# PREIMO 2014 — N (Mattiua)

Titolo nota 28/05/2014

$$\frac{xy^3}{x+y} = p^3 \qquad xy^3 = p^3(x+y)$$

$(x,y) = d$, $x = da$, $y = db$ $(a,b) = 1$.

$$d^4ab^3 = dp^3(a+b) \qquad d^3ab^3 = p^3(a+b)$$

$(ab^3, a+b) = 1$ Quindi $ab^3 \mid p^3$.

Quindi $b = 1$ o $b = p$

**Caso 1** $b = 1$ $a \mid p^3$ $d^3a = p^3(a+1)$

1a) $p \mid d$ $\left(\frac{d}{p}\right)^3 = \frac{a+1}{a}$ $\Rightarrow$ $a = 1$ $\left(\frac{d}{p}\right)^3 = 2$ $d^3 = 2p^3$

IMPOSSIBILE.

1b) $p \nmid d$ $\Rightarrow p^3 \mid a \Rightarrow a = p^3$

$d^3 \cdot p^3 = p^3(p^3+1)$ $d^3 = p^3 + 1$ IMPOSSIBILE

**Caso 2** $b = p$, $a = 1$.

$d^3 = p^3 = p^3(1+p)$ $d^3 = 1+p$

$d^3 - 1 = p$ $d^3 - 1 = (d-1)(d^2+d+1) = p$.

($d-1 = 1$, $d^2+d+1 = p$)

$d = 2$ $p = 7$

$x = a \cdot d = 2$ $y = bd = 14$.

**Es. 2** $1 \leq a_1 < a_2 < \ldots < a_n \leq 50$

Condizione: Dati $b_1, \ldots, b_n$ $\exists\, m, c_1, \ldots, c_n$ con $c_i^{a_i} = m b_i$.

**Oss.** $(a_i, a_j) = 1$ $\forall i \neq j$. Se no, supponiamo

$a_1 = du$, $a_2 = dv$ $d > 1$.

Posso scegliere, per esempio, $b_1 = 1$ $b_2 = 2$.

$(c_1^u)^d = m$ $(c_2^v)^d = 2m$ IMPOSSIBILE

Cerco $m$ della forma $m = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$,
dove $p_1, \dots, p_k$ sono tutti i fattori primi di $d_1, \dots, d_n$.

$$c_i^{a_i} = m\, b_i \qquad b_i' = p_1^{\beta_{i,1}} \cdots p_k^{\beta_{i,k}} \qquad \beta_{i,k} \geq 0$$

$\longrightarrow$ $\beta_{1,1} + \alpha_1 \equiv 0 \pmod{a_1}$ ... $\beta_{n,1} + \alpha_1 \equiv 0 \pmod{a_n}$

$\vdots$

$\longrightarrow$ $\beta_{1,k} + \alpha_k \equiv 0 \pmod{a_1}$ ... $\beta_{n,k} + \alpha_k \equiv 0 \pmod{a_n}$

CRT $\to \exists$ sol

\# primi $\leq 50 = 15$.

Con 16 $\quad a_i = p, 1$ eccetto

| | | | |
|---|---|---|---|
| $p=2$ | 2, 4, 8, 16, 32 | 5 poss | |
| $p=3$ | 3, 9, 27 | 3 poss. | 60 poss |
| $p=5$ | 5, 25 | 2 poss. | |
| $p=7$ | 7, 49 | 2 poss. | |

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Es. 3 (Da usare: LTE) $\quad p > 2$

$$\alpha \geq 1 \quad p^{\alpha} \parallel A^{B} + 1 \Rightarrow p^{\alpha + v_p(m)} \parallel A^{Bm} + 1$$

Tesi: $\left((n-1)^n + 1\right)^2 \mid n(n-1)^{(n-1)^n + 1} + n \qquad (n \text{ dispari})$

Considero $p \mid (n-1)^n + 1$.

Oss. $n \mid (n-1)^n + 1$

1° caso $p \mid n$, $p^{\alpha} \parallel n \qquad (n-1)^1 + 1 = n$

$$v_p(LHS) = 2(v_p(n) + v_p(n)) = 4\alpha.$$

$$v_p(RHS) = v_p(n) + v_p\left[(n-1)^{(n-1)^n + 1} + 1\right]$$

$$= v_p(n) + v_p(n) + 2v_p(n) = 4\alpha.$$

2° caso $p \nmid n \qquad p^{\alpha} \parallel (n-1)^n + 1 \qquad v_p(LHS) = 2\alpha$

A destra conta solo $(n-1)^{(n-1)^n+1}+1$

$n \mid (n-1)^n+1$

$v_p\left[(n-1)^{(n-1)^n+1}+1\right] = v_p\left((n-1)^n+1\right) + v_p\left((n-1)^n+1\right) - v_p(n)$

$= \alpha + \alpha - 0 = 2\alpha$

**Esercizio 4** Dato $k \geq 2$ Trovare $a_1, a_2, \dots$ t.c. $\forall m \geq 1$ $a_m \geq 3$ dispari

$t(a_1 \dots a_m) = k$

Esempio $\frac{2^k-1}{2-1} = 2^{k-1}+2^{k-2}+\dots+2+1$ ha $k$ cifre 1

$\frac{2^{ak}-1}{2^a-1} = 2^{a(k-1)}+2^{a(k-2)}+2^a+1$ "

$a_m = \frac{a_1 \dots a_m}{a_1 \dots a_{m-1}} = \frac{b_m}{b_{m-1}} = $ (tentativo) $= \frac{2^{ak}-1}{2^a-1} \cdot \frac{2^b-1}{2^{bk}-1}$.

**Oss** $(2^c-1, 2^d-1) = 2^{(c,d)}-1$.

$(x^c-1, x^d-1) = x^{(c,d)}-1$.

$b_m = \frac{2^{(k+1)^m k}-1}{2^{(k+1)^m}-1}$ $a = (k+1)^m$

$a_m = \frac{b_m}{b_{m-1}} = \frac{2^{(k+1)^m k}-1}{2^{(k+1)^m}-1} \cdot \frac{2^{(k+1)^{m-1}}-1}{2^{(k+1)^{m-1}k}-1}$

$\left(2^{(k+1)^m}-1, 2^{(k+1)^{m-1}\cdot k}-1\right) = 2^{(k+1)^{m-1}}-1$

DIVISIBILITA' OK.

**2ª parte** $\exists N$ t.c. $\forall m > N$ $t(3 \cdot 5 \dots (2m+1)) > k$

Basta vedere che $t(n(2^r-1)) \geq r$.

Induzione su $n$:

$n=1$ ovvio.

$n>1$

1° caso: $n$ pari $t(n(2^r-1)) = t\left(\frac{n}{2}(2^r-1)\right) \geq r$.

2° caso: $n$ dispari $n = 2j+1$.

$t\left((2j+1)(2^r-1)\right)$ $\qquad 2j+1 = 2j+2-1$

$= t\left((2j+2)(2^r-1) - 2^r + 1\right) = t\left((2j+2)(2^r-1)-2^r\right) + 1$

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$t(a-2^r) \geq t(a) - 1$ con $a > 2^r$

$\geq t\left((2j+2)(2^r-1)\right) - 1 + 1$

$= t\left(4(j+1)(2^r-1)\right) \geq r.$

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