PREIMO 2016 - ALGEBRA - POMERIGGIO

Titolo nota 26/05/2016

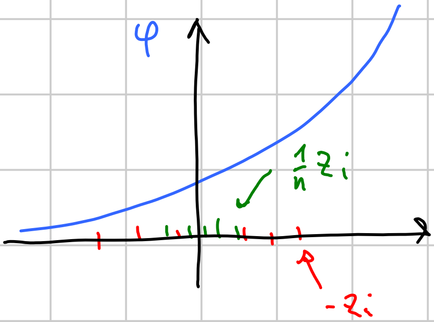
$\boxed{5}$ $x_1, x_2, \dots, x_{n+1} > 0 \qquad \prod x_i = 1 \qquad\qquad \sum_i z_i = 0$

$$\sum_i \sqrt[x_i]{n} \overset{?}{\geq} \sum_i n^{\sqrt[n]{x_i}}$$

$$\sum_i n^{x_i^{-1}} \overset{?}{\geq} \sum_i n^{x_i^{1/n}} \qquad\qquad \varphi(z) = n^{e^z}$$

$$\sum_i \varphi(-\log x_i) \overset{?}{\geq} \sum_i \varphi\left(\tfrac{1}{n}\log x_i\right) \qquad\qquad z_i := \log x_i$$

$$\sum_i \varphi(-z_i) \overset{?}{\geq} \sum_i \varphi\left(\tfrac{1}{n} z_i\right)$$



Disuguaglianza di Karamata

$a_i = -z_i \quad i = 1, \dots, n+1 \quad a_1 < a_2 < \dots < a_{n+1}$

$b_i = \frac{1}{n} z_{n-i+2} \quad i = 1, \dots, n+1 \quad b_1 < b_2 < \dots < b_{n+1}$

φ convessa

$\sum a_i = \sum b_i$ stesso baricentro

b maggiorizza a

basta $\longrightarrow$ $b_1 + b_2 + \dots + b_k \geq a_1 + a_2 + \dots + a_k \quad k = 1, \dots, n+1$

ts : $\sum \varphi(b_i) \leq \sum \varphi(a_i)$

$z_1 \geq z_i \ \forall i$

$k = 1 \quad b_1 \overset{?}{\geq} a_1 \qquad \frac{1}{n} z_{n+1} \overset{?}{\geq} -z_1 \qquad z_{n+1} + n z_1 \overset{?}{\geq} 0$

$\geq z_{n+1} + z_n + \dots + z_2 + z_1 = 0$ ✓

k generico

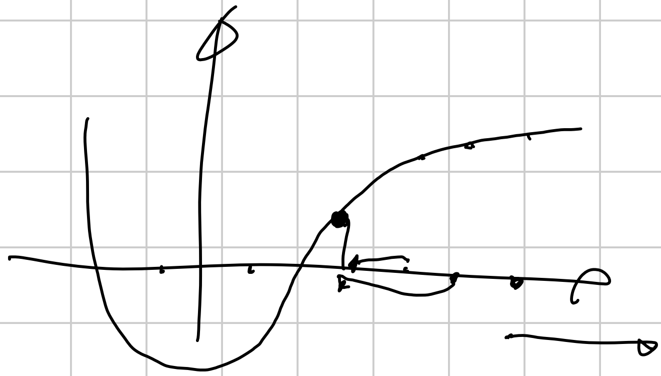
Oppure : AM-GM $\displaystyle\sum_{i \neq j} \sqrt[x_i]{n} \geq n \sqrt[n]{n^{\sum_{i\neq j} \frac{1}{x_i}}} \geq n \sqrt[n]{n^{\sqrt[n]{x_j}}} = n \cdot n^{\sqrt[n]{x_j}}$

Sommo su j e ho finito

$t_i \in \mathbb{R}$ $\qquad \sum t_i = 0 \qquad (t_i = \log x_i)$

$$\sum e^{\log n \, e^{-t_i}} - e^{\log n \, e^{\frac{t_i}{n}}} \overset{?}{\geq} 0$$

$$\psi(t) = e^{(\log n) e^{-t}} - e^{(\log n) e^{\frac{t}{n}}}$$



$x_1 = x_2 = \dots = x_n = x \qquad x_{n+1} = \frac{1}{x^n}$

$\boxed{6}$ n. Dire se $\exists\, a_0, \dots, a_n : \forall\, \varepsilon_0, \dots, \varepsilon_n \in \{-1, 1\}$

il poly $\sum a_i \varepsilon_i x^i$ ha n radici reali distinte.

