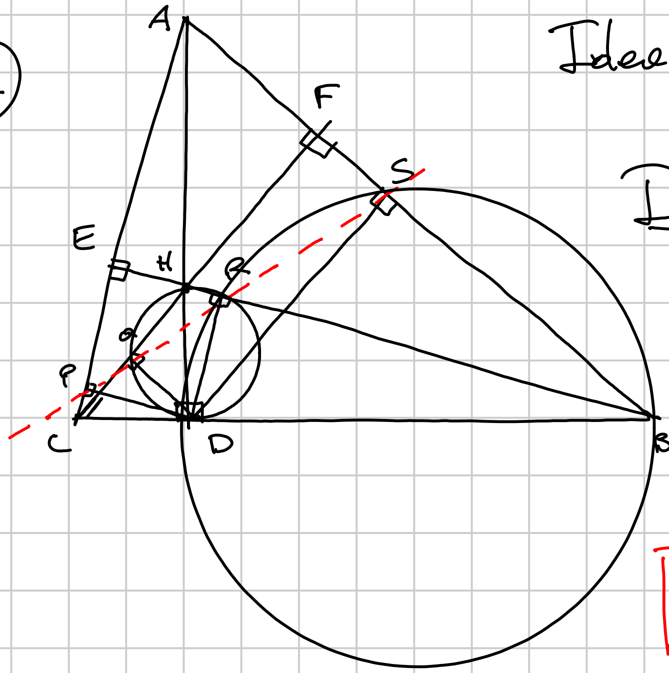
Pre IMO 2018 - Geometria - Morgen

Note Title 5/24/2018

(1)



Idea 1: DRSB, DQHR ciclici

Idea 2: Angoli :)

Ts $\iff \widehat{DRQ} + \widehat{SRD} = \pi$

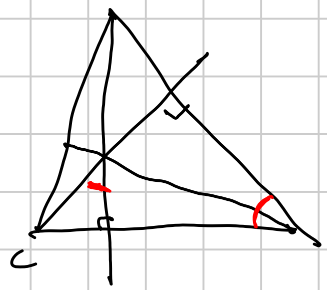
Q R S

R Q P

$\widehat{DRS} = \pi - \widehat{SBD}$

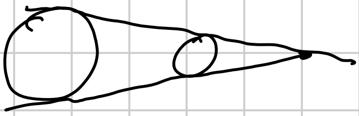
$\boxed{\widehat{SBD} = \widehat{DRQ}}$

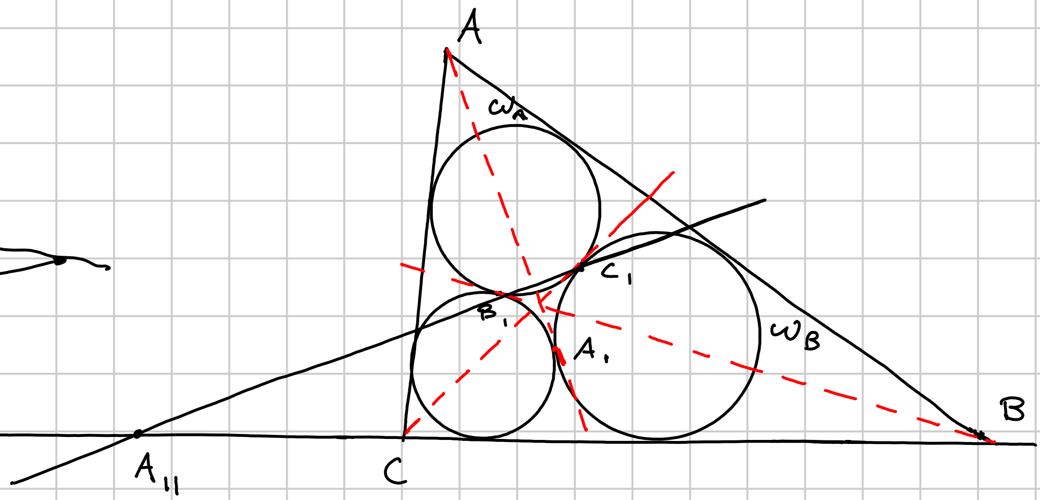
$\widehat{DRQ} = \widehat{DHQ} = \widehat{DHC}$



