PRE-IMO 2015

ALGEBRA - pomeriggio

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$\boxed{\text{A5.}}$ Determinare il massimo valore di $\alpha > 0$ tale che

$$a^2 + b^2 + c^2 + \alpha(ab + bc + ca) \geq (1 + \alpha)\sqrt{3(a^2b^2 + b^2c^2 + c^2a^2)}.$$

per ogni terna di reali positivi a, b, c.

$$\sum_{cyc} a^2 + \alpha \sum_{cyc} ab \geq (1+\alpha)\sqrt{3\sum_{cyc} a^2b^2}$$

omogenea + simmetrica : bunching + schur?

$$\left(\sum_{cyc} a^2\right)^2 + 2\alpha \sum_{cyc} a^2 \sum_{cyc} ab + \alpha^2\left(\sum_{cyc} ab\right)^2 \geq (1+2\alpha+\alpha^2)\cdot 3 \cdot \sum_{cyc} a^2b^2$$

$$\sum_{cyc} a^4 + 2\sum_{cyc} a^2b^2 + 2\alpha\sum_{cyc} a^3b + 2\alpha\sum_{cyc} ab^3 + 2\alpha\sum_{cyc} a^2bc + \alpha^2\sum_{cyc} a^2b^2 + \alpha^2\cdot 2\sum_{cyc} a^2bc$$

$$\geq (1+2\alpha+\alpha^2)\cdot 3\sum_{cyc} a^2b^2$$

moltiplico tutto per 2 $\qquad \sum_{sym} a^m b^n c^l = [m,n,l] \quad m \geq n \geq l$

$$[4,0,0] + 4\alpha[3,1,0] + (2\alpha^2+2\alpha)[2,1,1] \geq (2\alpha^2+6\alpha+1)[2,2,0]$$

Bunching : $[4,0,0] \geq [3,1,0] \geq [2,2,0] \geq [2,1,1]$

Schur : $[4,0,0] + [2,1,1] \geq 2[3,1,0] \geq 2[2,2,0]$

$1 \geq 2\alpha^2 + 2\alpha \qquad 2\alpha^2 + 2\alpha - 1 \leq 0$ condizione sufficiente

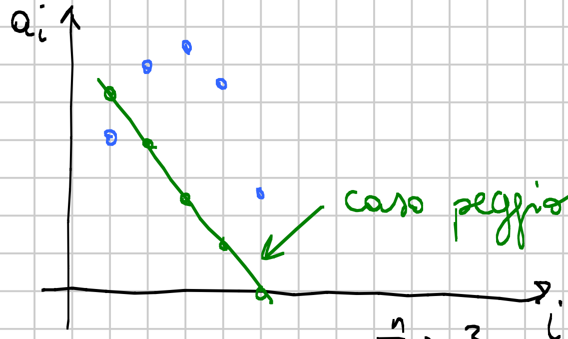
★ Unsmoothing. $a=b=1$ $\quad$ c molto piccolo $\quad c \sim 0$

$2+\alpha \geq (1+\alpha)\sqrt{3} \qquad \alpha \leq \dfrac{2-\sqrt{3}}{\sqrt{3}-1} = \dfrac{\sqrt{3}-1}{2}$ condizione necessaria

A6. Diciamo che una n-upla di reali $a_1, a_2, \ldots, a_n$ è *concava* se per ogni $2 \le i \le n-1$, si ha $a_i \ge \frac{a_{i-1}+a_{i+1}}{2}$. Determinare il massimo valore di $c > 0$ tale che per ogni n-upla concava di reali non negativi si abbia

$$\sum_{i=1}^n i a_i^2 \ge c \sum_{i=1}^n a_i^2$$

Smoothing



$a = (a_1, a_2, \ldots, a_n)$ $\quad r(a) = \dfrac{\sum_{i=1}^n i a_i^2}{\sum_{i=1}^n a_i^2}$ $\quad r(a) \geq c$ per ogni a concava

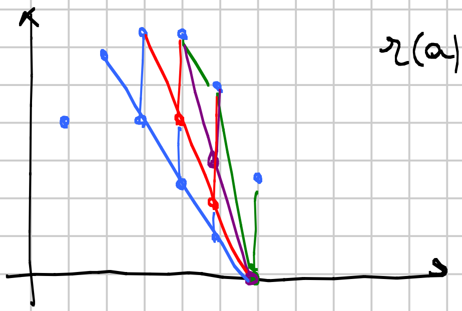
$c = \inf_a r(a)$

1) $a_i = k(n-i)$ $\quad r(a) = \dfrac{\sum_i i k^2 (n-i)^2}{\sum_i k^2 (n-i)^2} = \dfrac{\sum_{j=0}^{n-1} (n-j) j^2}{\sum_{j=0}^{n-1} j^2}$

$$= \frac{n \sum j^2 - \sum j^3}{\sum j^2} = \frac{\frac{\cancel{n}\cdot(\cancel{n-1})n(2n-1)}{6} - \frac{(n-1)^{\cancel{2}} n^{\cancel{2}}}{4}}{\frac{(\cancel{n-1})\cancel{n}(2n-1)}{6}} = \frac{n(4n-2-3n+3)}{4n-2}$$

$$= \frac{n(n+1)}{4n-2}$$

2)



