PREMIO 2017 AM

Note Title 24/05/2017

A1. $\boxed{(n-1)a_{n+1}=(n+1)a_n-2(n-1)}$

$$(n-1)\,b_{n+1}=(n+1)\,b_n$$

$$b_{n+1}=\frac{n+1}{n-1}\,b_n=\frac{n+1}{n-1}\,\frac{n}{n-2}\,b_{n-1}\cdots$$

$$b_n=C\,n(n-1)$$

Divido per $(n+1)n(n-1)$

$$\frac{a_{n+1}}{n(n+1)}=\frac{a_n}{n(n-1)}-\boxed{\frac{2}{(n+1)n}}$$

$$\frac{2}{(n+1)n}=\frac{2}{n}-\frac{2}{n+1}$$

$$\frac{a_{n+1}}{n(n+1)}-\frac{2}{n+1}=\frac{a_n}{n(n-1)}-\frac{2}{n}$$

$$\Rightarrow\quad \frac{a_n}{n(n-1)}-\frac{2}{n}=\text{costante}=C$$

$$\boxed{a_n=C\,n(n-1)+\underline{2(n-1)}}\qquad a_2=2C+2$$

(Oppure: provo i polinomi di grado 2 ...)

$$Q_{1999} = C \cdot 1999 \cdot 1998 + 2 \cdot 1998 \equiv 0 \quad (2000)$$

$$= C \cdot (-1) \cdot (-2) + 2 \cdot (-2) \equiv 0 \Rightarrow 2C \equiv 4 \quad (2000)$$

$$\underset{\equiv}{\overset{4}{\boxed{2C}}} \cdot \frac{n(n-1)}{2} + 2(n-1) \equiv 0 \quad (2000)$$

$$\overset{2}{\cancel{4}} \cdot \frac{n(n-1)}{\cancel{2}} + 2(n-1) \equiv 0 \quad (2000)$$

$$(2n+2)(n-1) \equiv 0 \quad (2000)$$

$$1000 \mid (n+1)(n-1)$$

$$\begin{cases} n \equiv \pm 1 & (125) \\ \boxed{n \equiv \pm 1 \quad (4)} \end{cases}$$

125 divide $n-1$ o $n+1$

2 divide uno tra $n-1$ e $n+1$
e 4 divide l'altro

$$n \equiv 249 \quad (1000)$$

A2 $$P(x)^3 + Q(x)^3 = x^{12} + 1$$

$$\left(P(x) + Q(x)\right)\left(P(x)^2 - P(x)Q(x) + Q(x)^2\right)$$

$$(x^{12}+1)(x^{12}-1)=x^{24}-1$$

un fattore per ogni ordine mod. 24

$$\Phi_k(x)=\prod_{\mathrm{ord}_{24}(i)=k}(x-\omega^i)$$

ω rad. primitiva

$$x^{24}-1=\underbrace{(x-1)}_{\mathrm{ord}\,1}\underbrace{(x+1)}_{\mathrm{ord}\,2}\underbrace{(x^2+x+1)}_{\mathrm{ord}\,3}\underbrace{(x^2+1)}_{\mathrm{ord}\,4}\ \cdots$$

In $x^{12}+1$ stanno ordine 8 e ordine 24

grado $\Phi(8)=4$ grado $\Phi(24)=8$

$$\underbrace{(x^4+1)}_{\Phi}\underbrace{(x^8-x^4+1)}_{\Phi}=x^{12}+1$$

irriducibili su $\mathbb{Q}[x]$

$P(x)+Q(x)=$	$\alpha(x^4+1)$	$\alpha(x^8-x^4+1)$	$\alpha(x^4+1)(x^8-x^4+1)$	α
$P^2(x)-P(x)Q(x)+Q^2(x)=$	$\frac{1}{\alpha}(x^8-x^4+1)$	$\frac{1}{\alpha}(x^4+1)$	$\frac{1}{\alpha}\,1$	$\frac{1}{\alpha}(x^{12}+1)$

E se

$P(x)$ ha grado 37, $Q(x)$ ha grado 37,

e tutto si cancella?