G3

AA_3

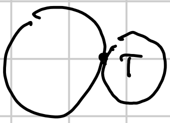




Desargues

$\triangle ABC$ $\quad \triangle A_1B_1C_1$

AA_1, BB_1, CC_1 concorrono $\iff$ $A_{11} = BC \cap B_1C_1$, B_{11}, C_{11} sono allineati



Monge su $\omega_A, \omega_B, \omega_C \Rightarrow B_1\ C_1$ c. sim int

c. d. sim esterno $\in BC$, $\in B_1C_1$

$\Rightarrow A_{11}$ c. d. sim ext di $\omega_B\ \omega_C$

B_{11} $\qquad \omega_A\ \omega_C$

C_{11} $\qquad \omega_A\ \omega_B$

Monge $\Rightarrow$ Tesi.

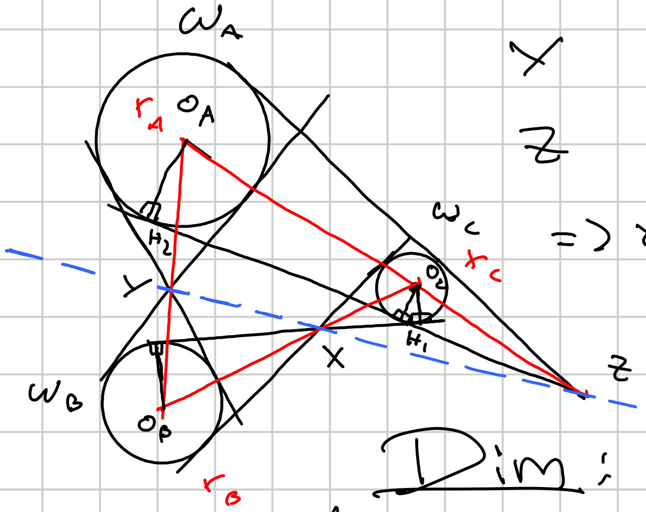
Monge

X c. sim. INT $\omega_B\ \omega_C$

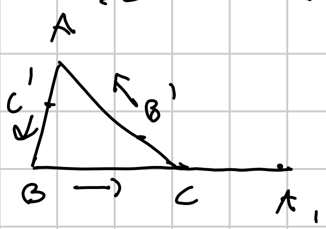
Y INT $\omega_A\ \omega_B$

Z EST $\omega_A\ \omega_C$

$\Rightarrow X, Y, Z$ sono allineati



Dim: $\triangle\ O_A O_B O_C$ e Menelao



$$\frac{\overline{AC'}}{\overline{C'B}} \cdot \frac{\overline{BA'}}{\overline{A'C}} \cdot \frac{\overline{CB'}}{\overline{B'A}} = -1$$

$$\boxed{AC' = -C'A}$$

$$\frac{ZO_C}{ZO_A}$$

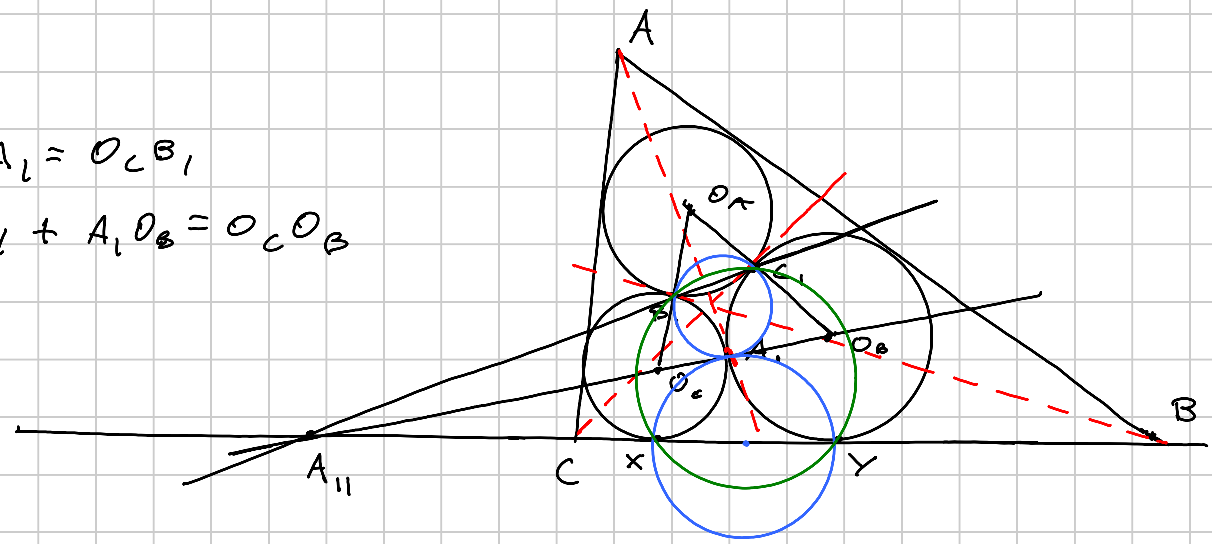
$$\boxed{\triangle ZO_CH_1 \sim \triangle ZO_AH_2}$$

