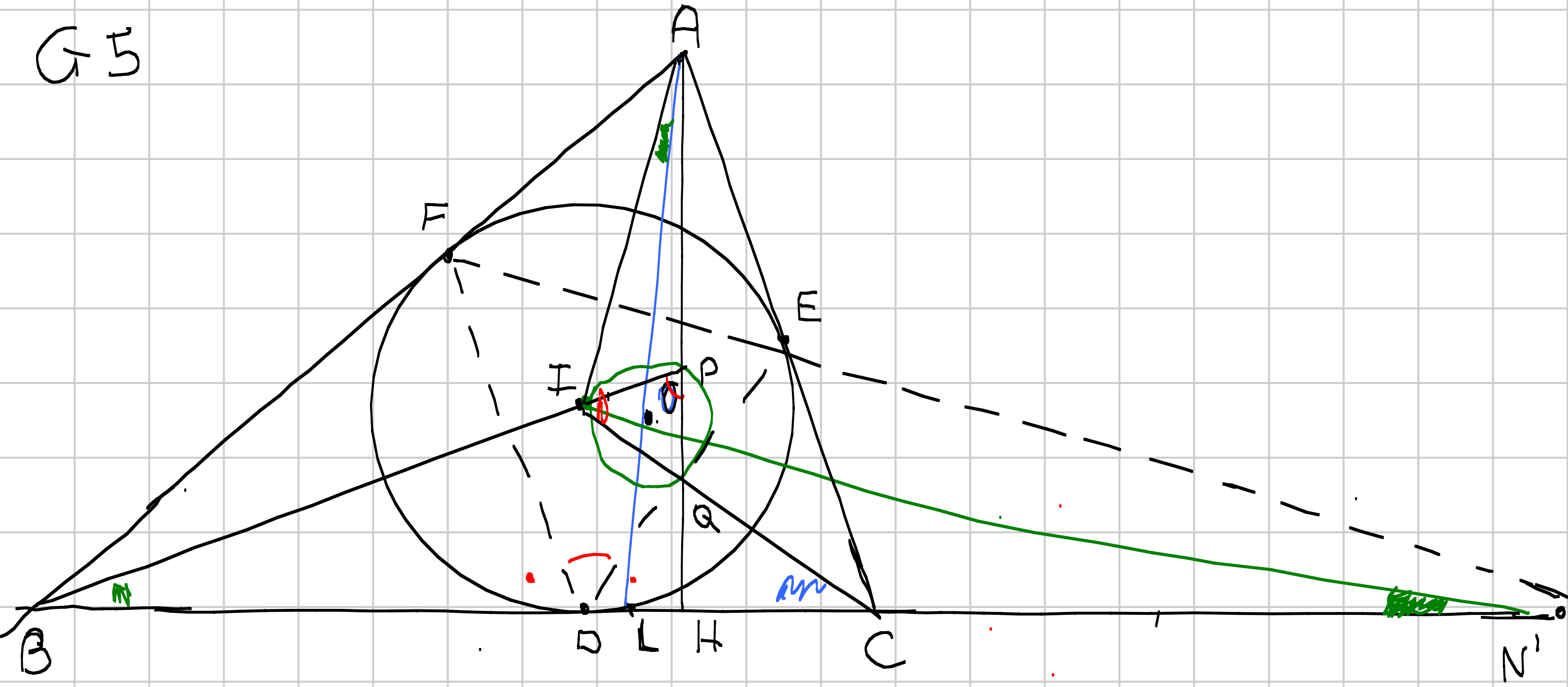
PreIMO 2017 - Geometria Pomeriggio

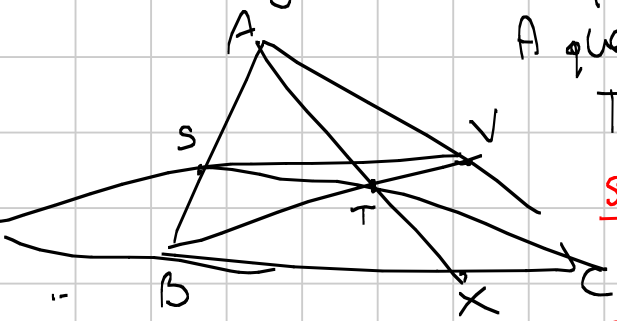
Note Title 5/22/2017

G5



Th Preso $N \in BC$ t.c. A, I, L, N ciclico, allora vale $\frac{BD}{DC} = \frac{BN}{NC}$ (*)

Sol (*) $\iff$ $(B, C; D, N) = -1$. Chi è $N \in BC$ t.c. valga (*)? Risposta: N sarà $EF \cap BC$.



