# PreIMO '16 - ALGEBRA (mattino)

Titolo nota 26/05/2016

$\boxed{1}$ $p(x), q(x) \in \mathbb{C}[x]$ di grado 2016

$$p(x) + (-1)^x q(x) = 2^x \qquad x = 1, 2, \dots, 4034$$

Coefficiente di $x^{2016}$ in $q(x)$

$d(x) = p(x) - q(x) = 2^x \qquad x = 1, 3, 5, \dots, 4033$

$s(x) = p(x) + q(x) = 2^x \qquad x = 2, 4, \dots, 4034$

Grado + 1 = # punti $\Rightarrow$ univocamente determinati

Metodo di interpolazione "tipo" polinomi di Newton

↳ funziona bene se conosco i valori per $x = 1, 2, 3, \dots, n+1$

(interi consecutivi)

$r(x) := s(2x) = 4^x \qquad x = 1, 2, \dots, 2017$

$d(x) = \frac{1}{2} r\left(\frac{x+1}{2}\right) \qquad s(x) = r\left(\frac{x}{2}\right)$

$$\boxed{q(x) = \frac{1}{2}s(x) - \frac{1}{2}d(x) = \frac{1}{2}r\left(\frac{x}{2}\right) - \frac{1}{4}r\left(\frac{x+1}{2}\right)}$$

$r(x) = \sum_i r_i x^i \qquad q(x) = \sum_i q_i x^i$

$q_{2016} = \frac{1}{2}\left(\frac{1}{2}\right)^{2016} r_{2016} - \frac{1}{4}\left(\frac{1}{2}\right)^{2016} r_{2016} = \frac{1}{4} 2^{-2016} r_{2016}$

Dato $f(x)$ polinomio sia $\Delta f(x) = f(x) - f(x-1)$

$f(x) = f_n x^n + f_{n-1} x^{n-1} + \dots \longrightarrow \boxed{\Delta f(x) = n f_n x^{n-1} + \dots} \dots$ (check!)

$\Delta^n f(x) = n! \, f_n$

$\Delta^{2016} r(x) \equiv 2016! \, r_{2016}$

| $x$ | 1 | 2 | 3 | $\dots n$ | 2017 |
| --- | --- | --- | --- | --- | --- |
| $r(x)$ | 4 | 16 | 64 | $\dots 4^n$ | $4^{2017}$ |
| $\Delta r(x)$ | – | 12 | 48 | $\dots 3 4^{n-1}$ | $3 \cdot 4^{2016}$ |
| $\Delta^2 r(x)$ | – | – | 36 | $9 \cdot 4^{n-2}$ | $3^2 4^{2015}$ |
| $\dots$ | | | | | |
| $\Delta^{2016} r(x)$ | – | – | – | | $3^{2016} \cdot 4^1$ |

$$q_{2016} = \dots = \frac{3^{2016}}{2^{2016} \cdot 2016!}$$

☆ Polinomi di Newton

$x_i = 1, 2, \dots, n$

$p_1(x) = (x-n)$ $\qquad p_0(x) = 1$

$p_2(x) = (x-n)(x-n+1)$

$\dots$

$p_k(x) = (x-n) \cdot (x-n+1) \cdot \dots \cdot (x-n+k-1)$

$p_{n-1}(x) = (x-n) \cdot \dots \cdot (x-2)$

$$p_{n-1}(x_i) = \begin{cases} \pm n! & x_i = 1 \\ 0 & x_i \neq 1 \end{cases}$$

$\exists\, a_k$ facili da trovare : $\sum_{k=0}^{n-1} a_k p_k(x_i) = y_i$

$\Delta p_k(x) = c \cdot p_{k-1}(x)$

$$r(x) = 4^x = (1+3)^x = \sum_{k=0}^{x} \binom{x}{k} 1^{x-k} 3^k$$

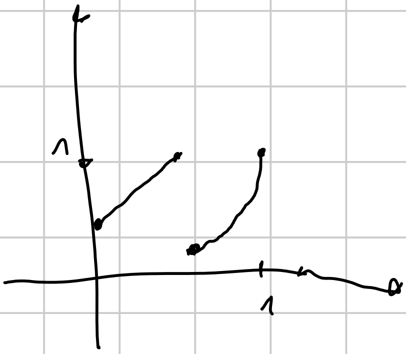
$$\binom{x}{k} = \frac{1}{k!} x(x-1)(x-2) \cdot \dots \cdot (x-k+1)$$

