ALGEBRA PREIMO 2017. POMERIGGIO

Note Title 24/05/2017

$\boxed{A5}$ $a,b,c>0$ $a+b+c=3$

$$\sum_{cyc}\frac{1}{a^2} \geq \sum_{cyc} a^2$$

Omogeneizzo? $\sum_{cyc}\left(\frac{1}{a^2}-a^2\right) = \sum\frac{1-a^4}{a^2} = 3^{-4}\sum\frac{(a+b+c)^4-(3a)^4}{a^2}$

Posso tentare Bunching + Shur facendo tutti i conti.

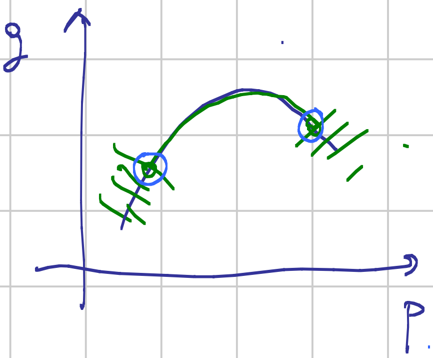
$$\boxed{3^{-4}\frac{1}{a^2b^2c^2}}\sum_{cyc}\left[(a+b+c)^4-(3a)^4\right]b^2c^2 \geq 0$$

$$f(a,b,c) = g(s,q,p) = g_1(s,q) + g_2(s,q)p + \underset{2}{g_3(s,q)}\underset{6}{p^2}$$

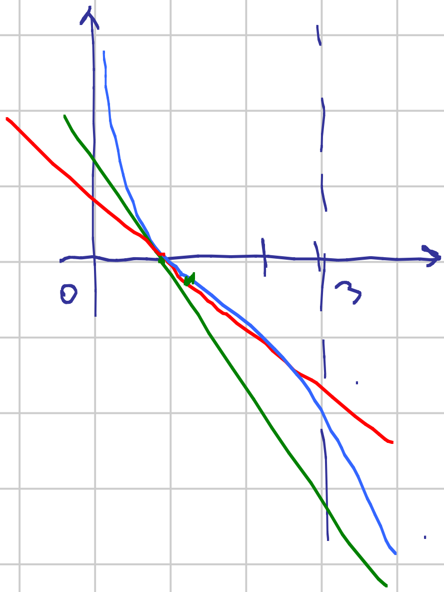
$s = a+b+c$

$q = ab+bc+c$

$p = abc$



$f(x) = \frac{1}{x^2} - x^2$ $\sum_{cyc} f(a) \geq 0$



Se fosse convessa: $\frac{1}{3}\sum_{c} f(a) \geq f\left(\frac{a+b+c}{3}\right) = f(1) = 0$

$f'(x) = -2x^{-3} - 2x$

$f''(x) = 6x^{-4} - 2$ $f''(x_0)=0$ $x_0 = \sqrt[4]{3} \approx 1{,}3$

Se fosse $f(x) \geq r(x)$ per $x \in [0,3]$ e $r(1)=0$

$$\sum_a f(a) \geq \sum_a r(a) = \sum_a (ma+q) = m(a+b+c)+3q = 3r(1) = 0$$

$r(x) = 4-4x \qquad r(3) = -8 \qquad f(3) = \frac{1}{9} - 9 < r(3)$

Se $a,b,c <$ punto di intersezione, ok, altrimenti a maggiore

$r\left(\frac{7}{3}\right) = 4 - 4\cdot\frac{7}{3} = -\frac{16}{3} \qquad f\left(\frac{7}{3}\right) = \frac{9}{49} - \frac{49}{9} = \frac{81-49^2}{49\cdot 9}$

$\frac{81}{49} - 49 \overset{?}{\gtrless} -48 \qquad\qquad f\left(\frac{7}{3}\right) > r\left(\frac{7}{3}\right)$

$a > \frac{7}{3}$ almeno uno tra b e c è $< \frac{1}{3}$ $\qquad b \leq \frac{1}{3}$