$$P(x)=\alpha x^d+\ldots$$

$$Q(x) = \beta x^d + \dots$$

$$P^2 - PQ + Q^2 = \underbrace{\alpha^2 x^{2d} - \alpha\beta x^{2d} + \beta^2 x^{2d}} + \dots O\left(x^{2d-1}\right)$$

$$\alpha^2 - \alpha\beta + \beta^2 \neq 0 \quad \text{per ogni } \alpha, \beta \in \mathbb{Q}$$

$$P + Q = x^4 + 1$$

$$P^2 - PQ + Q^2 = x^8 - x^4 + 1$$

$$3PQ = (P+Q)^2 - \left(P^2 - PQ + Q^2\right) = \dots = 3x^4$$

$$PQ = x^4 \qquad P = x^3 \quad Q = x$$ non va bene perché se no $P+Q$ ha grado troppo basso

Un fattore deve avere grado 4

$$P(x) = \alpha x^4 \qquad Q(x) = \frac{1}{\alpha}$$

$$\alpha x^4 + \frac{1}{\alpha} = x^4 + 1 \quad \text{sse} \quad \alpha = 1$$

soluzioni: $P(x) = x^4$, $Q(x) = 1$
$\qquad\qquad\quad = 1 \qquad\quad = x^4$

A3.

$$\sum_{n=0}^{m} a_n (-1)^n \binom{m}{n} = 0 \qquad (*)$$

Altra freccia: se $a_n = P(n)$ per un certo polinomio $P(x)$ di grado $d < m$, allora

$$\sum_{n=0}^{m} a_n (-1)^n \binom{m}{n} = 0$$

se P ha grado 0, allora $P(1) - P(0) = 0$

se P ha grado ≤ 1, allora $P(2) - 2P(1) + P(0) = 0$

se P ha grado ≤ 2, allora $P(3) - 3P(2) + 3P(1) - P(0) = 0$

⋮

valori assunti da $Q(x) = P(x+1) - P(x)$ di grado $\deg P - 1$

$P(0)$ $\quad P(1)$ $\quad P(2)$ $\quad P(3)$ $\quad P(4)$ $\quad P(5)$ $\quad \dots$

$P(1)-P(0)$ $\quad P(2)-P(1)$ $\quad P(3)-P(2)$ $\quad P(4)-P(3)$ $\quad \dots$

$P(2) - \boxed{2}P(1) + P(0)$ $\quad P(3) - \boxed{2}P(2) + P(1)$ $\quad -$

$$\boxed{P(3) - \boxed{3}P(2) + \boxed{3}P(1) - P(0)}$$

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supponiamo $(*)$ vera per $m \geq \bar{m}$

Sapendo

$a_0 \quad a_1 \quad a_2 \quad \dots \quad a_{\bar{m}-1}$

determino $a_{\bar{m}}$ dalla $(*)$ con $m = \bar{m}$

determino $a_{\bar m+1}$ dalla (*) con $m=\bar m+1$

$\vdots \qquad \vdots \qquad \vdots$

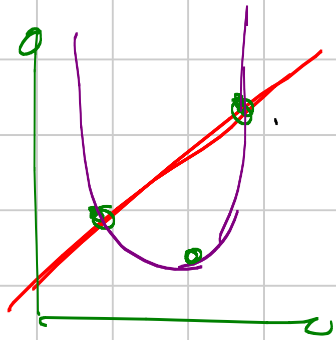
$\Rightarrow$ esiste una sola successione che soddisfa (*) per $m \geq \bar m$ e ha valori iniziali

$$a_0,\ a_1,\ a_2,\ \dots\ a_{\bar m-1}$$

Prendo il polinomio di grado $\leq \bar m$ tale che

$$p(0)=a_0,\ p(1)=a_1,\ \dots\ p(\bar m-1)=a_{\bar m-1}$$

(Interpolazione di Lagrange)