$\boxed{2}$

$$f:(0,1)\longrightarrow(0,1)$$

$$f(x)=\begin{cases} x+\frac{1}{2} & \text{se } x<\frac{1}{2}\\ x^2 & \text{se } x\geq\frac{1}{2}\end{cases}$$

$a_0\in(0,1)$ $\qquad a_n=f(a_{n-1})$




$a_0\neq b_0 \qquad a_0>b_0$

$a_n-b_n$

$$a_{n+1}-b_{n+1}=(a_n-b_n)\underbrace{(a_n+b_n)}_{\geq 1}$$

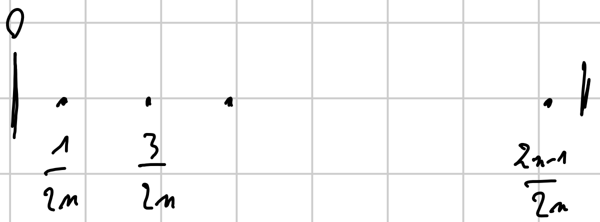
| $\geq\frac{1}{2}$ | $<\frac{1}{2}$ | | |
|---|---|---|---|
| $a_n$ | $a_{n+1}=a_n^2$ | $a_{n+2}=a_{n+1}+\frac{1}{2}$ | $a_{n+3}=a_{n+2}^2$ |
| $b_n$ | $b_{n+1}=b_n^2$ | $b_{n+2}=b_{n+1}+\frac{1}{2}$ | $b_{n+3}=b_{n+2}^2$ |

$\boxed{3}$ $f:\mathbb{R}_+\to\mathbb{R}_+$ $\quad a,b,c$ distinti $\quad a,b,c$ lati $\Delta$ $\iff$ $f(a),f(b),f(c)$ lati di $\Delta$

$$a<b+c \iff f(a)<f(b)+f(c)$$

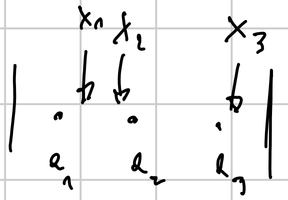
$\boxed{4}$ $f:[0,1]\longrightarrow\mathbb{R}$ crescente

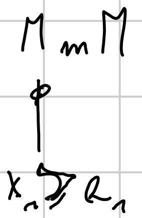
$$\max\sum_{i=1}^{n} f\left(\left|x_i-\frac{2i-1}{2n}\right|\right) \qquad 0\leq x_1\leq\dots\leq x_n\leq 1$$

