PRE-IMO 2015

ALGEBRA - mattino

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A1. Siano a, b, c numeri reali positivi tali che $a+b+c=1$ e sia n un intero positivo. Dimostrare che

$$\frac{(3a)^n}{(b+1)(c+1)} + \frac{(3b)^n}{(c+1)(a+1)} + \frac{(3c)^n}{(a+1)(b+1)} \geq \frac{27}{16}.$$

$$\sum \frac{(3a)^n}{(b+1)(c+1)} \overset{?\text{Hope}}{\geq} \sum \frac{(3a)}{(b+1)(c+1)}$$

WLOG $a \geq b \geq c$

$$\frac{3a}{(b+1)(c+1)} \geq \frac{3b}{\sim} \geq \frac{3c}{\sim}$$

$$\| $$

$$\sum (3a)^{n-1} \frac{3a}{(b+1)(c+1)} \geq \frac{1}{3} \underbrace{\left(\sum (3a)^{n-1}\right)}_{\text{si stima con AM-}(n-1)M} \left(\sum \frac{3a}{(b+1)(c+1)}\right)$$

$(3a)^{n-1} \geq (3b)^{n-1} \geq (3c)^{n-1}$

$$\frac{\sum a^{n-1}}{3} \geq \left(\frac{\sum a}{3}\right)^{n-1} = \frac{1}{3^{n-1}}$$

Quindi $\text{LHS} \geq \frac{1}{3} 3^{n-1} \sum a^{n-1} \sum \frac{3a}{(b+1)(c+1)} \geq \sum \frac{3a}{(b+1)(c+1)}$

Voglio mostrare $\sum \frac{3a}{(b+1)(c+1)} \geq \frac{27}{16}$

moltiplico per $\prod (a+1)$ e faccio i conti:

viene da dimostrare $7 \geq 27abc + 18 \sum ab$

$\sum ab \le \frac{(\sum a)^2}{3} = \frac{1}{3}$ e $abc \le \left(\frac{\sum a}{3}\right)^3 = \frac{1}{27}$

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Sol. b: Hölder

$$\left(\sum (b+1)(c+1)\right)\left(\sum \frac{(3a)^n}{(b+1)(c+1)}\right)\left(\sum 1\right)^{n-2} \overset{\text{Hölder}}{\ge} \left(\sum 3a\right)^n$$

$$\left(\sqrt[n]{(b+1)(c+1)},\ \text{cyc}\right)\left(\frac{3a}{\sqrt[n]{(b+1)(c+1)}},\ \text{cyc}\right)\overbrace{(1,1,1)}^{n-2}$$

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$$\sum \frac{(3a)^n}{(b+1)(c+1)} \ge \text{roba}$$

$a+b+c=1$

$b+c=1-a$

$$(b+1)(c+1) \le 1+b+c+\left(\frac{b+c}{2}\right)^2 = 1+1-a+\left(\frac{1-a}{2}\right)^2 = \frac{1}{2}(3-a)^2$$

$$\text{LHS} \ge \sum \frac{2(3a)^n}{(3-a)^2} \qquad f(x) = \frac{x^n}{(3-x)^2}$$

Il problema è diventato: cerco min di $f(a)+f(b)+f(c)$ con $a+b+c$ fissato

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$\sum \frac{(3a)^n}{(b+1)(c+1)} \ge \frac{27}{16}$: moltiplico per i denominatori e resta

$$\sum 3^n a^{n+1} + \sum 3^n a^n \ge \frac{27}{16} \prod (a+1)$$

$\sum a^{n+1}$ la stimo con $(n+1)\,\Pi - AM$

$\sum a^n$ idem

viene LHS $\geq 6 \geq$ RHS,

A2. È definita la successione di reali positivi seguente

$$a_1 = 1,$$
$$a_n = \frac{n^2+1}{(n-1)^2}a_{n-1}, \qquad n \geq 2.$$

Dimostrare che

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \leq 1 + \sqrt{1 - \frac{1}{a_n}}.$$