A questo punto sia $N' := EF \cap BC$.

Th $\iff$ A, I, L, N' ciclici. ← [STEP 1]

STEP 2 Chi sono gli angoli di $I\hat{P}Q$?

$P\hat{I}Q = \frac{\beta}{2} + \frac{\gamma}{2}$

$I\hat{P}Q = \frac{\alpha}{2} + \frac{\gamma}{2}$

$P\hat{Q}I = \frac{\alpha}{2} + \frac{\beta}{2}$

Noto che $F\hat{D}E = \frac{\beta}{2} + \frac{\gamma}{2}$ e omologhe $\Rightarrow$

$\Rightarrow I\hat{P}Q \sim D\hat{E}F$ [STEP 2]

AI tange $\odot IPQ \iff A\hat{I}P = I\hat{Q}P$. [STEP 3]

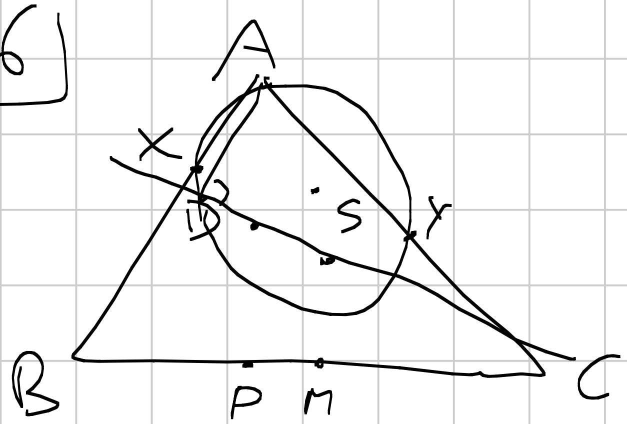
Ma $A\hat{I}P = \frac{\alpha}{2} + \frac{\beta}{2} = I\hat{Q}P$. ☒

Come procede la dim? STEP 1, STEP 2 ∧ STEP 3

Conclusione: STEP 2 ∧ STEP 3 $\Rightarrow$ $I\hat{A}O = I\hat{N}'D$ $\Rightarrow$ A, I, L, N' ciclico $\underset{\text{STEP 1}}{\Rightarrow}$ th. ☒

similitudine

Problema 6



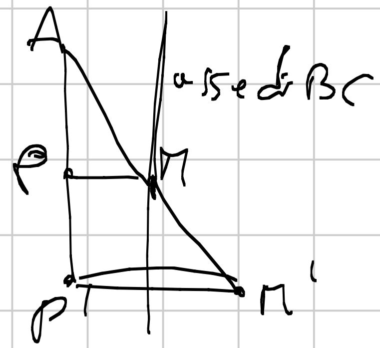
Omotetia di centro A e fattore 2

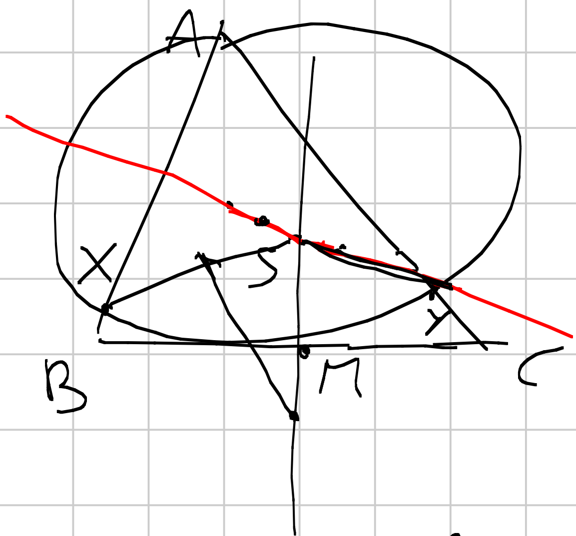
$S \mapsto S' \in OH$

$X \mapsto X' = 2X - A$ Y analogo

l'asse di PM $\mapsto$ asse di BC

$M \mapsto 2M - A$





Complessi con ⊙ABC circonferenza unitaria

$$\frac{z-o}{\bar z-\bar o} = \frac{h-o}{\bar h-\bar o} \qquad o=0 \quad h=a+b+c$$

$\bar a = \frac{1}{a}$ e $\sim$/c

$$OH: \frac{z}{\bar z} = \frac{\sum a}{\sum \frac{1}{a}} \qquad OH: \bar z = z \cdot \frac{\sum ab}{abc \sum a}$$