$$\frac{ZO_C}{ZO_A} = \frac{r_C}{r_A}$$

$$\frac{ZO_C}{ZO_A} \cdot \frac{O_AY}{YO_B} \cdot \frac{O_BX}{XO_C} = \frac{\cancel{r_C}}{\cancel{r_A}} \cdot \frac{-\cancel{r_A}}{\cancel{r_B}} \cdot \frac{\cancel{r_B}}{\cancel{r_C}} = -1$$

$O_C A_1 = O_C B_1$

$O_C A_1 + A_1 O_B = O_C O_B$



A_{11}, B_{11} C_{11} allineati

<u>Desargues</u> $\triangle ABC$ $\triangle O_A O_B O_C$ AO_A, BO_B, CO_C concorrono

$O_A O_B \wedge AB := A_{111}$, B_{111}, C_{111} sono allineati

<u>Claimone</u>: $A_{11} = A_{111}$

$\Longleftrightarrow$ BC, $O_B O_C$, $B_1 C_1$ concorrono

(I) $\odot(A_1 B_1 C_1)$ e' l'inscritta di $\triangle O_A O_B O_C$

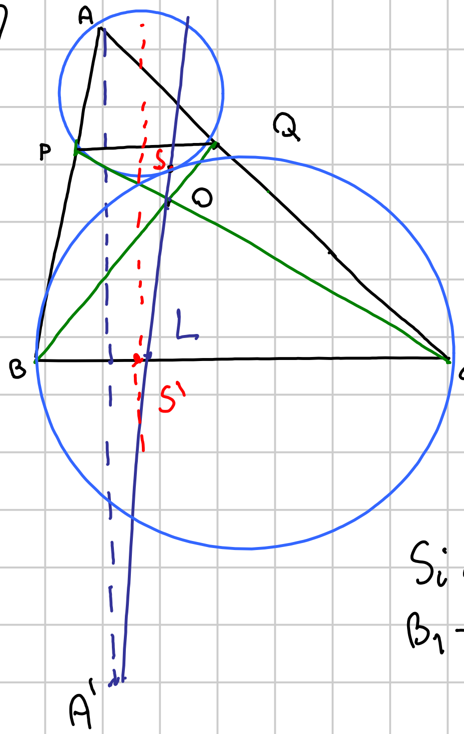
(II) $\odot(X A_1 Y)$ tange $O_B O_C$ in A_1

Di questi due cerchi $O_B O_C$ asse radicale

<u>Claimone finale</u> <u>$B_1 C_1 Y Z$ ciclici</u>

<u>Dim</u> Angoli su ω_B e ω_C

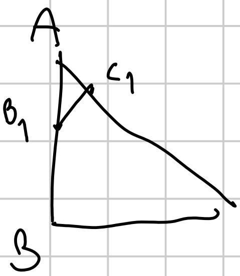
4



Tesi: $\bigcirc_{APQS}$ è tangente (in S) alla $\bigcirc_{BCS}$

Se inverto in A di raggio $AP \cdot AC$

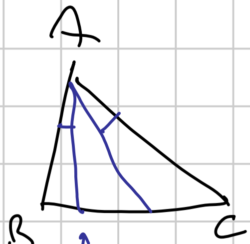
Dato tale $\frac{AP}{AB} = \frac{AQ}{AC} \Rightarrow AP \cdot AC = AQ \cdot AB$



$B \to B_1$ con $AB_1 = AQ$

$C \to C_1$ con $AC_1 = AP$

$\Rightarrow \Delta AB_1C_1 \cong \Delta APQ$



simmetrizzo

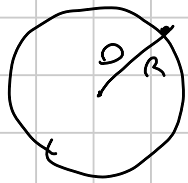
Simmetria rispetto alla bisettrice di $\widehat{BAC}$

$B_1 \to B'$ su AC con $AB' = AQ \Rightarrow B' = Q$

INVERSIONE di centro O e raggio R.