$$a_n = \frac{n^2+1}{(n-1)^2} \cdot \frac{(n-1)^2+1}{(n-2)^2} \cdots \frac{2^2+1}{1^2} = \frac{\prod_{k=2}^{n}(k^2+1)}{\prod_{k=1}^{n-1} k^2} \qquad n \geq 2$$

$$\sum_{i=1}^{n} \frac{1}{a_i} = 1 + \sum_{i=2}^{n} \frac{\prod_1^{i-1} k^2}{\prod_2^i (k^2+1)} = 1 + \frac{1}{\prod_2^n (k^2+1)} \sum_{i=2}^{n} \underbrace{\prod_1^{i-1} k^2 \prod_{i+1}^{n} (k^2+1)}_{\approx \frac{\text{roba}}{i^2}}$$

$$\text{LHS} \approx 1 + C \cdot \sum_{i=2}^{n} \frac{1}{i^2} \qquad \text{moralmente}$$

$$\sum_{i=1}^{n-1} \frac{1}{i(i+1)} = \sum_{i=1}^{n-1}\left(\frac{1}{i} - \frac{1}{i+1}\right) = 1 - \frac{1}{n} \qquad \sum_2^n \frac{1}{i(i-1)} = 1 - \frac{1}{n}$$

$$\sum_{i=2}^{n} \prod_1^{i-1} k^2 \prod_{i+1}^{n}(k^2+1)\, i(i-1) \frac{1}{i(i-1)}$$

$$= \sum_{i=2}^{n} \frac{1}{i(i-1)} 1^2 \cdot 2^2 \cdots (i-1)^2 (i-1) i \left((i+1)^2+1\right) \cdots (n^2+1)$$

$$(\text{LHS}-1)^2 \overset{?}{\leq} 1 - \frac{1}{a_n} \iff \left[a_n(\text{LHS}-1)\right]^2 \overset{?}{\leq} a_n(a_n - 1)$$

HOPE: $a_n(\text{LHS}-1) \overset{?}{\leq} a_n - 1 \iff \text{LHS}-1 \leq 1 - \frac{1}{a_n}$

$$LHS-1 = \frac{1}{\prod_2^n (k^2+1)} \sum_{i=2}^n \prod_1^{i-1} k^2 \prod_{i+1}^n (k^2+1)(i-1)i\frac{1}{(i-1)i} \overset{?}{\leq} 1 - \frac{\prod_1^{n-1} k^2}{\prod_2^n (k^2+1)}$$

$$\sum_{i=2}^n \prod_1^{i-1} \underset{\leq (k^2+1)}{k^2} \prod_{i+1}^n (k^2+1)\underset{i(i-1)\leq i^2+1}{(i-1)i}\frac{1}{(i-1)i} \overset{?}{\leq} \prod_2^n (k^2+1) - \prod_1^{n-1} k^2$$

$$\text{LHS ultimo} \leq \sum_{i=2}^n \prod_2^n (k^2+1) \cdot \frac{1}{i(i-1)} = \prod_2^n (k^2+1)\left(1-\frac{1}{n}\right)$$

$$\overset{?}{\leq} \prod_2^n (k^2+1) - \prod_1^{n-1} k^2$$

$$\prod_2^n (k^2+1) \overset{?}{\geq} n \prod_1^{n-1} k^2$$

$$n^2+1 > n$$

$$k^2+1 > k^2$$

ok

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$$b_i = \frac{1}{a_i} \qquad Hp: \; b_1 = 1 \qquad \boxed{(n-1)^2 b_{n-1} = (n^2+1)\, b_n}$$

$$\text{Tesi}: \quad b_1 + \dots + b_n \leq 1 + \sqrt{1-b_n}$$

$$b_n = (n-1)^2 b_{n-1} - n^2 b_n$$

$$b_1 + b_2 + \dots + b_n = b_1 + (1^2 b_1 - \underbrace{2^2 b_2}) + (\underbrace{2^2 b_2} - \underbrace{3^2 b_3}) + \dots + (\underbrace{(n-1)^2 b_{n-1}} - n^2 b_n)$$

$$= 2 - n^2 b_n$$

A3. Sia n un intero positivo e siano $a_1, a_2, \ldots, a_n$ numeri interi positivi. Si estenda questa n-upla ad una successione infinita periodica ponendo $a_{n+i} = a_i$ per ogni $i \geq 1$. Supponendo che