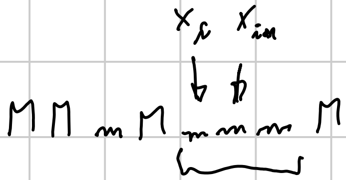


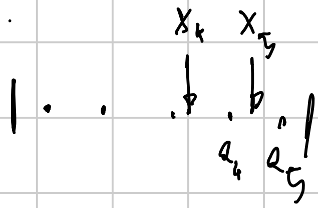

$$\frac{2i-1}{2n}=a_i$$

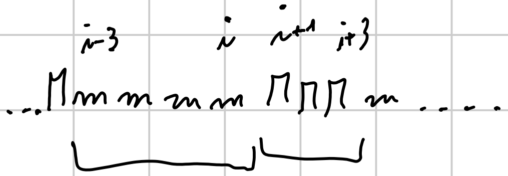
dati $x_1 \dots x_n$ $\qquad m \quad M$

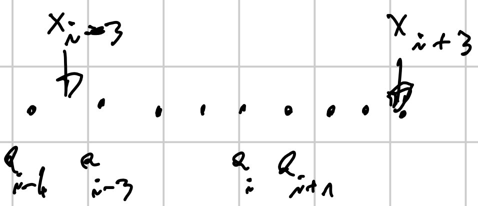


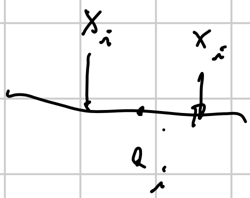












$$a_{i-4} \le x_{i-3} \le a_{i-3} \qquad x_{i+1} = x_{i+3} \le a_{i+4}$$

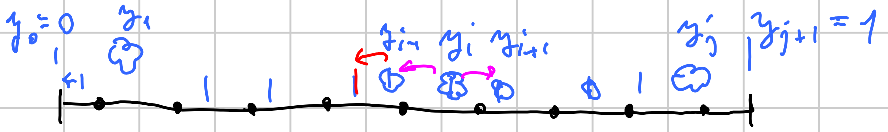
$$\Downarrow$$

$$2a_{i+1} - x_{i+1} \ge 2a_{i+1} - a_{i+4} = a_{i-2}$$

Alla fine mi ritrovo con tutti gli $x_i$ uguali

Idea: trovo una sequenza di mosse che manda qualunque n-uple in quella ottima

- aumentando sempre la somma
- numero di passi limitato




Considero che in un passo intermedio i punti siano raggruppati in $j \le n$ punti $y_1, y_2, \dots, y_j$ con molteplicità $k_1, k_2, \dots, k_j$

gruppo $i$ ha spazio $y_{i+1} - y_{i-1}$ e $k_i$ punti

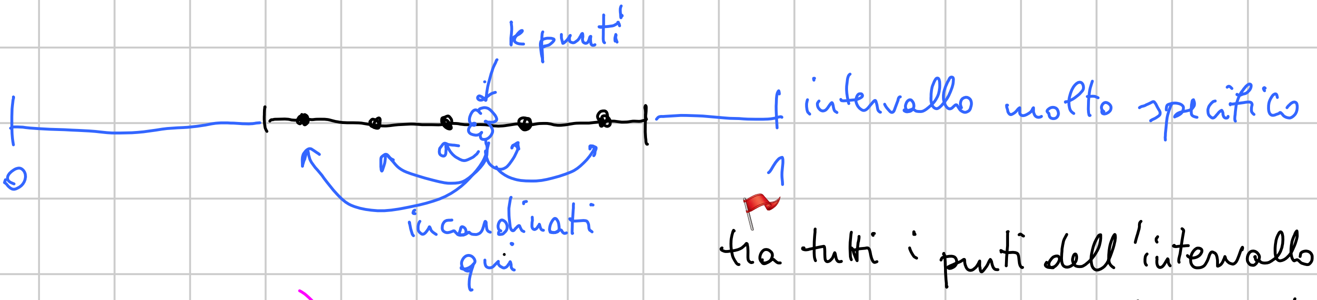
allora almeno uno ha spazio $\ge$ di $\frac{k_i}{n}$

RPA $\quad y_{i+1}-y_{i-1} < \frac{k_i}{n} \quad \forall i \Rightarrow \sum_{i=1}^{j}(y_{i+1}-y_{i-1}) < \frac{1}{n}\sum k_i = 1$

$$\sum_{2}^{j+1} y_i - \sum_{0}^{j-1} y_i = 1 + y_j - y_1 - 0 \ge 1 \quad \text{assurdo}$$

$y_j \ge y_1$

Considero un intervallo largo almeno $\frac{k}{n}$ con $k$ punti raggruppati




tra tutti i punti dell'intervallo il max è se il gruppo è ad un estremo