$\bar{S} = s\dfrac{\sum ab}{abc\sum a}$ $\qquad \bar{s}\, abc \sum a = s\sum ab$

T pt medio AX $\qquad ST \perp AB$

$$\frac{s-t}{\bar{s}-\bar{t}} = -\frac{a-b}{\bar{a}-\bar{b}} = ab \qquad s-t = ab\bar{s} - ab\bar{t} \quad (1)$$

$T \in AB \qquad \dfrac{a-t}{\bar{a}-\bar{t}} = \dfrac{a-b}{\bar{a}-\bar{b}} = -ab$

$$a-t = -b + ab\bar{t} \qquad ab\bar{t} = a+b-t \quad (2)$$

$(1)+(2) \Rightarrow x = 2t - a = b + s - ab\bar{s}$

Analogamente $\quad y = c + s - ac\bar{s}$

Tesi $\Leftrightarrow$ asse $XS \cap$ asse BC simm in b e c.

asse BC passa per M e O

$$\frac{z-0}{\bar{z}-\bar{0}} = \frac{m-0}{\bar{m}-\bar{0}} \qquad m = \frac{b+c}{2}$$

asse BC: $\dfrac{z}{\bar{z}} = bc \qquad z = \bar{z}bc$

asse SX : $(z-x)(\bar{z}-\bar{x}) = (z-s)(\bar{z}-\bar{s})$

$$z(\bar{s}-\bar{x}) + \bar{z}(s-x) = s\bar{s} - x\bar{x}$$

$$\bar{z}(ab\bar{s}-b) + z\left(\frac{s}{ab} - \frac{1}{b}\right) = s\bar{s} - (b+s-ab\bar{s})\left(\frac{1}{b} + \bar{s} - \frac{s}{ab}\right)$$

$$\bar{z}(ab\bar{s}-b) + z\left(\frac{s}{ab} - \frac{1}{b}\right) = -1 - b\bar{s} - \frac{s}{b} + \frac{s}{a} + a\bar{s} - s\bar{s} + ab\bar{s}^2 + \frac{s^2}{ab}$$

Moltiplico per $abc^2(\sum a)^2$ e volendolo intersecare con OM uso $\bar{z}bc = z$.

$$z\left(-abc(\textstyle\sum a)^2 + sa\sum a \sum ab\right) + z(s-a)\left(c^2(\textstyle\sum a)^2\right) =$$

$$= -abc^2(\textstyle\sum a)^2 - bc(\sum a)(\sum ab)s - sac^2(\sum a)^2 +$$

$+ ac \sum' a \sum' ab\, s + s b c^2 (\sum' a)^2 + s^2 (\sum' ab)^2 + c^2 (\sum' a)^2 s^2 +$
$- s^2 c \sum a \sum ab$

$$z\left[s(\sum' a)\left[a \sum' ab + c^2 \sum' a\right] - (\sum' a)(abc \sum' a + ac^2 \sum' a)\right]$$

$$a \sum' ab + c^2 \sum' a = (b+c)(a^2+ac+c^2)$$

$$z (b+c) \sum' a \left[s (a^2+ac+c^2) - (\sum a) ac \right]$$

$$(\sum' ab)^2 + c^2 (\sum' a)^2 - c \sum' a \sum' ab = (a^2+ac+c^2)(b^2+bc+c^2)$$

$$- bc \sum' a \sum' ab - ac^2 (\sum' a)^2 + ac \sum' a \sum' ab + bc^2 (\sum' a)^2 =$$
$$= (\sum' a)(a-b)(c \sum' ab - c^2 \sum' a) = c (\sum' a)(a-b)(ab - c^2)$$

$$z \text{ sym} \overset{\text{inbtc}}{\iff} z (b+c) \sum a \text{ sym}$$

$$\underbrace{\frac{s^2 (a^2+ac+c^2)(b^2+bc+c^2)}{0}} + \underbrace{s \;\underline{\quad}\; + \;\underline{\quad}}_{bc \sum' a \left(s(a^2+ac+c^2) - ac \sum' a\right)} \qquad \frac{s(a^2+ac+c^2) - ac \sum a}{s (b^2+bc+c^2)}$$

$$z (b+c) \sum' a = s (b^2+bc+c^2) + bc \sum' a$$

Simmetrico $\Rightarrow$ Tesi!