Allora i valori $b_n := p(n)$

soddisfano $\quad b_i = a_i$ per $i=0,1,\dots,\bar m-1$

e (*) per $m \geq \bar m$

$$\sum_{n=0}^{m} a_n (-1)^n \binom{m}{n} = 0 \qquad m > m_0$$

$$\sum_{m\geq 0} x^m \sum_{n=0}^{m} a_n (-1)^n \binom{m}{n} = P(x)$$

$$= \sum_{m\geq 0} \sum_{n=0}^{m} a_n (-1)^n x^m \binom{m}{n}$$

(Snake Oil)

$$= \sum_{n\geq 0} a_n (-1)^n \underbrace{\sum_{m\geq 0} x^m \binom{m}{n}}$$

$$\frac{x^{n-1}}{(1-x)^n} = \frac{1}{n!} D^n \frac{1}{(1-x)}$$

$$x P(x) = \sum_{n\geq 0} a_n \left[- \frac{x}{1-x} \right]^n$$

$$= \sum_{n\geq 0} a_n \left(\frac{x}{x-1} \right)^n = Q(x)$$

$$y = \frac{x}{x-1} \qquad x = \frac{y}{y-1}$$

$$\boxed{\sum_{n\geq 0} a_n y^n} = Q\left(\frac{y}{y-1}\right)$$

$$= \sum_{i=0}^{d} c_i \frac{y^i}{(y-1)^i}$$

$$= \sum_{i=0}^{d} c_i y (-1)^i \cdot \sum_{j\geq 0} y^j \binom{j}{i}$$

$$= \sum_{i=0}^{d} \sum_{j\geq 0} c_i (-1)^i y^{j+1} \binom{j}{i}$$

$$= \boxed{\sum_{j\geq 0} y^{j+1} \underbrace{\sum_{i=0}^{d} c_i (-1)^i \binom{j}{i}}_{a(j)}}$$

$$a_n = a(j-1)$$

Trovare α t.c. esiste $P(x)$ polinomio non nullo

$$\frac{P(1)+P(3)+\ldots+P(2n-1)}{n} = \alpha P(n) \quad (\cancel{**})$$

Provo i casi piccoli: $d = \deg(P)$

$\boxed{d=0}$ P costante, $P = a$

$$\frac{na}{n} = \alpha a \Rightarrow \boxed{\alpha = 1}$$

con P poli. qualunque di grado 0

$\boxed{d=1}$ $P(n) = bn+a$

$$\frac{\left(\overbrace{1+3+5+\dots+(2n-1)}^{n^2}\right) b + na}{n} = \alpha(bn+a)$$

vale per ogni a, b $\Rightarrow$ $\boxed{\alpha = 1}$

con P poli. qualunque di grado ≤ 1

$$P(n) = cn^2+bn+a$$

$$\frac{\left(\boxed{1^2+3^2+\dots+(2n-1)^2}\right) c + n^2 b + na}{n} = \alpha(cn^2+bn+a)$$

$$1^2+2^2+3^2+\dots+(2n)^2 = \frac{(2n)(2n+1)(4n+1)}{6} = \frac{1}{3}8n^3+\dots$$

$$-2^2(1^2+2^2+\dots+n^2) = -2^2\cdot\frac{n(n+1)(2n+1)}{6} = -\frac{4}{3}n^3+\dots$$

$$= \frac{4}{3}n^3+\square n^2+\square n$$

$$\frac{c\left(\frac{4}{3}n^{\cancel{3}\,2}+\Box n^{\cancel{2}}+\Box \cancel{n}\right)+bn^{\cancel{2}}+a\cancel{n}}{n}=\alpha\left(cn^2+bn+a\right)$$

$$\Rightarrow \boxed{\alpha=\frac{4}{3}} \qquad \frac{4}{3}c=\alpha c \quad \leftarrow (\text{wlog } c=1)$$

$$\Box c+b=\alpha b \quad \leftarrow \text{mi ricavo } b$$

$$\Box c+a=\alpha a \quad \leftarrow \text{mi ricavo } a$$

$\Rightarrow$ $\alpha=\frac{4}{3}$ va bene con P una certa famiglia di polinomi di grado 2, $P(n)=c\left(n^2+\Box n+\Box\right)$ $\forall c\in\mathbb{R}$