$O \to O$ $\quad P' \in OP$

$P \to P'$ $\quad OP \cdot OP' = R^2$



$\omega \to \omega$

dentro $\to$ fuori

fuori $\to$ dentro

Proprietà inversione:

1) Manda rette per O in rette per O
2) Manda rette NON passanti per O in circonferenze passanti per O (e viceversa)
3) Manda circonferenze NON passanti per O in altre circonferenze
4) Conserva gli angoli tra le curve

—o—o—

Applico Inversione + Simmetria!

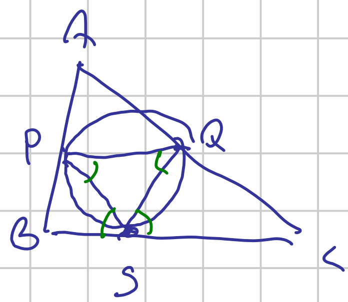
$B \leftrightarrow Q$ $\quad C \leftrightarrow P$ $\quad BC \Leftrightarrow \bigcirc_{APQ}$ $\quad S \to S'$ $\quad \bigcirc_{BSC} \Leftrightarrow \bigcirc_{QS'P}$

Tesi $\Leftrightarrow$ BC tange $\bigcirc_{QS'P}$

Oss $\quad S \in \bigcirc_{APQ} \Rightarrow S' \in BC$

Se la tesi è vera, $S' \in BC$ $\quad BC$ tange $\bigcirc_{QS'P}$

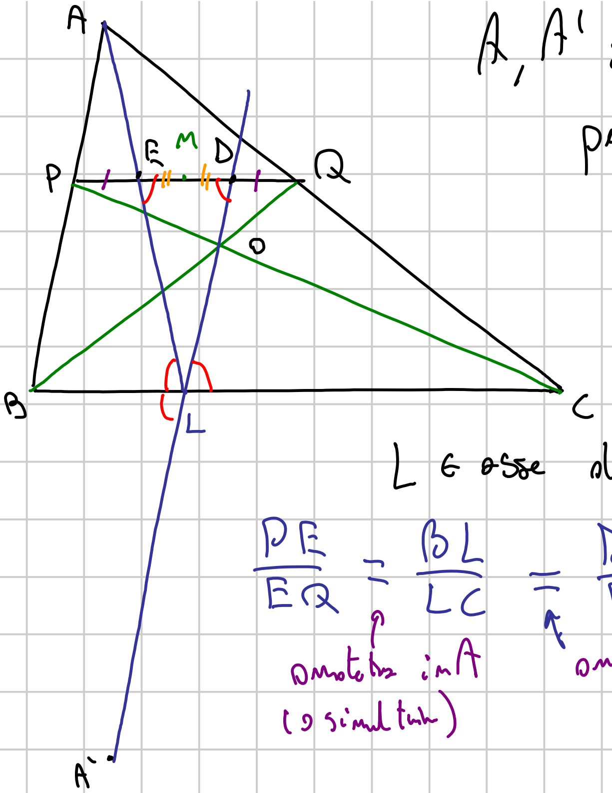
$\Rightarrow \angle PSB \underset{\text{tangenza}}{=} \angle PQS \underset{\text{parallele}}{=} \angle QSC \underset{\text{tangen.}}{=} \angle QPS$



Se la tesi è vera $\Rightarrow$ $\Delta PQS'$ è isoscele $\to$ $S' \in$ asse di PQ

