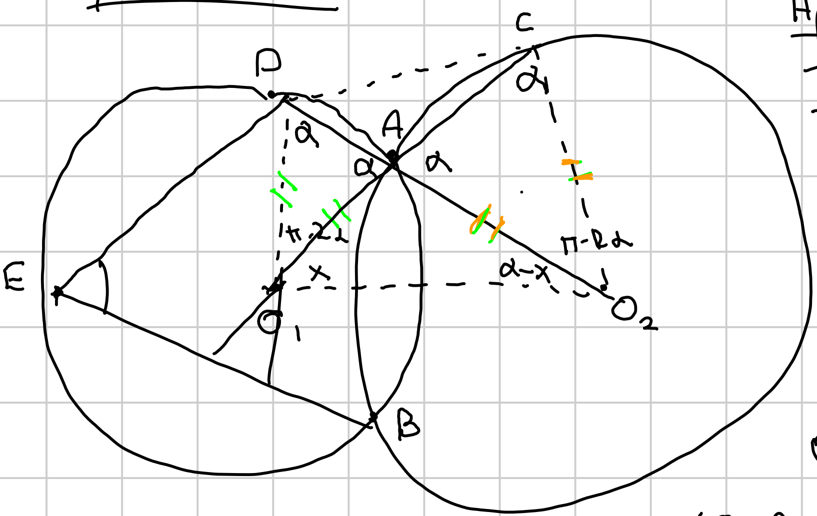
Pre-IMO Pomeriggio 2019 Geometria

Note Title 22/05/2019

Problema 1



Hp. $DE \parallel O_1A$, $BE \parallel O_2A$

Th. $\angle DCO_2 = \frac{\pi}{2}$

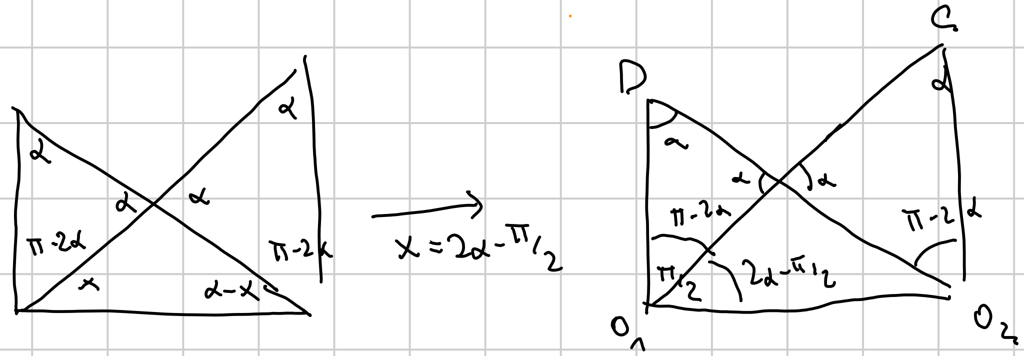
Sol.

[Idea] Voglio scrivere bene gli angoli nel quadrilatero DO_1O_2C, dove sappiamo $O_1A = O_1D$, $O_2A = O_2C$. Chiamo $\alpha \doteq \angle DAO_1$. $\angle DO_1A = \pi - 2\alpha = \angle CO_2A$, se chiamo $\angle AO_1O_2 \doteq x$, allora $\angle AO_2O_1 = \alpha - x$. Quale informazione in più mi dà l'ipotesi?

Sol. Hp. mi dice $\angle DEB = \alpha \Rightarrow \angle DO_1B = 2\alpha$. Siccome

$\angle DO_1B = \angle DO_1O_2 + \angle O_2O_1B = (\pi - 2\alpha + x) + x = \pi - 2\alpha + 2x$

