3\mu_2, \dots, \dots]$$

$$A = [1, 0, 0] \quad B = [0, 1, 0] \quad C = [0, 0, 1]$$

$$BC_1 = u \qquad AC_1 = v \qquad (u + v = c)$$

$$C_1 = [u, v, 0]$$

$$B\hat{C_1}A_1 \sim B_1\hat{C_1}A \sim B\hat{C}A$$

$$BA_1 = BC_1 \cdot \frac{c}{a} = \frac{uc}{a} \qquad CA_1 = a - BA_1 = a - \frac{uc}{a} = \frac{a^2 - uc}{a}$$

$$A_1 = [0, a^2 - uc, uc]$$

analogamente $\rightarrow$ $B_1 = [b^2 - vc, 0, vc]$

calcolo r_{AA_1}
$$\begin{matrix} 0 & a^2-\mu c & \mu c \\ 1 & 0 & 0 \end{matrix}$$

$$r_{AA_1}: \quad 0\cdot x + (\mu c)\,y + \left(-(a^2-\mu c)\right) z = 0$$

$$-\mu c\, y + (a^2-\mu c)\, z = 0$$

$$r_{BB_1}: \quad -vc\, x + (b^2-vc)\, z = 0$$

calcolo c_2
$$\begin{matrix} 0 & -\mu c & a^2-\mu c \\ -vc & 0 & b^2-vc \end{matrix}$$

$$c_2 = \left[-\mu c\,(b^2-vc),\ -vc\,(a^2-\mu c),\ -\mu v c^2\right] =$$

$$= \left[\mu\,(b^2-vc),\ v\,(a^2-\mu c),\ \mu v c\right]$$

calcolo $r_{c_1c_2}$
$$\begin{matrix} \mu & v & 0 \\ \mu(b^2-vc) & v(a^2-\mu c) & \mu v c \end{matrix}$$

$$r_{c_1c_2}: \quad (\mu v^2 c)\,x + (-\mu^2 vc)\,y + \left[\mu v\,(a^2-\mu c) - \mu v\,(b^2-vc)\right] z = 0$$

$$r_{c_1c_2}: \quad (vc)\,x + (-\mu c)\,y + (a^2-b^2-\mu c+vc)\,z = 0$$

"PASSAGGIO LOSCO"

$$\begin{matrix} r_{\mu=0}: & c^2 & 0 & a^2-b^2+c^2 \\ r_{v=0}: & 0 & -c^2 & a^2-b^2-c^2 \end{matrix}$$

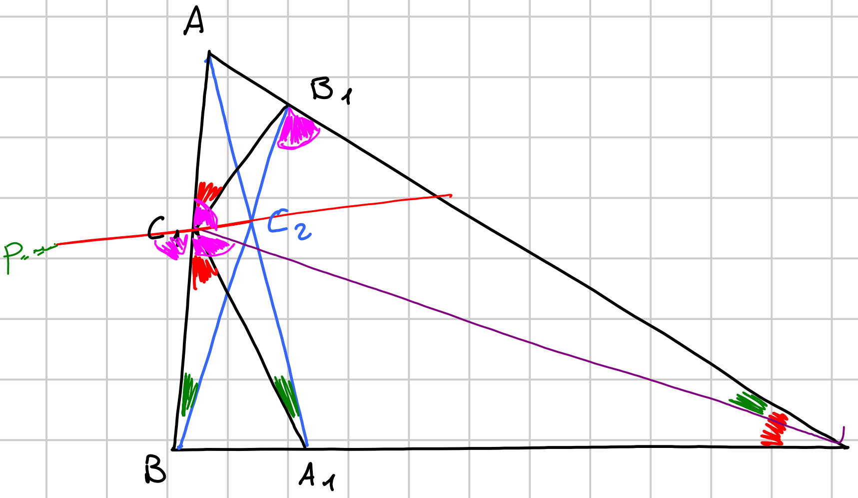
$$P = \left[c^2(a^2-b^2+c^2),\ -c^2(a^2-b^2-c^2),\ -c^4\right] =$$

$$= \left[a^2-b^2+c^2,\ b^2-a^2+c^2,\ -c^2\right]$$

$$vc\,(a^2-b^2+c^2) - \mu c\,(b^2-a^2+c^2) - c^2\,(a^2-b^2+vc-\mu c) \overset{?}{=} 0$$

$$vca^2 - vcb^2 + vc^3 - \mu cb^2 + \mu ca^2 - \mu c^3 - a^2c^2 + b^2c^2 - vc^3 + \mu c^3 \overset{?}{=} 0$$