$r(a) = \dfrac{a_1^2 + 2a_2^2 + \ldots + n a_n^2}{a_1^2 + a_2^2 + \ldots + a_n^2} = n \dfrac{\frac{a_1^2}{n} + \frac{2a_2^2}{n} + \ldots + 1 \cdot a_n^2}{a_1^2 + a_2^2 + \ldots + a_n^2}$

$\dfrac{a}{b} \to \dfrac{a-c}{b-c} \le \dfrac{a}{b}$ vero se $\dfrac{a}{b} < 1$

$r(a) < n \Rightarrow \dfrac{r(a)}{n} < 1 \Rightarrow$ posso farlo

azzero a_n $\quad r(a') < n-1$ posso diminuire a_{n-1} fino alla retta viola.

Voglio diminuire $a_j, a_{j+1}, \ldots, a_n$ tutti assieme in modo proporzionale

$$\frac{\sum_1^n i a_i^2 - \varepsilon \sum_j^n i a_i^2}{\sum_1^n a_i^2 - \varepsilon \sum_j^n a_i^2} = r_j(a) \cdot \frac{\frac{\sum i a_i^2}{r_j(a)} - \varepsilon \sum_j^n a_i^2}{\sum_1^n a_i^2 - \varepsilon \sum_j^n a_i^2}$$

$r_j(a) = \sum_j^n i a_i^2 / \sum_j^n a_i^2$

funziona se $\frac{r(a)}{r_j(a)} < 1 \iff r(a) < r_j(a)$

$$r(a) = \frac{\sum_1^{j-1} i a_i^2 + \sum_j^n i a_i^2}{\sum_1^n a_i^2} = \underbrace{\frac{\sum_1^{j-1} a_i^2}{\sum_1^n a_i^2}}_{\theta} \frac{\sum_1^{j-1} i a_i^2}{\underset{\leq j-1}{\sum_1^{j-1} a_i^2}} + \underbrace{\frac{\sum_j^n a_i^2}{\sum_1^n a_i^2}}_{1-\theta} \cdot \overbrace{\frac{\sum_j^n i a_i^2}{\sum_j^n a_i^2}}^{r_j(a)}$$

$r_j(a) \geq j$

$r(a) < r_j(a)$

A7. Sia $n \geq 2$ un intero assegnato. Determinare tutti i polinomi non costanti f a coefficienti complessi che soddisfano

$$1 + f(x^n + 1) = f(x)^n.$$

ω radice primitiva n-esima di 1,

$$f(\omega x)^n = 1 + f(\omega^n x^n + 1) = f(x)^n$$

$\forall x \quad \exists\, i(x) \quad$ t.c. $\quad f(\omega x) = \omega^{i(x)} f(x)$

c'è $\bar{i}$ t.c. $\exists \infty\ x$ con $f(\omega x) = \omega^{\bar{i}} f(x)$

$\exists\, \bar{i}$ t.c. $f(\omega x) = \omega^{\bar{i}} f(x) \implies \boxed{f(x) = x^{\bar{i}} g(x^n)}$

$f(x) = a_k x^k + \dots + a_0$

$$f(\omega x) = \underbrace{\omega^k a_k}\, x^k + \dots + a_0 = \omega^{\bar{i}} f(x) = \underbrace{\omega^{\bar{i}} a_k}\, x^k + \dots + \omega^{\bar{i}} a_0$$

CASO 1: $\bar{i} \neq 0$ cioè $f(0) = 0$ $\qquad f(x^n+1) = f(x)^n - 1$

CASO n pari: $f(0) = 0 \quad f(1) = -1 \quad f(2) = 0$

$f(2^n+1) = -1 \quad f((2^n+1)^n + 1) = 0$

CASO n dispari: $f(0) = 0 \quad f(1) = -1 \quad f(2) = -2$

$f(2^n+1) = -2^n - 1 \quad f((2^n+1)^n + 1) = -(2^n+1)^n - 1$

$\Rightarrow f(x) \equiv -x$,

CASO 2 : $\bar{\imath} = 0$ $\quad f(x) = g(x^n)$ $\quad 1 + g((x^n+1)^n) = g(x^n)^n$