allora $\pi - 2\alpha + 2x = 2\alpha \Rightarrow x = 2\alpha - \pi/2$

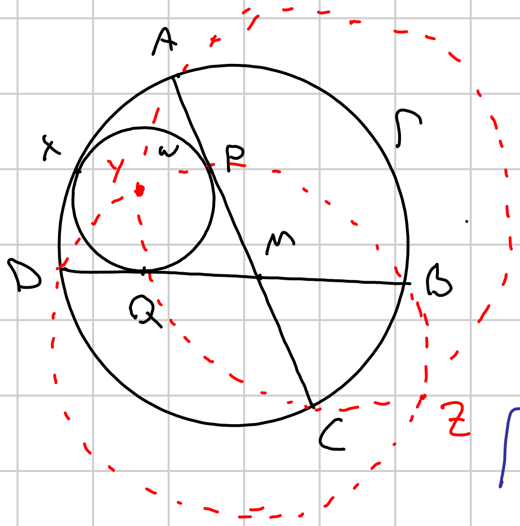


Dunque $\angle DO_1O_2 = \pi - 2\alpha + x = \pi - 2\alpha + 2\alpha - \pi/2 = \pi/2$

Inoltre $\angle DO_1C = \angle DO_2C \; (= \pi - 2\alpha) \Rightarrow DO_1O_2C$ ciclico

$\Rightarrow \angle DCO_2 = \pi - \angle DO_1O_2 = \pi - \pi/2 = \pi/2$ ∎

Problema 2



$\bigcirc_{ABC} = \omega_1$

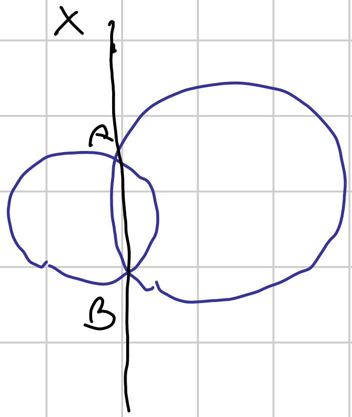
$\bigcirc_{BPD} = \omega_2$

$\omega_1 \cap \omega_2 = Y, Z$

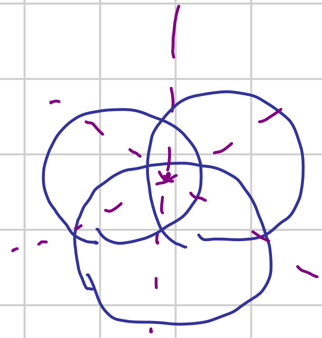
Tesi: $X \in YZ$

Reminder 1:

$X \in$ asse rad. $\iff$ $Pow_{\Gamma_1}(X) = Pow_{\Gamma_2}(X)$



$XA \cdot XB = Pow$



Th. su Γ

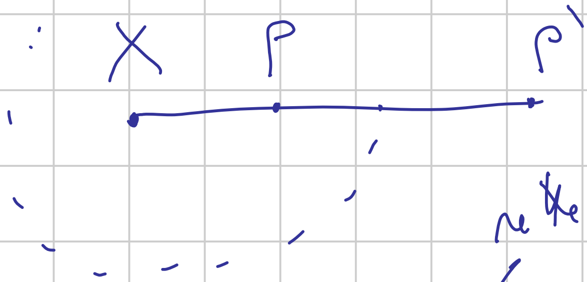
$MC \cdot MA = MB \cdot MD$

$Pow_{\omega_1} M = Pow_{\omega_2} M$

$\Rightarrow M \in YZ$

Inversione:

raggio r



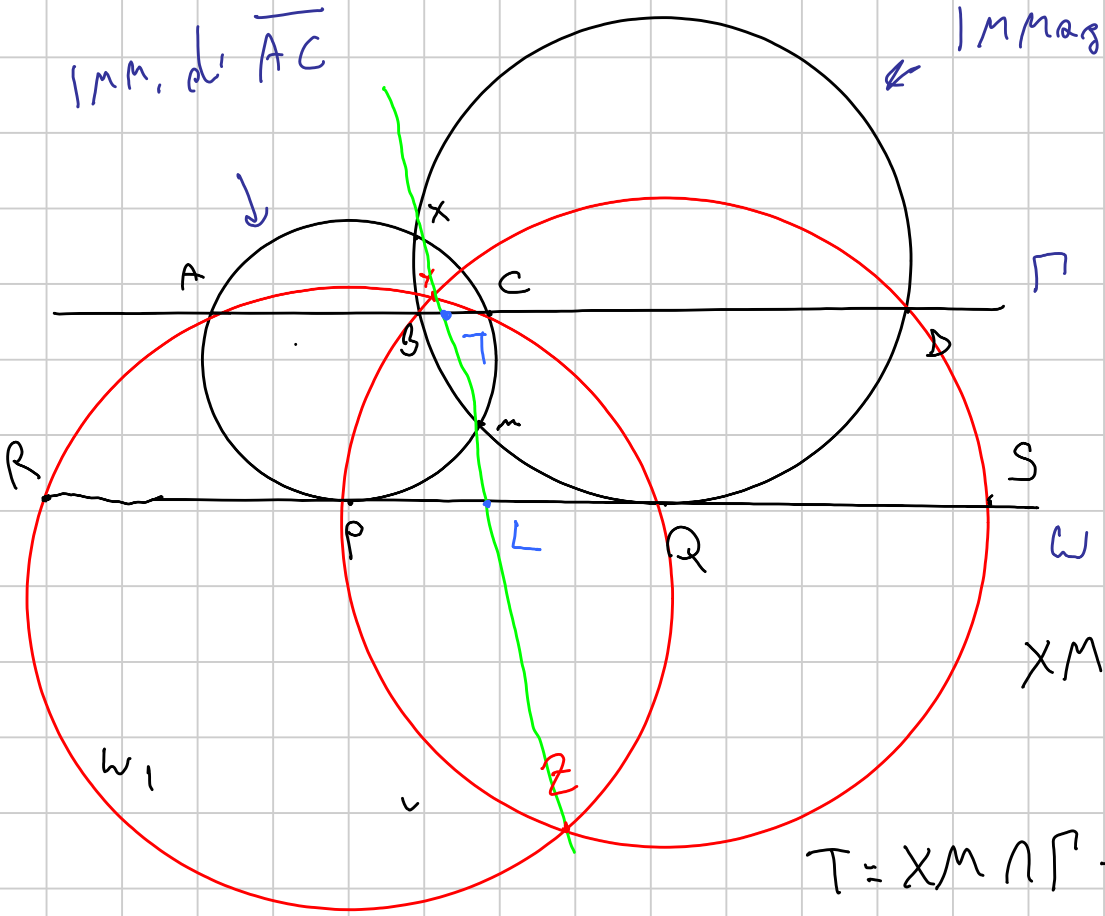
$XP \cdot XP' = r^2$

rette per X → rette per X

/ non per X → ○ per X

○ per X → / non per X

○ non per X → ○ non per X



Tesi X Y Z

Oss M sta su questa retta

YZ = asse di ω_1 e ω_2

XM è asse di O_{APC} e di O_{BDQ}

$T = XM \cap \Gamma = XM \cap \overline{ABCD}$

$Pow_{O_{ACP}}(T) = TX \cdot TM = TC \cdot TA = pow_{\omega_1}(T)$

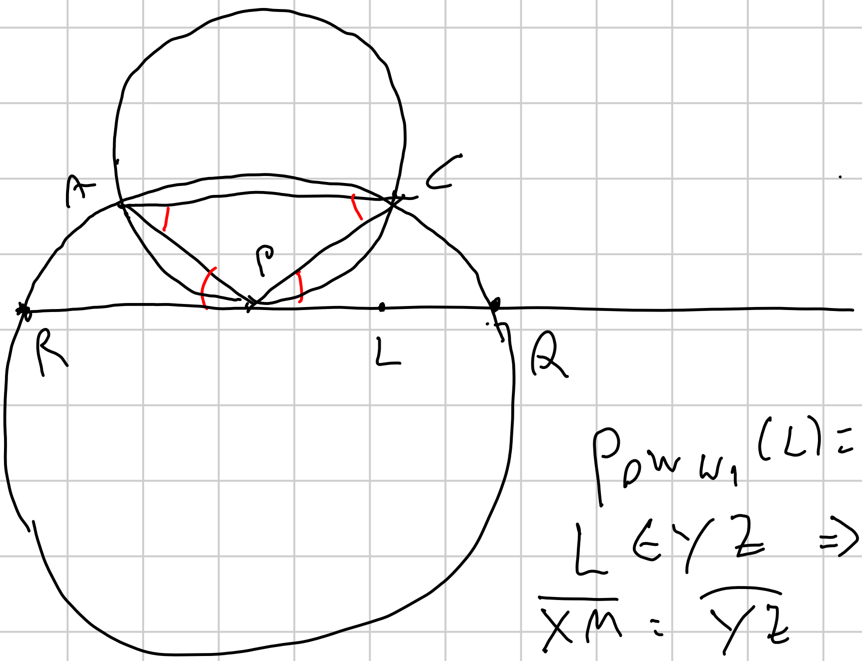
$\qquad\qquad\qquad\qquad\qquad \parallel$