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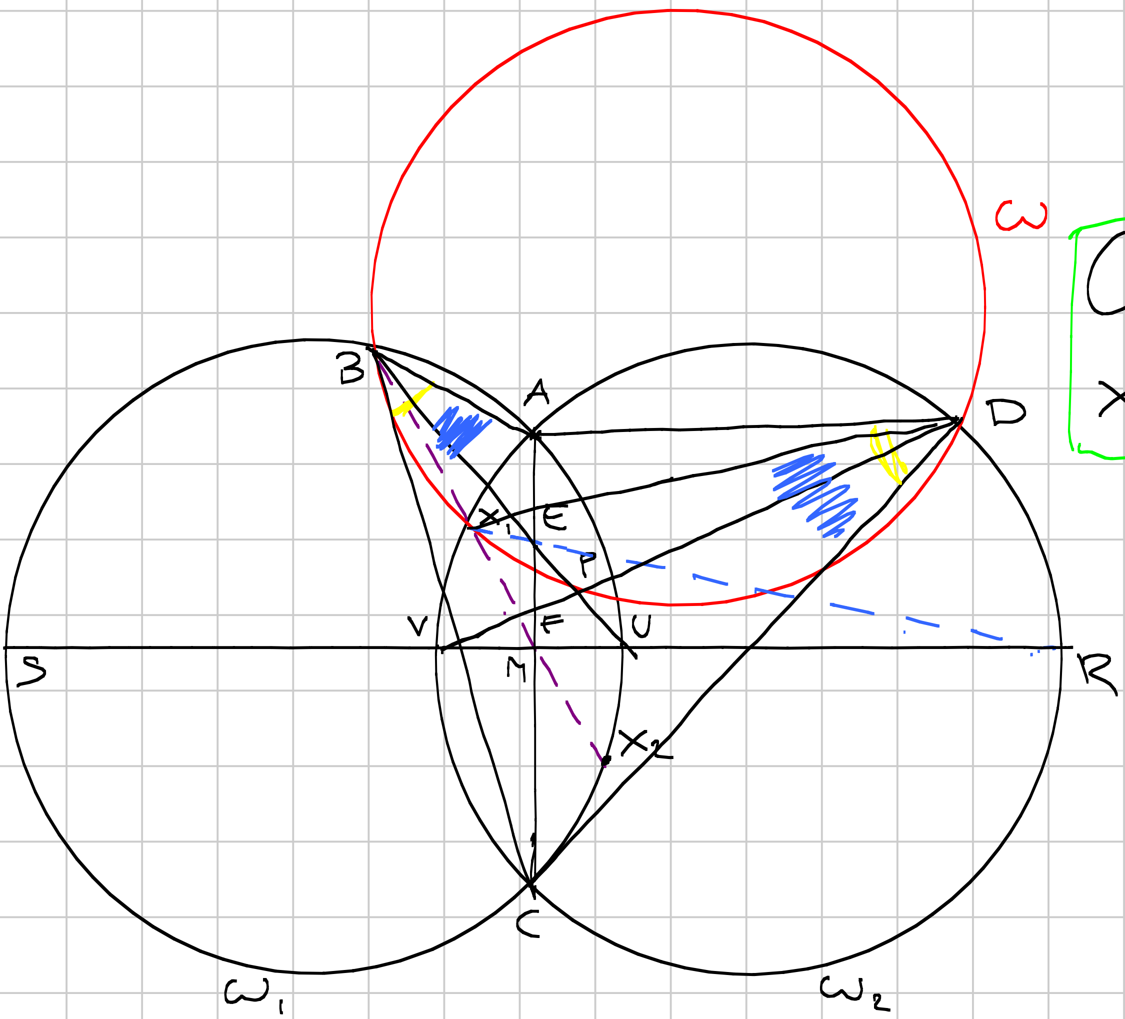


Claim:

$X \in \odot(\widehat{ADC})$

Supponiamo vero il Claim.

$X_2 = BM \cap \odot(ABC) \quad (\neq B)$

BM e' mediana in $\widehat{ABC}$

$$\frac{AM}{\sin\varphi} = \frac{BM}{\sin \widehat{BAC}}$$

$$\frac{AM}{BM} = \frac{\sin\varphi}{\sin\widehat{BAC}} \qquad \frac{CM}{BM} = \frac{\sin\theta}{\sin\widehat{ACB}}$$

$AM = CM$

$$\frac{\sin\varphi}{\sin\theta} = \frac{\sin\widehat{BAC}}{\sin\widehat{ACB}} = \frac{BC}{AB} \qquad \frac{AX_2}{CX_2} = \frac{BC}{AB} \qquad \begin{matrix} AX_1 = CX_2 \\ CX_1 = AX_2 \end{matrix}$$

$$\frac{AX_1}{CX_1} = \frac{AB}{BC} \qquad \frac{AE}{EC} = \frac{AB}{BC} \quad \text{(T d bisettrice)}$$

