PREMIO $\quad a_n$ limitata

Note Title 21/05/2019

A1. $\quad a_n < \sum_{k=n}^{2n+2018} \frac{a_k}{k+1} + \frac{1}{2n+2019} \qquad (*)$

Dietro: $\quad \sum_{k=a}^{b} \frac{1}{k(k+1)} = \sum_{k=a}^{b} \frac{1}{k} - \frac{1}{k+1} = \frac{1}{a} - \frac{1}{b+1}$

(serie di Mengoli)

Se potessi utilizzare la tesi:

$$a_n < \sum_{k=n}^{2n+2018} \frac{1}{k(k+1)} + \frac{1}{2n+2019} = \frac{1}{n} - \cancel{\frac{1}{2n+2019}} + \cancel{\frac{1}{2n+2019}}$$

Ma l'induzione non funziona così...

Idea: se $a_n > \frac{1}{n}$, ne trovo uno più avanti a_m tale che $a_m > \frac{1}{m}$ usando $(*)$, e spero che divergano a ∞

$$a_n - \frac{1}{n} = \varepsilon \geq 0 \qquad na_n \geq 1+\varepsilon$$

Ipotesi: $\quad na_n \geq 1+\varepsilon$ per un qualche n

Tesi: c'è un qualche a_m, $m \geq n$, con $ma_m \geq 1+\delta$

Definiamo $\quad \delta := \max_{m \in \{n, n+1, \dots 2n+2018\}} m \cdot a_m - 1$

cioè $\delta \leq m \cdot a_m - 1$ per tutti gli $m \in \dots$

$$\frac{1+\varepsilon}{n} < RHS \leq \sum_{k=n}^{2n+2018} \frac{1+\delta}{k(k+1)} + \frac{1}{2n+2019}$$

$$\frac{1+\varepsilon}{n} \leq (1+\delta)\left[\frac{1}{n} - \frac{1}{2n+2019}\right] + \frac{1}{2n+2019}$$

$$\frac{\varepsilon}{n} \leq \delta\left[\frac{1}{n} - \frac{1}{2n+2019}\right]$$

$$(2n+2019)\,\varepsilon \leq (2n+2019-n)\,\delta$$

$$\delta \geq \varepsilon \cdot \frac{2n+2019}{n+2019} > 1.5\,\varepsilon$$

se $n a_n \geq 1+\varepsilon$ per qualche n, allora

$m \cdot a_m \geq 1+\varepsilon \cdot \frac{2n+2019}{n+2019}$ per qualche $m \in \{n, \dots, 2n+2019\}$

Esiste n_0 t.c. $\frac{2n+2019}{n+2019} > 1.5$ per $n > n_0$?

Sì:
$$2n+2019 > 1.5n + 1.5 \cdot 2019$$
$$0.5n > 0.5 \cdot 2019$$
$$n > 2019$$

se n suff. grande, $2n+2019 < 3n$

$$\boxed{n \cdot a_n \geq 1+\varepsilon \Rightarrow m \cdot a_m \geq 1+1.5\,\varepsilon \text{ per qualche } m \leq 3n}$$