$$P(n)=zx^d+yx^{d-1}+\dots+bx+a$$

$$\frac{z\left(1^d+3^d+\dots+(2n-1)^d\right)+y\left(1^{d-1}+3^{d-1}+\dots+(2n-1)^{d-1}\right)+\dots+b\left(1+3+\dots+(2n-1)\right)+an}{n}=$$

$$=P(n)$$

$$\left[\begin{aligned}1^d+3^d+\dots+(2n-1)^d&=\left(1^d+2^d+\dots+(2n)^d\right)-2^d\left(1^d+2^d+\dots+n^d\right)=\\&=\frac{1}{d+1}(2n)^{d+1}+O(n^d)-\left[2^d\cdot\frac{1}{d+1}n^{d+1}+O(n^d)\right]\\&=\boxed{\frac{2^d}{d+1}n^{d+1}}+O(n^d)\end{aligned}\right]$$

$$= \frac{z\left(\frac{2^d}{d+1}n^{d+1}+\dots\right)+y\left(\frac{2^{d-1}}{d}n^{d}+\dots\right)+\dots+bn^2+an}{n} = P(n)$$

$$z\frac{2^d}{d+1} = \alpha z \quad \Leftarrow \quad \boxed{\alpha = \frac{2^d}{d+1}} \qquad \text{wlog } \boxed{z=1}$$

$$y\boxed{\frac{2^{d-1}}{d}}+\square z = \boxed{\alpha}\, y \qquad y = \boxed{\text{roba}}$$

$$x\boxed{\frac{2^{d-2}}{d-1}}+\square y+\square z = \boxed{\alpha}\, x \qquad x = \boxed{\text{roba}}$$

$$b+\square c+\dots+\square y+\square z = \alpha b \qquad b = \boxed{\text{roba}}$$

$$a+\square b+\square c+\dots+\square z = \alpha a \qquad a = \boxed{\text{roba}}$$

$\Rightarrow$ riesco effettivamente a trovare un polinomio che va bene per ogni grado d

⚠ riesco a risolverle perché

$$\frac{2^a}{a+1} \neq \frac{2^d}{d+1} \quad \text{per } a<d$$

(e $d \geq 2$, ma tanto il caso $d \leq 1$ l'ho fatto a mano)

$\Rightarrow$ gli unici α che vanno bene sono quelli della forma $\alpha = \frac{2^d}{d+1}$, e per ognuno di questi posso trovare un poly. di grado d

che funzione

2^a parte: $\alpha = 2 \Leftrightarrow d = 3$

uso lo stesso metodo di $d=2$ per farlo

$$c(n^3 - n) = P(n) \quad \text{va bene per } \alpha = 2$$

$$P(1) + P(3) + \dots + P(2n-1) = n \alpha P(n)$$

$$P(1) + P(3) + \dots + P(2n-1) + P(2n+1) = (n+1) \alpha P(n+1)$$

$$\Rightarrow P(2n+1) = (n+1)\alpha P(n+1) - n\alpha P(n) \quad (\cancel{***})$$

$$P(n) = \underbrace{z n^d + y n^{d-1}} + O(n^{d-1})$$

$$z(2n)^d + O(n^{d-1}) = \alpha\left(\cancel{z n^{d+1}} + z d n^d + z n^d + \cancel{y n^d} - \cancel{z n^{d+1}} - \cancel{y n^d} + O(n^{d-1})\right)$$

$$\Rightarrow \alpha = \frac{2^d}{d+1} \quad (\text{confrontando coeff. di } n^d)$$

nel caso $\alpha = 2$, $d = 3$:

$$P(2n+1) = 2\left[(n+1) P(n+1) - n P(n)\right]$$

vera come identità tra polinomi in n

$\boxed{n=-1} \Rightarrow P(-1) = 2P(-1) \Rightarrow P(-1)=0$

$\boxed{n=0} \Rightarrow P(1) = 2P(1) \Rightarrow P(1)=0$

Mi dice che i polinomi che cerco per $\alpha=2$, $d=3$ sono della forma

$$p(n) = d(n+1)(n-1)\underbrace{(\boxed{q}n+\boxed{r})}_{q=1,\ r=0}$$