Dimostriamo il claim

Definiamo $X_1 = \omega \cap \omega_1$

TS: B, X_1, M allineati

$\Longleftrightarrow$ (Simmetria centrale in M)

$$\widehat{ABX_1} = \widehat{X_1DC}$$

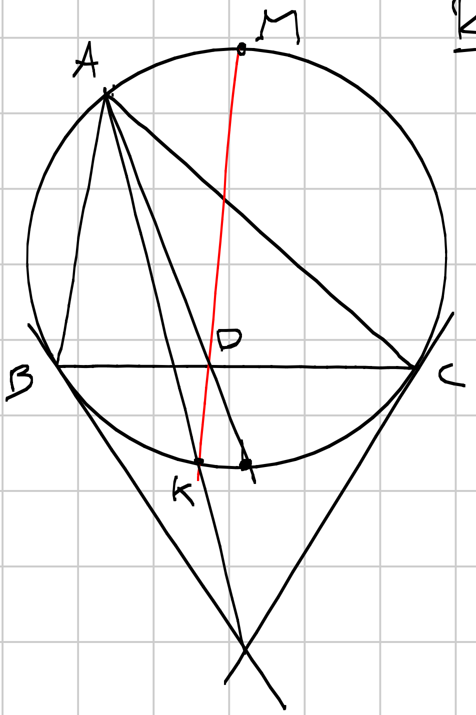
$$\widehat{BPD} = 2\pi - \alpha - \frac{\beta}{2} - \frac{\delta}{2} = 2\pi - \alpha - \beta$$

$$\parallel$$

$$\widehat{BX_1D}$$

$$\widehat{X_1BA} + \cancel{\alpha} + \underbrace{\widehat{ADX_1}}_{\cancel{\beta} - \widehat{X_1DC}} + \underbrace{\widehat{BX_1D}}_{\cancel{2\pi} - \cancel{\alpha} - \cancel{\beta}} = \cancel{2\pi}$$