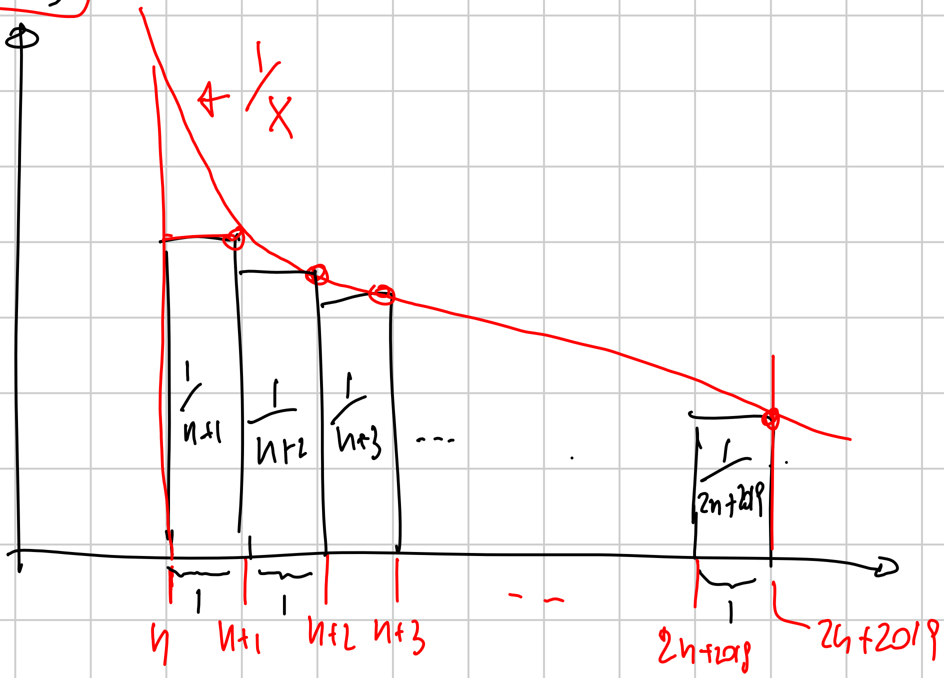
Induzione:

$m \cdot a_m \geq 1+(1.5)^k \varepsilon$ per qualche $m \leq 3^k n$

$a_n - \frac{1}{n} = \varepsilon \qquad \max_{m\in[n,n+1,\dots,2n+2018]} a_m - \frac{1}{m} =: \delta$

$$\cancel{\frac{1}{n}} + \varepsilon < RHS \le \sum_{k=n}^{2n+2018} \frac{1}{k+1}\left(\cancel{\frac{1}{k}} + \delta\right) + \cancel{\frac{1}{2n+2019}}$$

$$\varepsilon < \underbrace{\left(\sum_{k=n}^{2n+2018} \frac{1}{k+1}\right)}\cdot \delta \le$$



$\sum_{k=n}^{2n+2018} \frac{1}{k+1}$ = somma aree dei rettangolini

$$\le \int_n^{2n+2019} \frac{1}{x}\,dx$$

$$\Big[\ln x\Big]_n^{2n+2019} = \ln(2n+2019) - \ln(n)$$

$$= \underbrace{\ln(2n+2019) - \ln(2n)} + \underbrace{\ln(2n) - \ln(n)}_{=\ln(2) < 1}$$

$$= \ln\left(\frac{2n+2019}{2n}\right)$$

$$\downarrow$$

$\ln(1) = 0$ per $n \to \infty$

A2: $\sum_{cyc} \sqrt{\frac{a+b}{2} - ab} \ge \sqrt{2}$ $\qquad a+b+c=2$

IDEA 1 : omogeneità

$$\text{RADICALE} = \frac{2(a+b)-4ab}{4} = \frac{(a+b+c)(a+b)-4ab}{4} =$$

$$= \frac{a^2+2ab+b^2+c(a+b)-4ab}{4} = \frac{c(a+b)}{4} + \underbrace{\frac{(a-b)^2}{4}}_{\Delta_{ab}}$$

$$\sum_{cyc}\sqrt{\frac{a+b}{2}-ab} = \sum_{cyc}\sqrt{\frac{c(a+b)}{4}+\frac{(a-b)^2}{4}} \geq \sum_{cyc}\sqrt{\frac{c(a+b)}{4}} \overset{\ddot{\frown}}{\leq} \sqrt{2}$$

IDEA 2 : eleva al quadrato

$$\sum_{c}\cancel{\frac{a+b}{2}} - ab + \sum_{cyc} 2\sqrt{\frac{a+b}{2}-ab}\sqrt{\frac{b+c}{2}-bc} \overset{?}{\geq} \cancel{2}$$