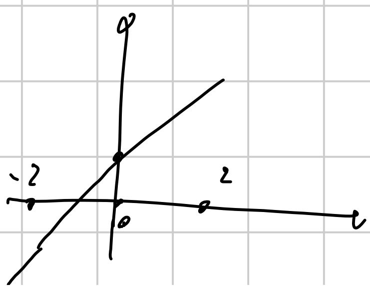
Fisso $\varepsilon = \frac{1}{100}$

$a_0 = 1 \qquad r_0 = 0 \qquad$ Il segno di $p(r_0)$ è determinato da ε_0.

$a_1 = 1 \qquad r_1 = 2 \qquad$ Il segno di $p(\pm r_1)$ è determinato da ε_1.

Voglio a_2 t.c. $a_2 r_1^2 \ll a_1 r_1$

chiedo $a_2 < \frac{\varepsilon}{n} a_1 \frac{1}{r_1}$

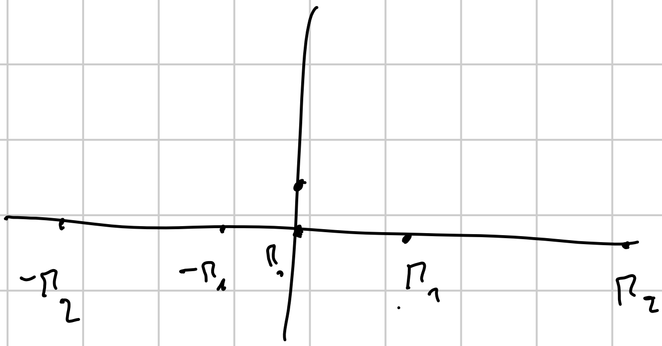


r_2 t.c. $a_2 r_2^2 \gg a_1 r_2$

$\phantom{r_2 \text{ t.c. } a_2 r_2^2 \gg} a_0$

$$> \frac{n}{\varepsilon} a_1 r_2 \qquad > \frac{n}{\varepsilon} a_0$$

Il segno di $p(\pm r_2)$ è determinato da e_2



$p(-r_1)$	$p(r_0)$	$p(r_1)$
$-a_1 e_1 r_1$	$a_0 e_0$	$a_1 e_1 r_1$

$p(-r_2)$	$p(-r_1)$	$p(r_1)$	$p(r_2)$
$a_2 e_2 r_2^2$	$-a_1 e_1 r_1$	$a_1 e_1 r_1$	$a_2 e_2 r_2^2$