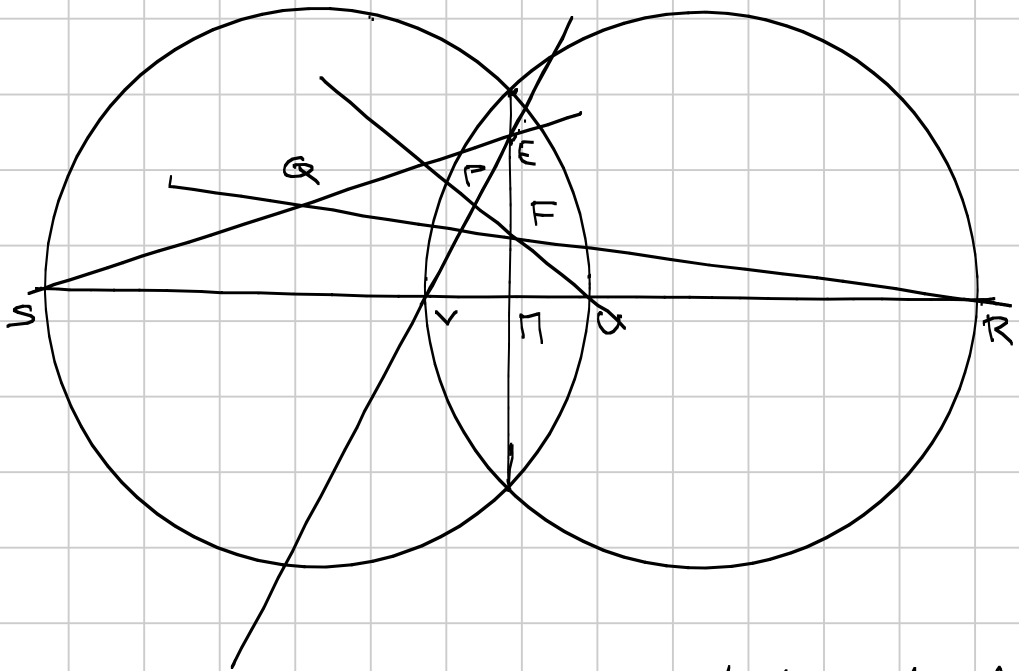
Ricordo



$$\frac{AB}{AC} = \boxed{\frac{BK}{KC} = \frac{BD}{DC}}$$

T_A bisettrice di $\widehat{BKC}$

$\Rightarrow$ Tesi



$E = A_1 \quad V = A_2 \quad S = A_3 \qquad A_1A_2 \cap A_4A_5 = P$

$F = A_4 \quad U = A_5 \quad R = A_6 \qquad A_2A_3 \cap A_5A_6 = ?$

$A_3A_4 \cap A_6A_1 = Q$

$C: xy = \alpha$ $\boxed{\lim_{\alpha \to 0}}$ $\alpha > 0$

r_1 $x = x_U$ $\cap$ C P_1 $P_1P_2 \,//\, P_3P_4$

$x = x_V$ $\cap$ C P_2

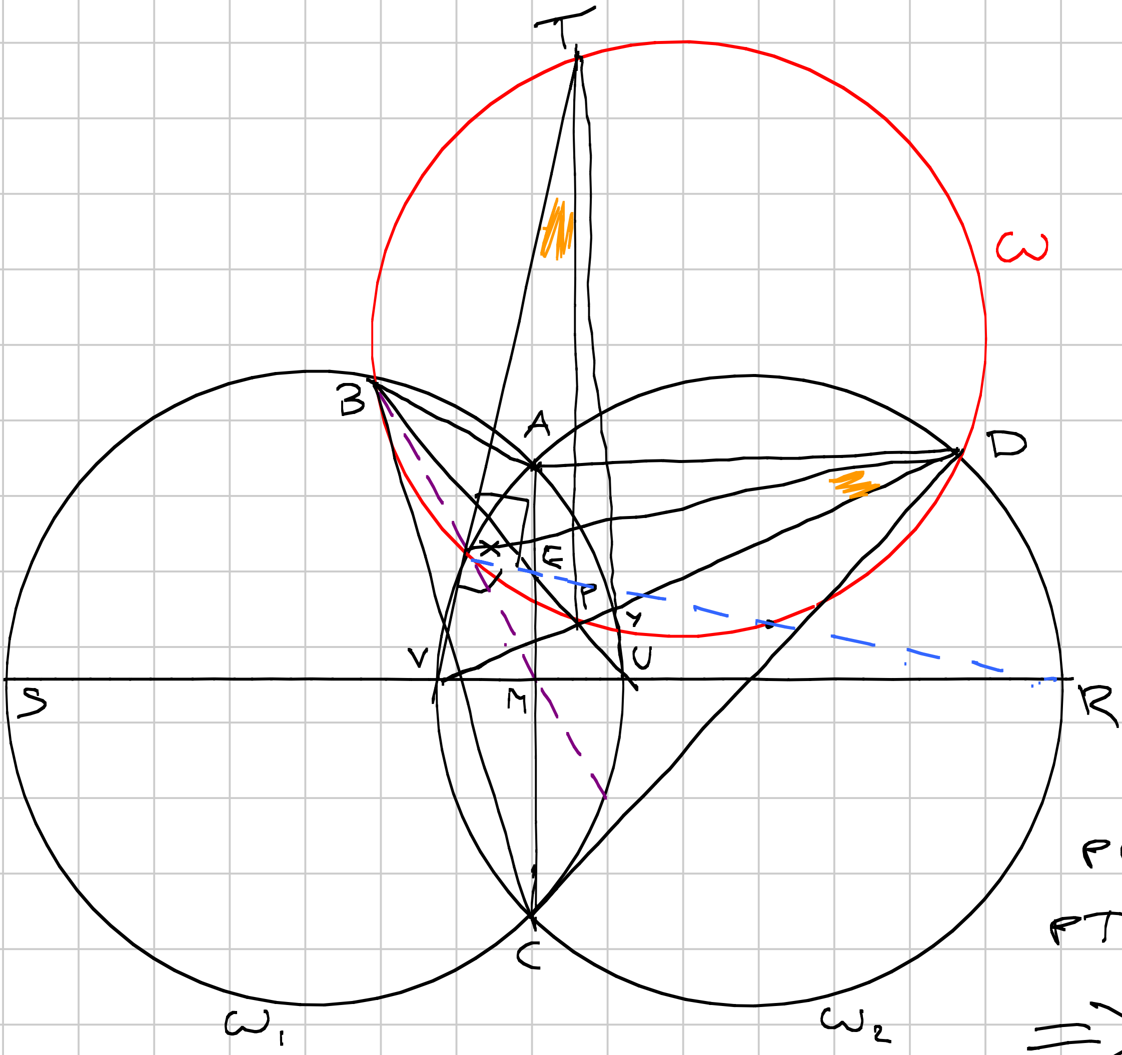
$x = x_R$ $\cap$ C P_3

$x = x_S$ $\cap$ C P_4

$\forall \alpha > 0$ $P_1P_2 \cap P_3P_4 \in r_\infty$

"Per continuità"

$\alpha = 0 \Rightarrow RS \cap UV \in r_\infty$



$TX \perp XE$

$TY \perp YF$

Q è il diametralmente opposto di T in ω.