$f(b) \geq 9 - \frac{1}{9} \qquad f(a) \geq \frac{1}{9} - 9$

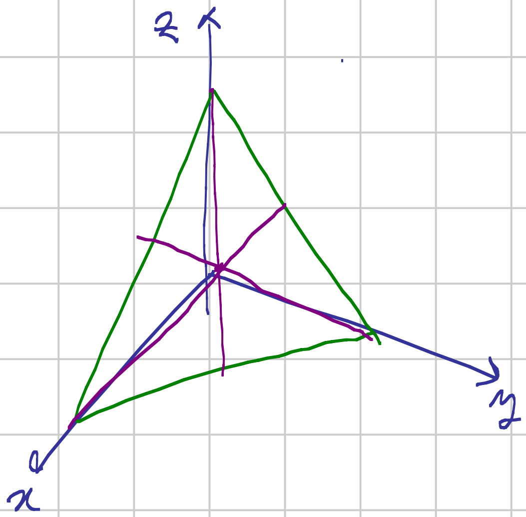
Modo 2: Convex-concave

vincolo somma $x+y+z = c$

f t.c. f'' cambia se a volta sola

$(f')^{-1}$ al più 2 controimmagini per ogni punto

Allora i punti stazionari di $f(x)+f(y)+f(z)$ sono tali per cui almeno due tra x, y, z sono uguali



Va controllato il bordo e $x = y$

$2f(x) + f(3-2x) \overset{?}{\geq} 0$

derivo $2f'(x) - 2f'(3-2x)\big|_{x=1} = 0$

derivo $2f''(x) + 4f''(3-2x)$

$\frac{3}{x^4} - 2 + \frac{6}{(3-2x)^4} - 4 > 0 \quad \forall x$

non banale

$$\sum \frac{1}{a^2} \geq \sum a^2$$

$$\sum_c$$

$$\sum_s$$

$$\sum a^2 b^2 \geq \sum a^4 b^2 c^2$$

$$(a+b+c)^4 = \sum_c a^4 + 6\sum_c a^2b^2 + 12 \sum_c a^2bc + 4\sum_s a^3 b$$

$$2[6,2,0] + 13[4,2,2] + 6[4,4,0] + 20[3,3,2]$$
$$+ 24[4,3,1] + 8[5,3,0] + 8[5,2,1] \overset{?}{\geq} 81[4,2,2]$$

$$[6,2,0] + 4[5,3,0] + 4[5,2,1] + \underline{3[4,4,0]}$$
$$+ \underline{12[4,3,1]} + 10[3,3,2] \overset{?}{\geq} 34[4,2,2]$$

$$\sum_s a^3b^3c^2 \quad \cancel{+} \sum_s a^5 b^1 c^2$$

$$\geq 2 \quad {}_s \; a^4 b^2 c^2$$

$$\sum_s a^6 b^2 + 2\sum_s a^3 b^3 c^2$$

$$\geq 3 \sum_s a^4 b^{8/3} c^{4/3}$$

$$\geq 3 \sum_s a^4 b^2 c^2$$

$\boxed{A6}$ $f: \mathbb{R}_+ \to \mathbb{R}$

$f(x)+f(y) \le \frac{f(x+y)}{2}$ $\qquad \frac{f(x)}{x} + \frac{f(y)}{y} \geq \frac{f(x+y)}{x+y}$

$g(x) := \frac{f(x)}{x}$ $\qquad \frac{x}{x+y} g(x) + \frac{y}{x+y} g(y) \le \frac{1}{2} g(x+y)$ $\qquad g(x)+g(y) \geq g(x+y)$

$\frac{x}{x+y} g(x) + \frac{y}{x+y} g(y) \le \frac{1}{2} g(x+y) \le \frac{1}{2} g(x) + \frac{1}{2} g(y)$