$Pow_{O_{BDQ}}(T) \qquad TB \cdot TD = pow_{\omega_2}(T)$

T è centro radicale $\quad T \in XM, YZ$

$L = XM \cap PQ$. $\quad pow_{O_{ACP}}(L): LM \cdot LX = LP^2 = LQ^2$

$Pow_{\omega_1}(L) = LQ \cdot LR \qquad pow_{\omega_2}(L) = LP \cdot LS$



$RP = PQ$

$LP = LQ = \frac{1}{2} PQ$

ω. $LQ \cdot LR = LQ \cdot 3LQ$

$= 3LQ^2$

$Pow_{\omega_1}(L) = 3LP^2 = 3LQ^2 = Pow_{\omega_2}(L)$

$L \in YZ \Rightarrow T, L \in YZ$

$\overline{XM} = \overline{YZ} \Rightarrow X, Y, Z$ allineati

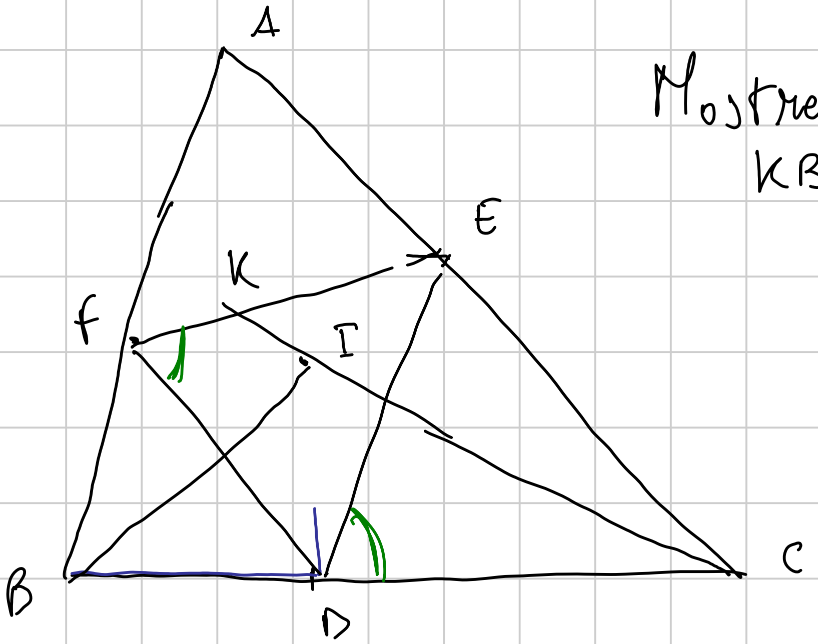
Problema 3

Lemma: $\triangle ABC$, D, E, F pti di tangenza dell'incerchio con i lati

I incentro di ABC

$K = CI \cap EF$

Tesi: $\angle BKC = 90°$



Mostreremo che KBIF ciclico

$\angle KIB = \beta/2 + \gamma/2$

$\angle BFK = \angle BFD + \angle DFE$

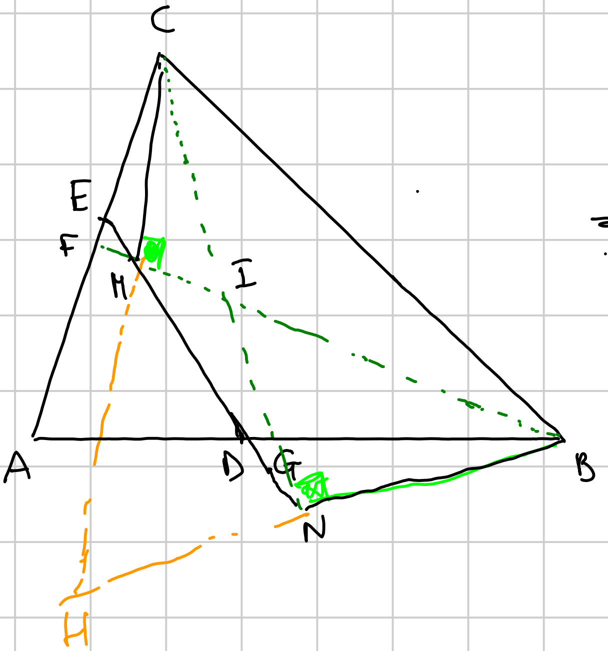
$= \frac{\pi}{2} - \beta/2 + \frac{\pi}{2} - \frac{\gamma}{2}$

$= \pi - \beta/2 - \gamma/2 = \pi - \angle KIB$

$\Rightarrow$ BDKF è ciclico

$\Rightarrow \angle BKC = 90°$

$\square$



Fatto: $\angle BNC = 90^\circ$

$\angle BMC = 90^\circ$