$PQ \perp PT$

$PT \parallel AC$

$\Rightarrow$ Tesi.

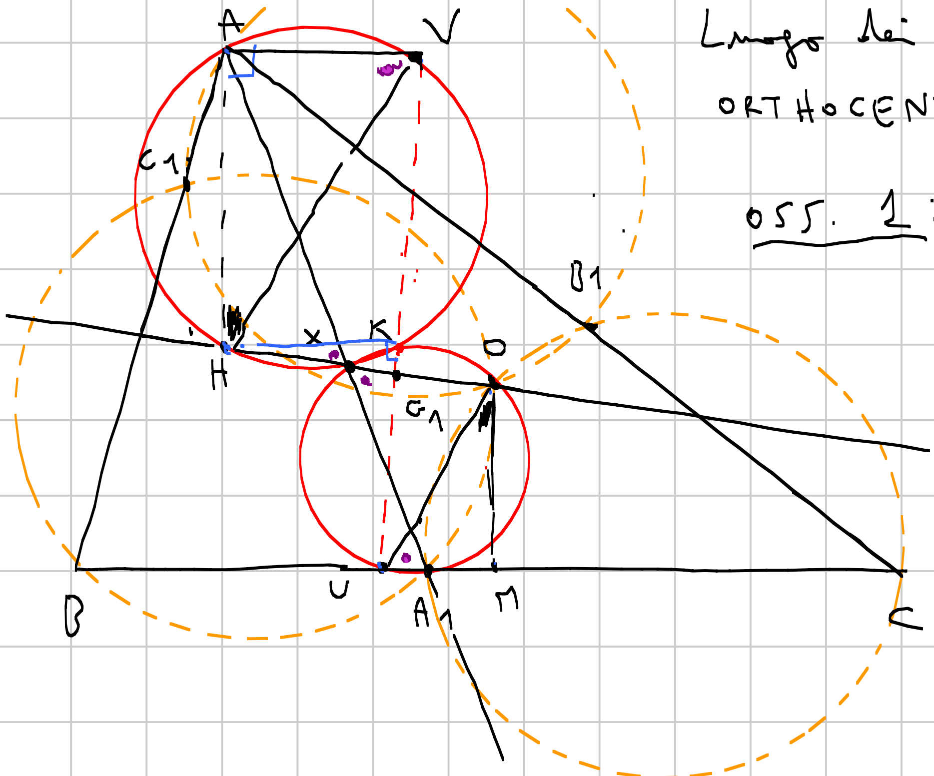
r parallela ad AC per P

$T = VX \cap r$

$\widehat{PTX} = 90 - \widehat{XUM} = \widehat{XRU} = \widehat{XDV}$

$\Rightarrow T \in \omega$.

8)



Luogo dei centri di similitudine

ORTHOCENTROIDAL CIRCLE

<u>OSS. 1</u>: il pt. di Miquel di A_1, B_1, C_1 risp. ad ABC è O (il circocentro di ABC) e O è l'ortocentro di $A_1B_1C_1$

H ortoc. di ABC

Sia $X = OH \cap AA_1$; allora $K = \odot(OXA_1) \cap \odot(HXA)$ $(\neq X)$

Se V è l'intersezione di $\odot(AHX)$ con la parallela a BC per A e U è la seconda intersezione di $\odot(OXA_1)$ con BC $\Rightarrow$ U, K, V sono allineati

$$\angle(VK, KX) + \angle(XK, KU) = \angle(AV, AA_1) + \angle(AA_1, AU) = 0$$

Inoltre $\angle(AVH) = \angle(AXH) = \angle(A_1XO) = \angle(A_1UO)$

$\Rightarrow VH \parallel UO \Rightarrow \dfrac{OG_1}{HG_1} = \dfrac{OU}{VH} = \dfrac{OM}{AH}$ (perché $A\hat{V}H = M\hat{U}O$)