$\frac{x-y}{x+y}(g(x)-g(y)) \le 0$ $\qquad \boxed{g \text{ non crescente}}$

$x=y$ $\qquad 2g(x) = g(2x)$ $\qquad g(2^n x) = 2^n g(x)$ $\qquad n \in \mathbb{Z}$

siccome è non crescente, $g(x) \le 0$

$\mathbb{N} \ni y = \sum_{i \in A} 2^i$ $\qquad A \subseteq \{0,1,2,\dots,n\} = S$ $\qquad \bar{y} = \sum_{i \in \bar{A}} 2^i + 1$

$2^{n+1} g(1) = g(2^{n+1}) = g(y+\bar{y}) \le g(y) + g(\bar{y}) = g\left(\sum_A 2^i\right) + g\left(\sum_{\bar{A}} 2^i + 1\right) \le \sum_A 2^i g(1) + \dots = y g(1) + \bar{y} g(1) = 2^{n+1} g(1)$

$\sum_A g(2^i)$ $\qquad \sum_{\bar{A}} g(2^i) + g(1)$

$g(y) = y g(1)$

$y > 0$ qualsiasi $\quad y = \sum_A 2^i$ $\qquad A \subseteq \{\dots,-3,-2,-1,0,1,\dots,n\} = S$

$y_m = \sum_{\substack{i \in A \\ i \geq m}} 2^i$ $\qquad \bar{y}_m = \sum_{\substack{i \in \bar{A} \\ i \geq m}} 2^i + 2^m$ $\qquad y_m + \bar{y}_m = 2^{n+1}$ $\qquad y_m \le y \le y_m + 2^m$

$2^{n+1} g(1) = \dots = g(y_m + \bar{y}_m) \le \dots \le y_m g(1) + \bar{y}_m g(1) \le 2^{n+1} g(1)$

$g(y_m) = y_m g(1)$

$(y+2^m) g(1) \le (y_m + 2^m) g(1) = g(y_m + 2^m) \le g(y) \le g(y_m) = y_m g(1) \le (y - 2^m) g(1)$

$|g(y) - y g(1)| \le 2^m g(1) \quad \forall m \qquad m \to -\infty$

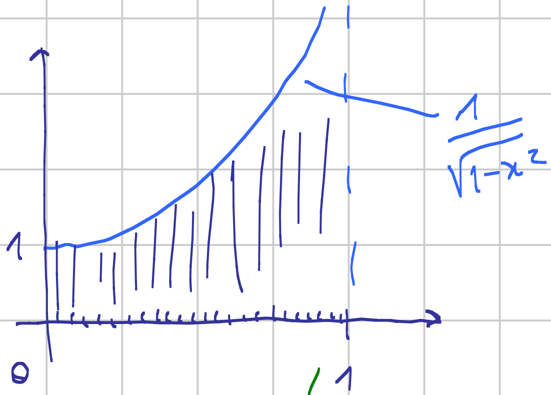
$\boxed{A7}$ $\sum_{k=1}^{n} x_k = 1 \qquad x_k > 0$

$$\sum_{k=1}^{n} \frac{x_k}{\sqrt{1-\left(\sum_{i<k} x_i\right)^2}}$$

$x_k = \frac{1}{n} \qquad \frac{1}{n}\sum_{k=0}^{n-1} \frac{1}{\sqrt{1-\left(\frac{k}{n}\right)^2}}$

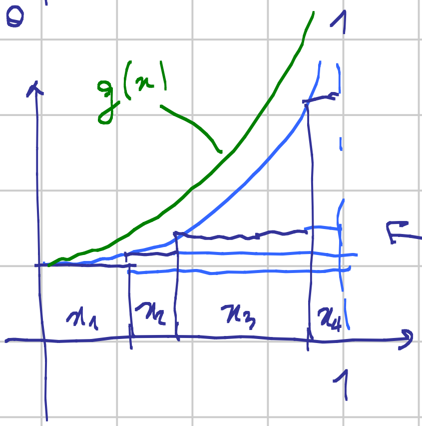
$\downarrow$

per $n \to \infty$ tende a $\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = \frac{\pi}{2}$