SOL 2 (sintetica, Sala)



BC_1B_1C ciclico

ACA_1C_1 ciclico

$C_1\hat{B}B_1 = C_1\hat{C}B_1 = C_1\hat{A}_1C_2$

$BA_1C_2C_1$ ciclico

$AB_1C_2C_1$ ciclico

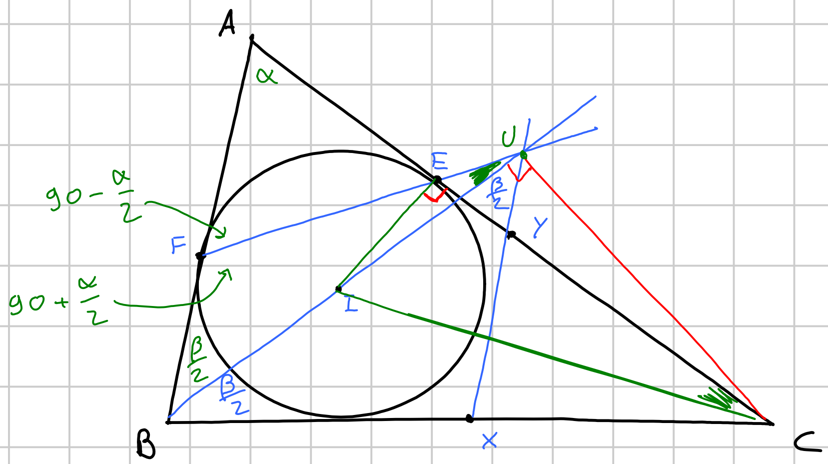
definisco P simmetrico di C rispetto AB

voglio dimostrare che P, C_1, C_2 allineati

DIM

$$A\hat{C}_1C_2 = B\hat{B}_1C = B\hat{C}_1C = B\hat{C}_1P$$

C_1, C_2, P allineati



PUNTO 1 → baricentriche

↘ sintetica

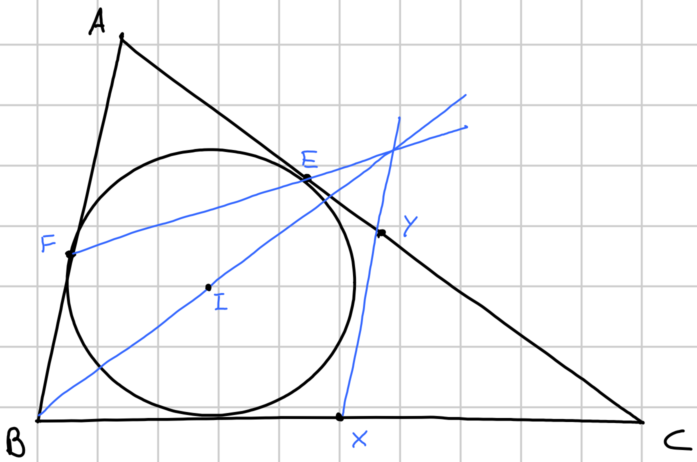
$U \overset{def}{=} EF \cap BI$ Th $\Leftrightarrow$ X, Y, U allineati