$\Rightarrow$ BNMC è ciclico

Allora $\Delta HNM \sim \Delta HBC$

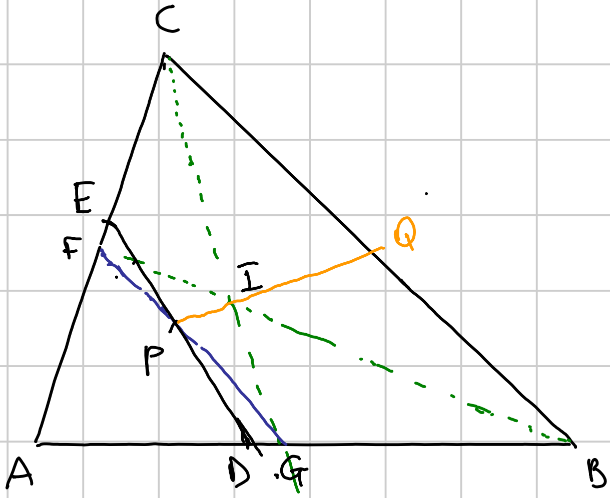
In particolare, $\dfrac{MN}{BC} = \dfrac{HM}{BH} = |\cos \angle BHC|$

$$\angle BHC = \pi - \angle HBC - \angle HCB$$
$$= \pi - \left(\frac{\pi}{2} - \angle ICB\right) - \left(\frac{\pi}{2} - \angle IBC\right)$$
$$= \angle ICB + \angle IBC$$
$$= \pi - \angle BIC$$
$$= \pi - \left(\frac{\pi}{2} + \frac{\alpha}{2}\right) \qquad \alpha = \angle BAC$$
$$= \frac{\pi}{2} - \frac{\alpha}{2}$$

$$\frac{MN}{BC} = \left|\cos\left(\frac{\pi}{2} - \frac{\alpha}{2}\right)\right| = \left|\sin \frac{\alpha}{2}\right|$$

$\Rightarrow 2MN = BC \iff \angle BAC = 60^\circ$

Dimostriamo $2IP = IQ \iff \angle BAC = 60°$



Baricentriche su $\triangle ABC$

Aside: Le coordinate baricentriche di P rispetto al triangolo $\triangle ABC$ sono

$$([PBC], [APC], [ABP])$$ o un suo multiplo

Coord esatte $\left(\frac{[PBC]}{[ABC]}, \ldots \right)$

$D = [s-a : s-b : 0]$

$E = [s-a : 0 : s-c]$

$F = [a, 0, c]$

$G = [a, b, 0]$

$s = \frac{1}{2}(a+b+c)$

Calcoliamo P!