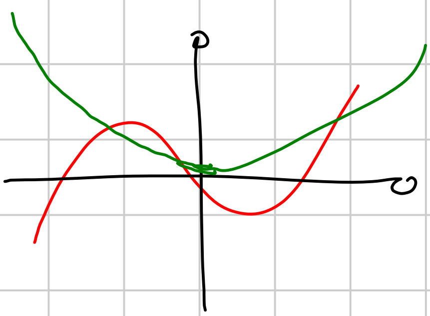
PIANO: INDUZIONE

Suppongo ok per grado n,

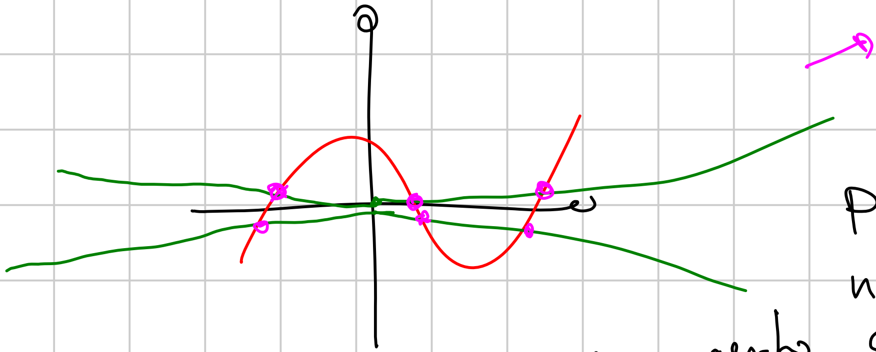
2^n polinomi

Passo di $\quad e_n a_n X^n + e_{n-1} a_{n-1} X^{n-1} + \dots + e_0 a_0$

$\pm e_{n+1} a_{n+1} X^{n+1} + \quad " \quad " \quad \dots$



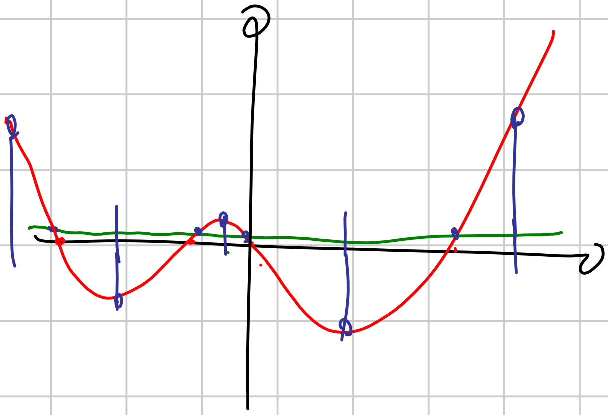
$-e_{n+1} a_{n+1} X^{n+1}$



Per ogni poli. di grado n che va bene, esiste un certo ε

Alternativamente: dato $p(x)$ di grado n che funziona, $\varepsilon x^{n+1}+p(x)$ e uno

Francesco: $x\,p(x)+\varepsilon$



$\boxed{7}$ $f:\mathbb{R}_+\to\mathbb{R}$

$$f(x+y)^2 = f(x)^2+2f(xy)+f(y)^2$$

$$f\equiv 0 \qquad f\equiv -2$$

$$f(\underbrace{x}+\underbrace{y+z}_{y'})^2 = f(x)^2+2f(xy+xz)+f(y+z)^2$$

$$= f(x)^2+f(y)^2+f(z)^2+2f(xy+xz)+2f(yz)$$

simmetrizzo $\longrightarrow$ $= \quad '' \quad '' \quad '' \quad +2f(yz+yx)+2f(xz)$

$$\forall x,y,z>0 \quad f(\overset{c}{xy}+\overset{b}{xz})+f(\overset{a}{yz}) = f(\overset{a}{yz}+\overset{c}{xy})+f(\overset{b}{xz})$$ senza quadrati!

$$\boxed{f(a+b)+f(c)=f(a)+f(b+c) \qquad \forall a,b,c>0}$$

dati $a,b,c>0$ $\quad (a=yz \quad b=xz \quad c=xy)$ hope

$\sqrt{\dfrac{ab}{c}} =: z$ e cicliche definisco x,y,z

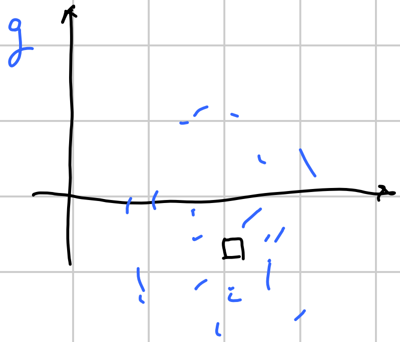
potrebbe essere una retta non per l'origine

$\forall x>0 \quad g(x) = f(a+x) - f(a)$; $a>0$ fissato per sempre

$g(x+y) = f(a+x+y) - f(a) \overset{?}{=} f(a+x) - f(a) + f(a+y) - f(a)$

$$\underbrace{f(a+y}_{A}+\underbrace{x}_{B}) + f(\underbrace{a}_{C}) = f(\underbrace{a}_{B}+\underbrace{x}_{C}) + f(\underbrace{a+y}_{A}) \quad \checkmark$$

Cerco una disuguaglianza



$\forall x,y>0 \quad 2f(xy) \leq f(\underbrace{x+y}_{=s})^2$

$=s$ fissato maggiore di $10\sqrt{a}$

(x',y') : $x'+y'=s \quad x',y'>\sqrt{a}$

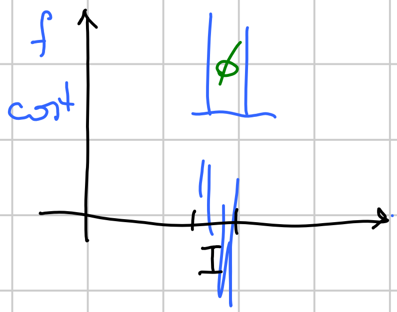
c'è un intervallo di valori di x' che soddisfano

$\rightarrow$ " " " " " " $x'y'>a$ che soddisfano

$\exists I : \forall u \in I, \; u>a \quad f(u) \leq cost$

$g(x) = f(x+a) - f(a)$

$g(u-a) \leq cost_1 \quad u \in I$



Allora $g(x) = \alpha x \implies f(x) = \alpha x + \beta$

Sostituisco : quattro soluzioni $f(x) = 0, \; -2, \; x, \; x-2$

$\boxed{8}$

$x_i \in \mathbb{R} \qquad \sum_1^n x_i = 0 \qquad \sum_1^n x_i^2 = 1$

$\forall A \subseteq T = \{1,2,\dots,n\} \qquad S_A := \sum_{i\in A} x_i \qquad S_\emptyset = 0$