$$a_1 \leq a_2 \leq \cdots \leq a_n \leq a_1 + n$$

e che

$$a_{a_i} \leq n + i - 1 \quad \text{per} \quad i = 1, 2, \ldots n,$$

dimostrare che necessariamente si ha

$$a_1 + \cdots + a_n \leq n^2.$$

$a_i \leq 2n \quad \forall i$

$a_1 \leq a_{a_1} \leq n \qquad a_i \leq a_n \leq a_1 + n \qquad a_i \leq 2n \quad \forall i$

Sia $k_h := \inf\{i : a_i \geq h\} \wedge (n+1)$

$k_1 \leq k_2 \leq \ldots \leq k_h \leq \ldots$

$k_h - 1 = \#\{i \leq n : a_i \leq h-1\} = \sum_{i=1}^{n} \mathbb{1}_{a_i \leq h-1}$

$a_{a_i} \leq n+i-1 \iff \begin{cases} \text{se } a_i \leq n & \text{la condiz.} \iff a_i \leq k_{n+i} - 1 \\ \text{se } a_i > n & \text{" } \quad \text{" } \iff a_i - n \leq k_{n+i} - 1 \end{cases}$ $i = 1, 2, \ldots, n$

$\sum_h (n - k_h + 1) = \sum_h \sum_{i=1}^{n} (1 - \mathbb{1}_{a_i \leq h-1}) = \sum_{i=1}^{n} \sum_h \mathbb{1}_{h \leq a_i}$

eventi complementari

$\sum_{h=1}^{\cancel{\infty}\, 2n} \mathbb{1}_{h \leq a_i} = a_i$

$n - k_h + 1 \leq n - a_{h-n}$

$k_{n+i} \geq a_i + 1 \qquad i = 1, \ldots, n \qquad k_h \geq a_{h-n} + 1 \qquad \underbrace{h = n+1, \ldots, 2n}_{\text{sommo su questi } h}$

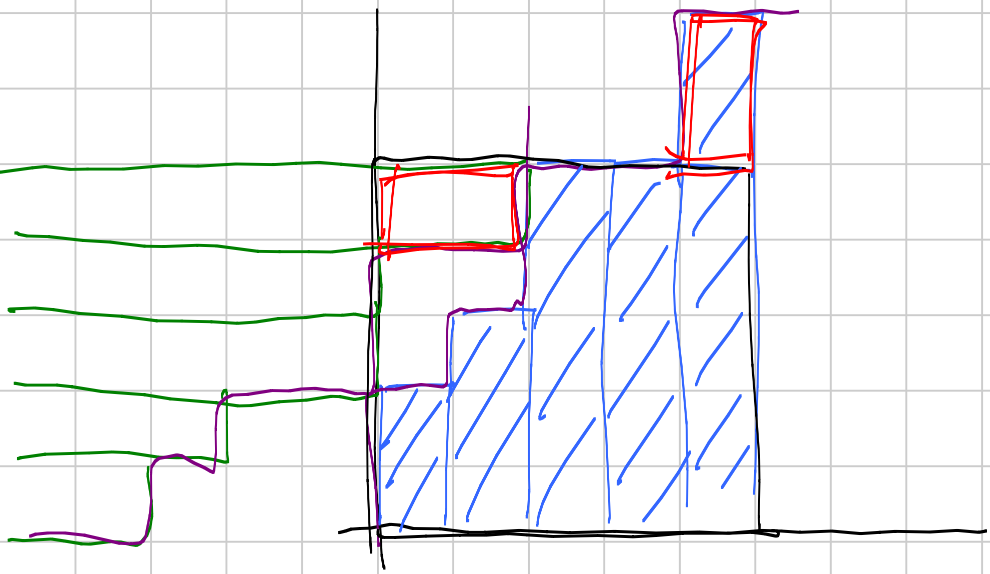
$\sum_{h=n+1}^{2n} (n - k_h + 1) = \sum_{i=1}^{n} (a_i - n) \vee 0$