CLAIM: $S' \overset{?}{=} A'O \cap BC$

Provo a dimostrare che $L = BC \cap A'O$, $L \in$ asse di PQ



A, A' simmetrici $\angle ALB = \angle BLA'$

per il parallelismo

$\angle DEL = \angle EDL \Rightarrow LED$ è isoscele

M pt medio di ED, $ML \perp ED$

$\Rightarrow L \in$ asse di ED

$L \in$ asse di $PQ \iff PE = DQ$

$$\frac{PE}{EQ} = \frac{BL}{LC} = \frac{DQ}{PD}$$

omotetia in A (o similitudine) — omotetia in D

$PE + EQ = PD + DQ = PQ$

$x = PE \Rightarrow EQ = PQ - x$ $\qquad y = DQ$

$$\frac{PE}{EQ} = \frac{x}{PQ-x} = \frac{y}{PQ-y}$$

conti $\Rightarrow$ equazione di 1° grado $\Rightarrow x = y$
$PE = DQ$

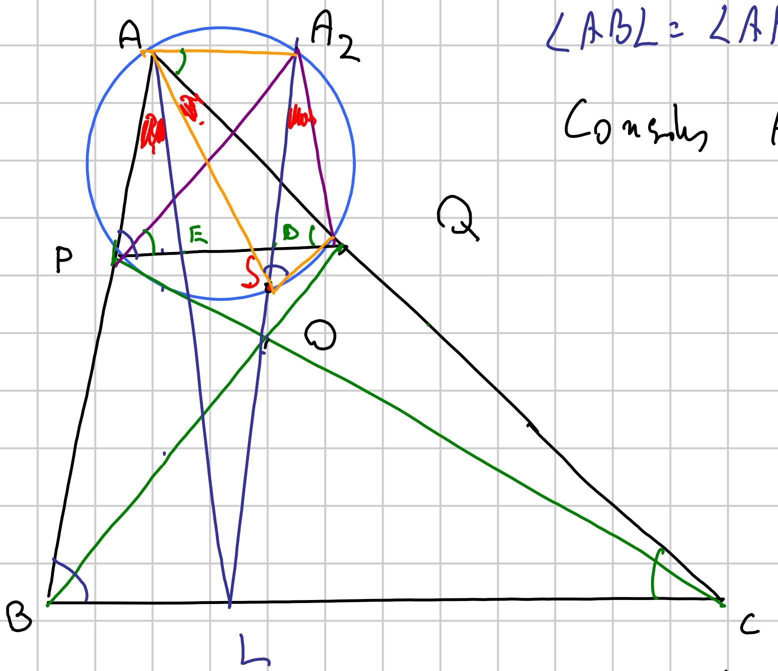
Dunque $PM = PE + EM = DQ + DM = QM \Rightarrow M$ punto medio di $PQ \Rightarrow L \in$ asse di PQ

Tesi $\iff S' = L$

Per costr. $S', L \in BC$ $\qquad S' = L \iff \angle BAS' = \angle BAL$

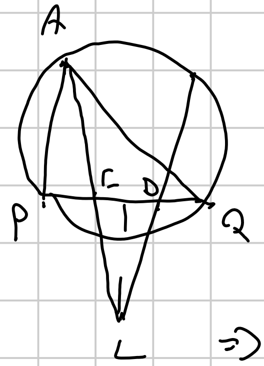
Dato che bisr + sym $S \to S'$ ho $\angle BAS' = \angle SAC$

Per la tesi basta $\angle SAC = \angle BAL$



$\angle ABL = \angle APQ = \angle ASQ$

Considero $A_2 = LS \cap \Omega_{APQ}$



LE, LD simmetrici rispetto all'asse di PQ

A e A_2 simmetrici rispetto asse PQ

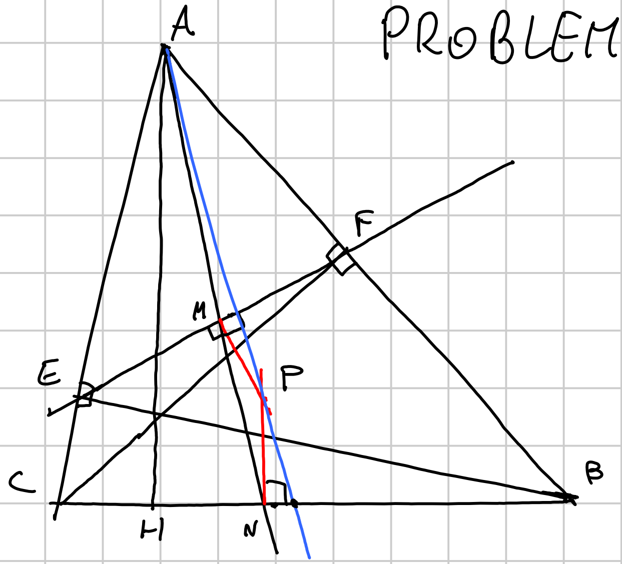
$\Rightarrow AA_2 \parallel PQ$

$\Rightarrow \angle A_2AQ = \angle AQP = \angle ACL = \angle A_2PQ$

$\triangle APL$ e $\triangle A_2QL$ sono congruenti ($L \in$ asse di AA_2, PQ)