$f(x) = \frac{1}{\sqrt{1-x^2}} \qquad f^{-1}(y) = \sqrt{1-\frac{1}{y^2}}$

$$E = \sum_{k=1}^{n} x_k f\left(\sum_{i<k} x_i\right)$$



$\leftarrow$ area rettangoli $= E$

$g^{-1}(y) = 1 - \frac{1}{y^2}$

$$f(x) = \frac{1}{\sqrt{1-x^2}} \le \frac{1}{\sqrt{1-x}} = g(x)$$

$z_k := 1 - \sum_{i<k} x_i \qquad z_1 = 1 \ldots z_2 = 1 - x_1 \ldots z_n = x_n \qquad z_{n+1} = 0$

$x_k = z_k - z_{k+1}$

$$E = \sum_{k=1}^{n} (z_k - z_{k+1}) \frac{1}{\sqrt{z_k}} = \sum_{k=1}^{n} (z_k - z_{k+1}) \sum_{m=1}^{k-1}\left(\frac{1}{\sqrt{z_{m+1}}} - \frac{1}{\sqrt{z_m}}\right) + 1 \underbrace{\sum_{k=1}^{n}(z_k - z_{k+1})}_{1}$$

$$= \sum_{m=1}^{n-1}\left(\frac{1}{\sqrt{z_{m+1}}} - \frac{1}{\sqrt{z_m}}\right) \underbrace{\sum_{k=m+1}^{n} (z_k - z_{k+1})}_{z_{m+1} \le \sqrt{z_m}\sqrt{z_{m+1}}} + 1$$

$$\le \sum_{m=1}^{n-1}\left(\sqrt{z_m} - \sqrt{z_{m+1}}\right) + 1 = 1 - \sqrt{x_n} + 1 = 2 - \sqrt{x_n} < 2$$

$\boxed{A8}$ $f:\mathbb{R}\to\mathbb{R}$

(a) $f(x)\le M \quad \forall x$

(b) $\forall x,y\in\mathbb{R}$

$$f(xf(y))+yf(x)=xf(y)+f(xy)$$

$x=0 \qquad 0=yf(0)$

$f(0)=0$

$y=1 \quad f(xf(1))=xf(1)$

$f(1)=0$ oppure illimitato

$x=1 \qquad f(f(y))+0=f(y)+f(y)=2f(y)$

$f(\dots f(f(y))..)=2^n f(y) \qquad \Rightarrow \; f(y)\le 0 \quad \forall y$

simmetrizzo $\quad f(xf(y))+f(yf(x))=2f(xy)$

$x\neq 0 \quad x,\frac{1}{x} \qquad \underbrace{f(xf(\tfrac{1}{x}))}_{\ge 0}+\underbrace{f(\tfrac{1}{x}f(x))}_{\ge 0}=0$ nulli entrambi

$x\neq 0 \Rightarrow f\left(\frac{f(x)}{x}\right)=0$

se y è t.c. $f(y)=0 \qquad \boxed{0+yf(x)=0+f(xy)}$

$\boxed{f\equiv 0}$ è soluz $\qquad$ se uno $x: f(x)<0$

$\Rightarrow f(y)=0 \Rightarrow y\ge 0$

$\frac{f(x)}{x}\ge 0$

$$\frac{f(x)^2}{x}=f(f(x))=2f(x)$$

$f(x)=0,\,2x$

$\forall x: f(x)\neq 0 \;\Rightarrow f(x)=2x \quad \Rightarrow x\le 0 \quad \Rightarrow f\equiv 0$ sui positivi

$x : f(x) = 0 \Rightarrow x \geq 0 \qquad \forall x < 0 \quad f(x) = 2x$

Verifico!!!

Due soluzioni : $f(x) \equiv 0$

$f(x) = \min(0, 2x)$