$\forall \lambda > 0 \quad \#\{A \subseteq T : S_A \geq \lambda\} \leq 2^{n-3}\lambda^{-2}$

$S_{\bar{A}} = -S_A$ $\qquad$ $\#\{S_A \geq \lambda\} = \frac{1}{2}\#\{|S_A| \geq \lambda\} = \frac{1}{2}\#\underbrace{\{S_A^2 \geq \lambda^2\}}_{T_\lambda}$

$\sum_{A\subseteq T} S_A^2 \geq \sum_{A \in T_\lambda} S_A^2 \geq \lambda^2 \# T_\lambda$ $\qquad$ $\# T_\lambda \leq \frac{1}{\lambda^2}\sum_{A\subseteq T} S_A^2$

non negative

$$\sum_{A\subseteq T} S_A^2 = \sum_{A\subseteq T}\Big(\sum_{i\in A} x_i\Big)^2 = \sum_{A\subseteq T}\Big(\sum_{i\in A} x_i^2 + 2\sum_{\substack{i<j\\ i,j\in A}} x_i x_j\Big)$$

$$= \sum_{i=1}^n x_i^2 \underbrace{\sum_{A\ni i} 1}_{2^{n-1}} + 2\sum_{i<j} x_i x_j \underbrace{\sum_{A\ni i,j} 1}_{2^{n-2}} = 2^{n-1}\sum_i x_i^2 - 2^{n-2}\sum_i x_i^2 = 2^{n-2}$$

$$0 = \Big(\sum x_i\Big)^2 = \sum_i x_i^2 + 2\sum_{i<j} x_i x_j$$

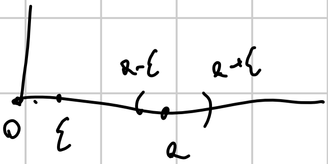
Metto assieme tutto e chiudo

3

$f : (0, +\infty) \longrightarrow (0, +\infty)$

$\forall a, b, c$ distinti $\quad a, b, c$ lati di $\Delta \iff f(a)\ f(b)\ f(c)$ lati di Δ

STEP 1 $\quad$ Prendo a $\quad$ scelgo $\varepsilon < \frac{a}{2}$



$\forall b \in (a-\varepsilon, a+\varepsilon)$ ho $\quad f(b) < f(a) + f(\varepsilon)$

STEP 2 $\quad$ Prendo a, ε come sopra

$\forall b \in (0, \varepsilon) \qquad a, b, a+\frac{b}{2}$ sono lati di Δ

$\Longrightarrow \quad f(b) < f(a) + f\left(a + \frac{b}{2}\right)$

STEP 3 $\quad f \longrightarrow 0$ per $x \longrightarrow 0$

se per assurdo $f \not\to 0$, trovo $x_i \to 0$
con $f(x_i) \to \lambda \neq 0$

$f(x_i), f(x_j) \in \left(\lambda\left(1-\frac{1}{10}\right), \lambda\left(1+\frac{1}{10}\right) \right)$

$x_k < |x_i - x_j| \quad f(x_k) \in \left(\lambda\left(1-\frac{1}{10}\right), \dots \right)$

ASSURDO perché x_i, x_j, x_k non Δ
$f(x_i)\ f(x_j)\ f(x_k)$ sì

STEP 4 $\quad$ Fissato a $\quad$ prendo degli $\varepsilon \ll a$
per $b \in (a-\varepsilon, a+\varepsilon)$
ε, a, b sono $\Delta \Rightarrow \underbrace{|f(a)-f(\varepsilon)|}_{f(a)} < f(b) < \underbrace{f(a)+f(\varepsilon)}_{f(a)}$

$\Rightarrow f$ continua

STEP 5 $\quad$ Supp $a \neq b \quad f(a) = f(b)$
prendo ε tale che $\varepsilon < |a-b|$
e $f(\varepsilon) < f(a)$

ε, a, b non Δ $\quad$ ma $f(\varepsilon), f(a), f(b)$

$\Rightarrow f$ iniettiva

STEP 6 $\quad f$ crescente (segue da 4 e 5)

STEP 7

$a, \; b+\varepsilon, \; a+b \quad \Delta \quad \Rightarrow \quad f(a+b) < f(a) + f(b+\varepsilon)$

$f(b)$

ottengo $f(a+b) \leq f(a) + f(b)$

$a, \; b-\varepsilon, \; a+b \quad$ non $\Delta \quad \Rightarrow \quad f(a+b) \geq f(a) + f(b-\varepsilon)$

$\Rightarrow \quad f(a+b) \geq f(a) + f(b)$