$\angle PAL = \angle LA_2Q = \angle SA_2Q = \underset{\text{ciclicità di } APSQ}{\angle SAQ} \Rightarrow \angle BAL = \angle SAQ \Rightarrow$ Tesi

PROBLEMA 2



Coordinate Baricentriche su ABC.

$E=(S_C, 0, S_A)$ $\quad F=(S_B, S_A, 0)$ $\qquad S_A = \dfrac{b^2+c^2-a^2}{2}$

$EF: -xS_A + yS_B + zS_C = 0$

$BC: x=0$

$AMN: zb = yc$

$N=(0,b,c)$

Ultimo fatto noto: la mediana è $y=z$.

$AM \cap EF$

$-xbS_A + ybS_B + ycS_C = 0$

$M=(bS_B+cS_C, bS_A, cS_A)$

$H=(0, S_C, S_B)$

$AH: yS_B = zS_C$ $\qquad AH_\infty = (-a^2, S_C, S_B)$

$$NP: \det\begin{pmatrix} x & y & z \\ 0 & b & c \\ -a^2 & S_C & S_B \end{pmatrix} = 0$$

$NP: x(bS_B - cS_C) - ya^2c + za^2b = 0$

$t_A: zb^2 + yc^2 = 0$

$$\det\begin{pmatrix} 0 & c^2 & b^2 \\ -S_A & S_B & S_C \\ 1 & 1 & 1 \end{pmatrix} \overset{?}{=} 0$$

$\Leftrightarrow c^2S_C - b^2S_A + c^2S_A - b^2S_B \overset{?}{=} 0$ $\qquad S_A+S_C=b^2 \quad S_A+S_B=c^2$ ✓

$\Rightarrow EF \parallel t_A \rightsquigarrow MP \parallel AO$

$O=(a^2S_A, b^2S_B, c^2S_C) \rightsquigarrow AO_\infty = (-b^2S_B - c^2S_C, b^2S_B, c^2S_C)$

$$\mathcal{M}P: \quad \det\begin{pmatrix} x & y & z \\ bS_B+cS_C & bS_A & cS_A \\ -b^2S_B-c^2S_C & b^2S_B & c^2S_C \end{pmatrix}=0$$

$$xbcS_A[cS_C-bS_B]-y[\underbrace{b^2cS_AS_B+c^3S_AS_C+bc^2S_BS_C+c^3S_C^2}]$$
$$+z\,(\text{roba simmetrica})=0$$

$$=b^2c^3S_C$$

$$=bc\left(bS_AS_B+bc^2S_C+cS_BS_C\right)$$

$$\mathcal{M}P: 0=xS_A(cS_C-bS_B)-y(bS_AS_B+bc^2S_C+cS_BS_C)+z(cS_AS_C+b^2cS_B+bS_BS_C)$$

Test iff:
$$\det\begin{pmatrix} S_A(cS_C-bS_B) & -(\quad) & (\quad) \\ 0 & 1 & -1 \\ bS_B-cS_C & -a^2c & a^2b \end{pmatrix}\overset{!}{=}0$$

Mi basta:
$$\det\begin{pmatrix} S_A & -(\quad) & (\quad) \\ 0 & 1 & -1 \\ -1 & -a^2c & a^2b \end{pmatrix}\overset{?}{=}0$$

$$a^2bS_A-(\quad)_y-a^2cS_A+(\quad)_z\overset{?}{=}0$$

$$a^2S_A(b-c)+(b-c)(S_BS_C)+cS_AS_C+cS_B(S_A+S_C)-\left(bS_AS_B+bS_C(S_A+S_B)\right)\overset{!}{=}0$$

$$(b-c)\left[a^2S_A+\cancel{S_BS_C}-S_AS_C-S_AS_B-\cancel{S_BS_C}\right]\overset{?}{=}0$$

$$S_B+S_C=a^2 \implies \text{fine!}$$