DISUG. EQUIVALENTE A

$$2\sum_{cyc}\sqrt{\frac{c(a+b)}{4}+\frac{(a-b)^2}{4}}\sqrt{\frac{a(b+c)}{4}+\frac{(b-c)^2}{4}} \geq \sum_{cyc} ab$$

$$\sum_{cyc}\sqrt{c(a+b)+(a-b)^2}\sqrt{a(b+c)+(b-c)^2} \overset{?}{\geq} 2\sum_{cyc} ab \qquad (*)$$

HOPE: non è che basta

$$\sum_{cyc}\sqrt{c(a+b)a(b+c)} \overset{\text{HOPE}}{\geq} 2\sum_{cyc} ab$$

$$\sum_{cyc}(a-b)^2 = [2,0,0]-[1,1,0]$$

$$\sqrt{(a+b)(b+c)} = \sqrt{b^2+(a+c)b+ac} \geq \sqrt{b^2+2\sqrt{ac}\,b+ac} = \sqrt{(b+\sqrt{ac})^2}$$

$$= b+\sqrt{ac}$$

$$\sum_{cyc} \sqrt{c(a+b)a(b+c)} \geq \sum_{cyc} \sqrt{ac}\left(b+\sqrt{ac}\right) \overset{\text{HOPE}}{\geq} 2\sum_{cyc} ab$$

$$\left[1,\tfrac{1}{2},\tfrac{1}{2}\right]+[1,1,0] \overset{\text{HOPE}}{\cancel{\geq}} 2[1,1,0] \quad \text{no!}$$

(VAIN)

$$\sqrt{\underset{c(a+b)}{(M}+\underset{(a-b)^2}{\varepsilon)}\underset{a(b+c)}{(N}+\underset{(b-c)^2}{\delta)}} \geq \sqrt{MN} + \sqrt{\varepsilon\delta}$$

$$(M+\varepsilon)(N+\delta) = MN + \underbrace{\varepsilon N + \delta M}_{\text{AM-GM}} + \varepsilon\delta \geq MN + 2\sqrt{\varepsilon\delta MN} + \varepsilon\delta$$

$$= \left(\sqrt{MN}+\sqrt{\varepsilon\delta}\right)^2$$

(*) diventa $\displaystyle\sum_{cyc} \sqrt{c(a+b)a(b+c)} + |a-b||b-c| \overset{\text{HOPE}}{\geq} 2\sum_{cyc} ab$ (**)

LHS di (**) $\displaystyle\geq \sum_{cyc} \sqrt{ac}\left(b+\sqrt{ac}\right) + b^2 + ac - ab - bc \overset{\text{HOPE}}{\geq} 2\sum_{cyc} ab$

$$[2,0,0]+\left[1,\tfrac{1}{2},\tfrac{1}{2}\right] \geq 2[1,1,0]$$

Schur ✓

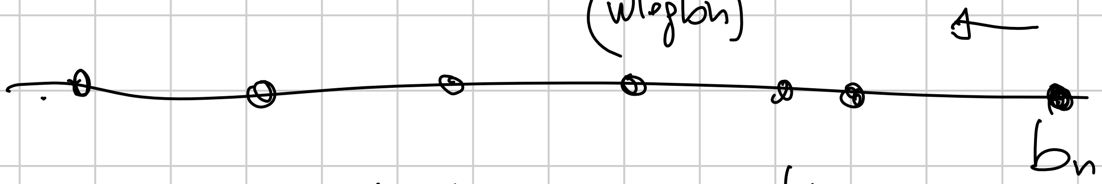
A4:

$$\sum_{i<j} |a_i-a_j| + |b_i-b_j| \leq \sum_{i,j} |a_i-b_j|$$

wlog $a_1 \le a_2 \le a_3 \le \dots \le a_n$

$\ b_1 \le b_2 \le \dots \le b_n$

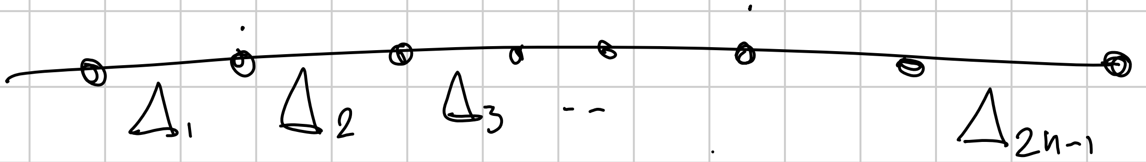
Idea 1: se sposto a sx l'ultimo valore, la disuguaglianza diventa più stretta (wlog b_n)



perché ci sono n intervalli del tipo $|b_n - \text{roba}|$ al RHS e $n-1$ al LHS

Allora, posso assumere che $b_n = \max(a_n, b_{n-1})$

Versione migliore: prendo i $2n$ punti in ordine



e chiamo $\Delta_1, \dots, \Delta_{2n-1}$ le differenze tra uno e il successivo

Ognuno dei termini al RHS e al LHS, $|\text{roba} - \text{roba}|$, è la somma di un po' dei Δ_i

"Double-counting": quante volte compare Δ_i?

se

h dei termini a_i stanno a sx — $n-h$ a dx

k dei termini b_i stanno a sx — $n-k$ a dx

allora Δ_i compare nella somma quante volte:

in $\sum_{i<j} |a_i - a_j|$ compare $h(n-h)$ volte

in $\sum_{i<j} |b_i - b_j|$ compare $k(n-k)$ volte

in $\sum_{ij} |a_i - b_j|$ compare $k(n-h)+h(n-k)$

$$h(\cancel{n}-h)+k(\cancel{n}-k) \le k(\cancel{n}-h)+h(\cancel{n}-k)$$

$$-h^2-k^2 \le -kh-hk \qquad \checkmark$$

Caso di uguaglianza: quando $h=k$ oppure $|\Delta_i|=0$, cioè quando coincidono a gruppetti della stessa dimensione

$a_1=a_2=\dots=a_{k_1}= \dots$ $\quad$ $a_{k_1+1}=a_{k_1+2}=\dots a_{k_2}$ $\quad$ ---

$=b_1=b_2=\dots b_{k_1}$ $\quad$ $=b_{k_1+1}=b_{k_1+2}=\dots=b_{k_2}$

$\square$

$a_1=b_1$ $\quad$ $a_2=b_2$ $\quad | \quad$ $a_1=b_1=a_2=b_2$.

$\boxed{A3}$ $f : \mathbb{R}^+ \to \mathbb{R}^+$ $\quad f(x+y) \geq y\, \underbrace{f(f(x))}$

" $f(y) \geq$ retta in y "

Impongo le f uguali: voglio $f(x+y) = f(f(x))$

mi basta $x+y = f(x)$

ovvero $y = f(x) - x$.

Formalmente: se $f(x) > x$ pongo $y = f(x) - x$ e

ottengo $f(f(x)) \geq (f(x) - x)\, f(f(x))$

da cui $1 \geq f(x) - x$ $\quad f(x) \leq x+1$.

Quindi ho mostrato che $\forall x$ $f(x) \leq x+1$.

$f(x+y) \geq y\, f(f(x))$

$x = 1 \Rightarrow$ $f(y+1) \geq y \cdot a$ $\quad a = f(f(1))$

per $z > 1$ $\quad f(z) \geq (z-1)a$

$(z = y+1)$

Se prendo z tale che $(z-1)a > 1$

ho anche $f(z) > 1$

da cui $f(f(z)) \geq (f(z)-1)a \geq ((z-1)a-1)a$

Prendo z tale che $f(f(z)) > 1$.

$$f(y+z) \geq y \, f(f(z))$$

$$\wedge \\ y+z+1$$

ASSURDO