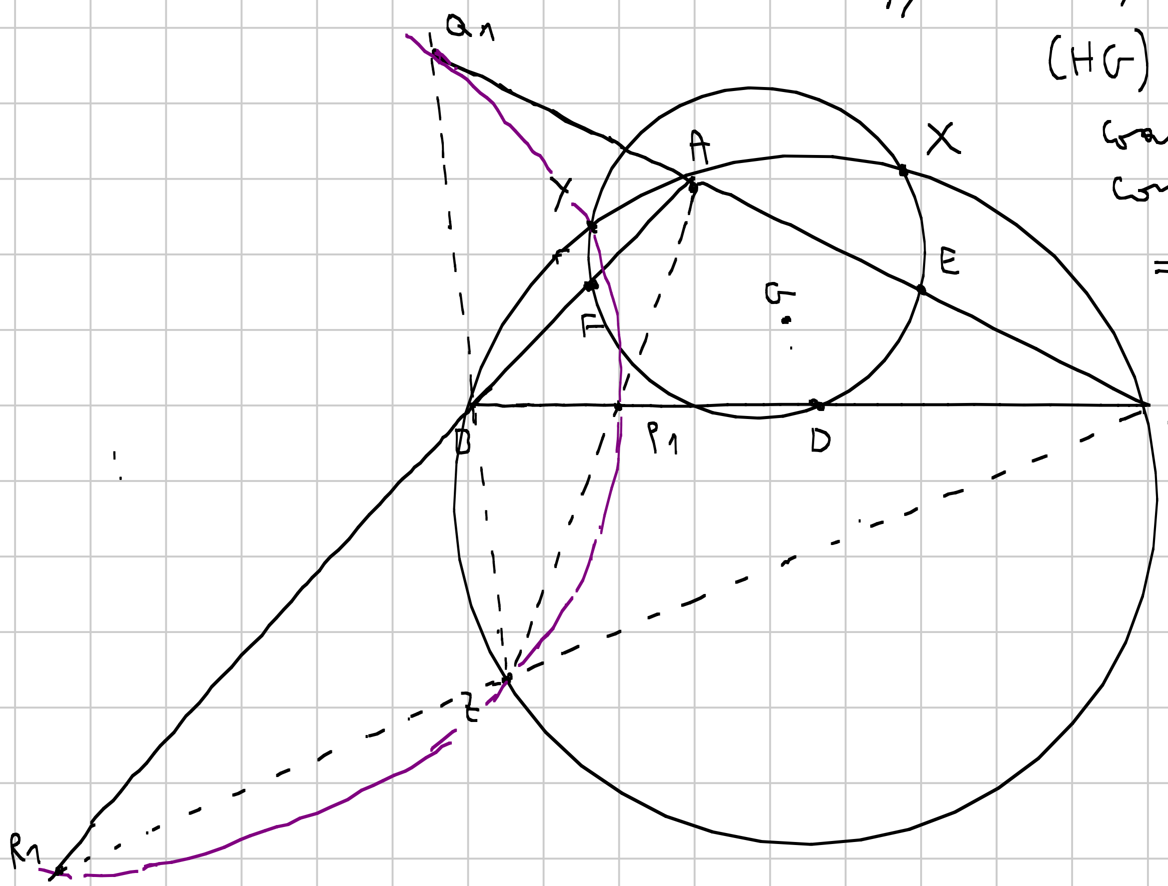
$= \dfrac{1}{2}$

$\Rightarrow$ G_1 è il baricentro di ABC e

$H\hat{K}G_1 = H\hat{A}V = 90°$

<u>OSS</u>: il luogo è tutto il cerchio (HG)

Ora H, G sono i centri di similitudine di $\odot(ABC)$, $\odot(DEF)$ $\Rightarrow$ (HG) è coassiale con essi $\Rightarrow$ passa per X, Y



per quanto detto prima esistono P_1, Q_1, R_1 su AB, BC, CA tali che $P_1\hat{Q_1}R_1 = A\hat{B}C$ e il loro centro di similitudine sia Y.

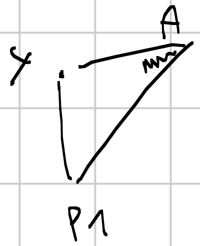
$Y \in \odot(ABC) \Rightarrow Y \in \odot(P_1Q_1R_1)$

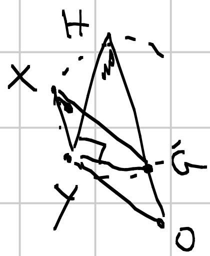
Se $Z = \odot(ABC) \cap \odot(P_1Q_1R_1)$ $(\neq Y)$ allora

$\angle(ZP_1, ZY) = \overset{\frown}{P_1Y}$ in $\odot(P_1Q_1R_1)$ $=$

$= \overset{\frown}{AY}$ in $\odot(ABC)$ $= \angle(ZA, ZY)$

$\Rightarrow Z, P_1, A$ allineati e analoghe





$\Rightarrow \angle(ZXY) = \angle(ZAY)$

$= \angle(P_1AY) = \angle(OHY)$

$= \angle(GXY)$

$\Rightarrow Z, G, X$ sono allineati

$\Rightarrow$ TS.