$x^n = t$ $\quad \boxed{1 + g((t+1)^n) = g(t)^n}$

$t+1 = y$ $\quad 1 + g(y^n) = g(y-1)^n$

$g(y-1) = y^J h(y^n)$

CASO $J \neq 0$ $\quad g(-1) = 0$ $\quad g((t+1)^n) = g(t)^n - 1$

CASO n pari : $\quad g(-1) = 0 \quad g(0) = -1 \quad g(1) = 0$

$g(2^n) = -1 \quad g((2^n+1)^n) = 0$ ASSURDO

CASO n dispari $\quad g(x) = -x-1$

CASO $J = 0$ $\quad g(y-1) = h(y^n)$ $\quad g(y) = h((y+1)^n)$

$1 + h((y^n+1)^n) = h(y^n)^n$

$\boxed{1 + h((t+1)^n) = h(t)^n}$

$g(x) = \quad -x-1 \quad -(x+1)^n - 1 \quad -((x+1)^n+1)^n - 1$

$-(((x+1)^n+1)^n+1)^n - 1 \quad \dots$

$f(x) = \quad -x \quad -x^n - 1 \quad -(x^n+1)^n - 1 \quad \dots$

A8. Determinare tutte le funzioni $f : \mathbb{R} \to \mathbb{R}$ tali che

$$f(1+xy) - f(x+y) = f(x)f(y) \quad \text{per ogni } x, y \in \mathbb{R},$$

$P(x,y)$

e tali che $f(-1) \neq 0$.

niente fuori dalla $f \to$ Cauchy

Sostituzioni facili

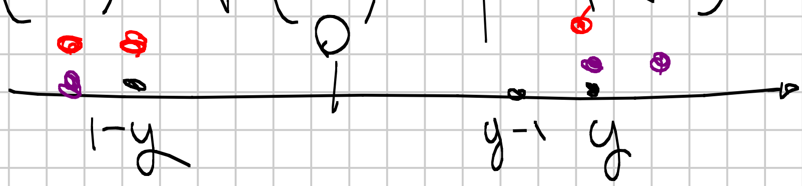
$$f(-1)=\beta$$
$$f(0)=-1$$
$$f(1)=0$$
$$f(2)=1$$

$$P(-1,y):\quad f(1-y)-f(-1+y)=\beta f(y) \qquad (*)\quad G(y)$$

non consecutivi

Ricorda ricorrente a tre termini sugli interi, del tipo $c_0 f(y)+c_1 f(y+1)+c_2 f(y+2)=0$

$$P(-1,3):\quad f(-2)-f(2)=\beta f(3)$$



$$f(1-y)-f(-1+y)=\beta f(y) \qquad G(y)$$
$$f(-y)-f(y)=\beta f(y+1) \qquad G(y+1)$$
$$G(-y)$$

$$f(y)+\beta f(y+1)+f(y+2)=0$$

$$f(-1)=\beta$$
$$f(0)=-1$$
$$f(1)=0$$
$$f(2)=1$$
$$f(3)=-\beta-1$$
$$f(4)=\beta(\beta+1)-1$$

cose derivate da $G(\ ) \to f(5)$:

Voglio un'equazione per B:

$$P(2,2): \quad f(5) = f(4) + f(2)^2$$

mi darà un'equazione per B

Due soluzioni: 1, -2

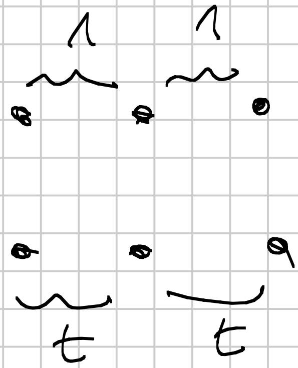
$$f(y) + f(y+1) + f(y+2) = 0$$
$$f(y) - 2f(y+1) + f(y+2) = 0$$

$\downarrow$

lineare, diff. finite

x	$f_1(x)$	$f_2(x)$
0	-1	-1
1	0	0
2	1	1
3	2	-1
4	3	0
5	4	1
6	⋮	-1
7	⋮	0
⋮		1

non si estenderà ai razionali



$$P(x,y) \quad f(1+xy) = f(x+y) + f(x)f(y)$$

$$P(x,y+1) \cdot\beta \quad \beta f(1+xy+x) = \beta f(x+y+1) + \beta f(x)f(y+1)$$

$$P(x,y+2) \quad f(1+xy+2x) = f(x+y+2) + f(x)f(y+2)$$

$$0 \qquad\qquad 0$$

$$f(\underbrace{1+xy}_{z}) + \beta f(\underbrace{1+xy}_{z}+x) + f(\underbrace{1+xy}_{z}+2x)$$

$$f(z) + \beta f(z+x) + f(z+2x) = 0$$

Relazione tra tre punti a distanza x

Da qui si esclude $\beta = 1$ (prendendo $x=0$)

Rimane $\beta = -2$ (funzioni lineari)

$g(x) := f(x) + 1$ soddisfa la stessa relazione

$$H(x,z) \quad g(x) - 2g(x+z) + g(x+2z) = 0 \quad \forall x, z$$

sostituendo $H(0,z)$ ho $g(2x) = 2g(x)$

$H(x,$ la cosa che fa venire $x+2z=y)$

ho $g(x)+g(y)=g(\cancel{x}+y)$

Mi manca l'ipotesi per passare da $\mathbb{Q}$ a $\mathbb{R}$

$P(x,y)$: $f(1+xy)=f(x+y)+f(x)f(y)$

$P(x,x)$ non funziona $g(-x)=-g(x)$

$P(x,-x)$: $f(1-x^2)=-1-[f(x)]^2+\dots$

Se $z\leq 1$, allora $f(z)\leq -1$.