$n - k_h + 1 \leq 2n - a_{h-n}$

$\sum_{h=n+1}^{2n} (n - a_{h-n}) \geq \sum_{i=1}^{n} (a_i - n) \vee 0$

$a_i = f(i)$ $\qquad f(f(i)) \leq n+i-1$ $\qquad \exists g \approx f^{-1}$

$$\text{"} f(i) \leq g(n+i-1) \text{"}$$



$a_i \leq n \implies n - a_i \geq n - k_{n+i} + 1$

$i \leq a_1 \leq n \implies a_i \leq a_{a_1} \leq n$

$i > a_1 \implies n+i > n+a_1 \geq a_j \quad \forall j \qquad n+i > \max_j a_j \implies k_{n+i} = n+1$

$$\sum_{j=1}^{n} (a_j - n) \vee 0 = \sum_{h=n+1}^{2n} (n - k_h + 1) = \sum_{i=1}^{n} (n - k_{n+i} + 1) = \sum_{i=1}^{a_1} (n - k_{n+i} + 1) \leq \sum_{i=1}^{a_1} (n - a_i) \leq \sum_{i=1}^{n} (n - a_i) \vee 0$$

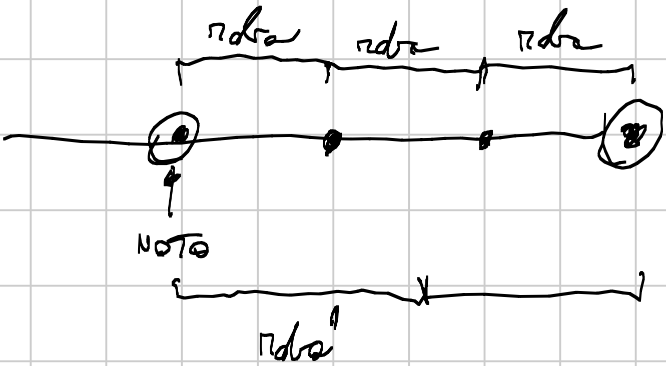
$$\sum_{j=1}^{n} (a_j - n) \vee 0 \leq \sum_{i=1}^{n} (n - a_i) \vee 0$$

$$\sum_{a_j > n} (a_j - n) \leq \sum_{a_i \leq n} (n - a_i) \qquad \sum_{1}^{n} a_i \leq \sum_{1}^{n} n = n^2$$

A4. Sia n un numero intero assegnato. Trovare tutte le funzioni $f : \mathbb{Z} \to \mathbb{Z}$ tali che per ogni scelta degli interi x e y si abbia

$$f\left(x + y + f(y)\right) = f(x) + ny$$

$$f(x + \underbrace{\text{roba}}_{y+f(y)}) = f(x) + \text{altra roba}$$



$$f\left(x + (y+f(y))(z+f(z))\right) = f(x) + ny(z+f(z))$$
$$= f(x) + nz(y+f(y))$$

$$f(x) + nyz + ny\,f(z) = f(x) + nyz + n\,f(y)\,z$$

$$ny\,f(z) = nz\,f(y)$$

se $n \neq 0$ $\quad y = 1 \implies f(z) = z\,f(1)$ (sostituisco nel testo ...)

se $n = 0$: il testo diventa $f(x + y + f(y)) = f(x)$

$g(x) = x + f(x)$: sostituisco e viene $g(x + g(y)) = g(x) + g(y)$

se $g(x+a) = g(x) + a$ e $g(x+b) = g(x) + b$ allora

$$g(x+(a,b)) = g(x) + (a,b)$$

6) $g(x) \equiv 0$ oppure definisco $m = MCD(\mathrm{Im}(g))$

$g(x+g(y)) = g(x)+g(y)$ vuol dire $g(x+m) = g(x)+m$

$g(x) = m\,h(x)$ $\quad h(x+m) = h(x)+1$

pongo $h(0) = a_0$ $\quad h(1) = a_1$ $\quad \ldots$ $\quad h(m-1) = a_{m-1}$

viene $h(qm+r) = q + a_r$ $\quad g(qm+r) = mq + m a_r$

$$f(qm+r) = mq + ma_r - (qm+r) = ma_r - r$$