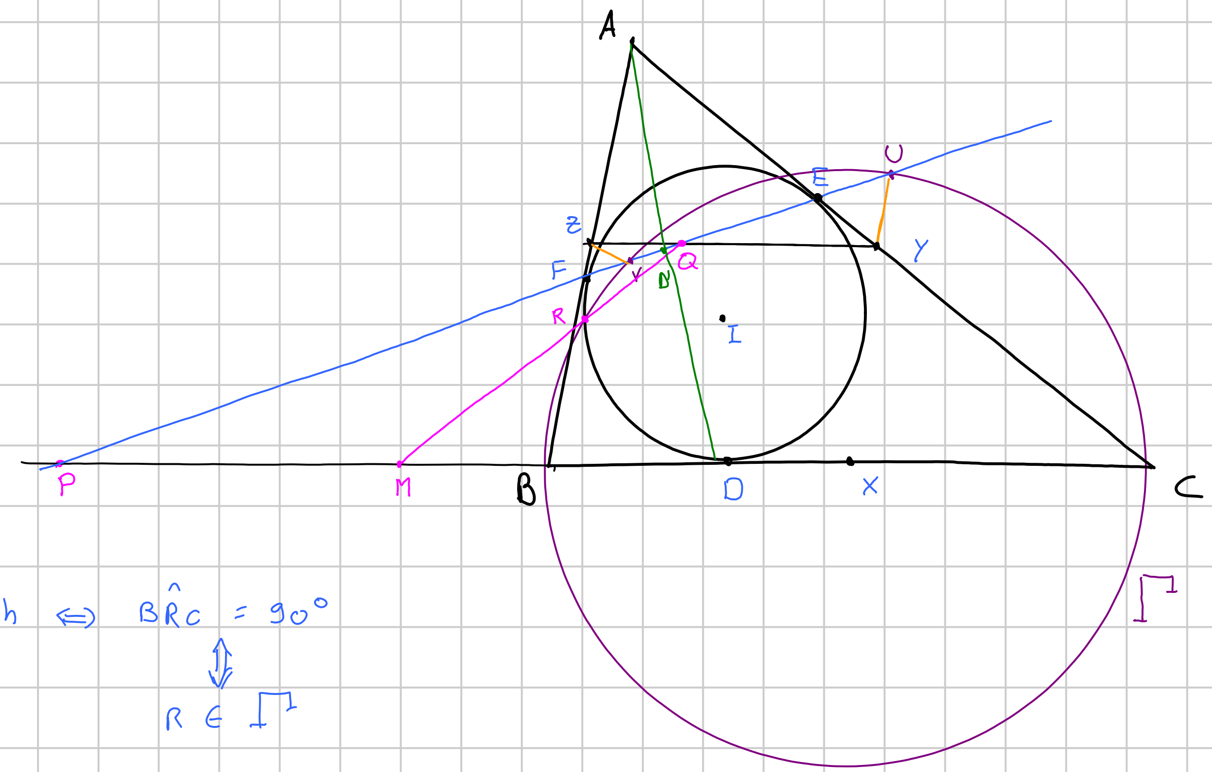
$E\hat{U}I = 180 - \left(90 + \frac{\alpha}{2}\right) - \frac{\beta}{2} = \frac{\gamma}{2} = I\hat{C}E$

ICUE ciclico $\to I\hat{U}C = B\hat{U}C = 90° \to U \in \Gamma^1$

(Γ^1 cfr diametro BC)

$B\hat{X}U = 180 - \beta = B\hat{X}Y$ CVD





Th $\Leftrightarrow$ $B\hat{R}C = 90°$

$\Updownarrow$

$R \in \Gamma$

GUESS : QM asse radicale di ω e Γ

(se è vero implica la tesi)

$M \in$ asse radicale

$\Updownarrow$

$MD^2 = MB \cdot MC$ $\Leftrightarrow$ B e C inversi rispetto alla cfr di diametro PD

$\Updownarrow$

[$D' = AD \cap EF$]

$(P, D', F, E) = -1$ $\Leftrightarrow$ $(P, D, B, C) = -1$

lemma della polare $\Updownarrow$

$D' \in pol_\omega(P)$

$\begin{cases} D \in pol_\omega(P) \ ✓ \\ A \in pol_\omega(P) \end{cases}$ $\Leftrightarrow$ $P \in pol_\omega(A)$ ✓

$AD = pol_\omega(P)$

$Q \in$ asse radicale

$\Updownarrow$

$QF \cdot QE = QU \cdot QV$ $\Leftrightarrow$ $\dfrac{QF}{QU} = \dfrac{QV}{QE}$

$$\left[\begin{array}{ll} \dfrac{QV}{QE} = \dfrac{QZ}{QY} & (Q\overset{\triangle}{Z}V \sim Q\overset{\triangle}{Y}E) \\ \dfrac{QZ}{QY} = \dfrac{QF}{QU} & (\overset{\triangle}{QFZ} \sim Q\overset{\triangle}{Y}U) \end{array}\right]$$