$$FG: \quad \det \begin{pmatrix} x & y & z \\ a & 0 & c \\ a & b & 0 \end{pmatrix} = 0$$

$$\boxed{FG: \quad -bcx + acy + abz = 0}$$

$$DE: \quad \det \begin{pmatrix} x & y & z \\ s-a & s-b & 0 \\ s-a & 0 & s-c \end{pmatrix} = 0$$

$$\boxed{DE: \quad (s-b)(s-c)x - (s-a)(s-c)y - (s-a)(s-b)z = 0}$$

$$P = \left[\frac{a(s-a)}{bc}, \frac{(a-c)(s-b)}{c(b-c)}, \frac{(b-a)(s-c)}{b(b-c)} \right]$$

$$I = \left[\frac{a}{2s}; \frac{b}{2s}, \frac{c}{2s} \right]$$

$$Q = [0, \text{roba}_1, \text{roba}_2]$$

$$\vec{IP} = \left[\frac{a(s-a)}{bc} - \frac{a}{2s}, \dots, \dots \right]$$

$$\vec{IQ} = \left[-\frac{a}{2s}, \text{altra roba}_1, \text{altra roba}_2 \right]$$

$$\left|\frac{\overrightarrow{IP}}{\overrightarrow{IQ}}\right| = \left| \frac{\frac{a(s-a)}{bc} - \frac{a}{2s}}{-\frac{a}{2s}} \right|$$

$$= \left| \frac{\frac{2sa(s-a) - abc}{2sbc}}{-\frac{a}{2s}} \right|$$

$$= \left| \frac{4s(s-a) - 2bc}{2bc} \right|$$

$$= \left| \frac{4s^2 - 4sa - 2bc}{2bc} \right|$$

$$= \left| \frac{(a+b+c)^2 - 2a(a+b+c) - 2bc}{2bc} \right|$$

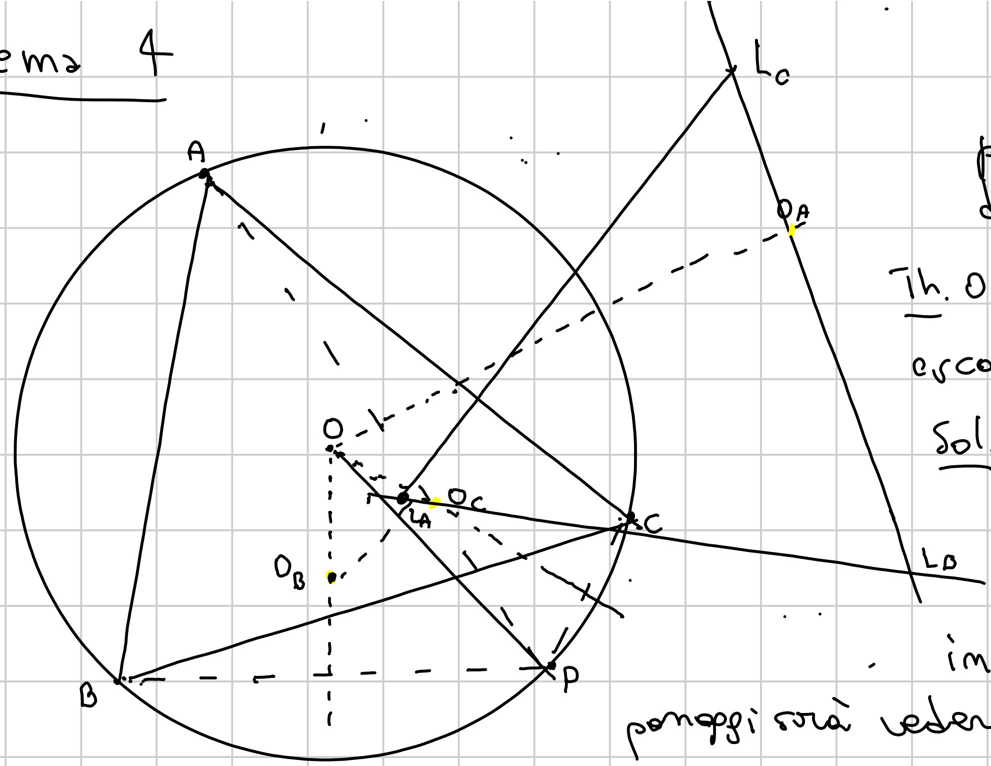
$$= \left| \frac{b^2 + c^2 - a^2}{2bc} \right|$$

$$= \left| \cos \angle BAC \right|$$

$$\frac{IP}{IQ} = \frac{1}{2} \iff \cos \angle BAC = \frac{1}{2}$$

$$\iff \angle BAC = 60^\circ$$

Problema 4

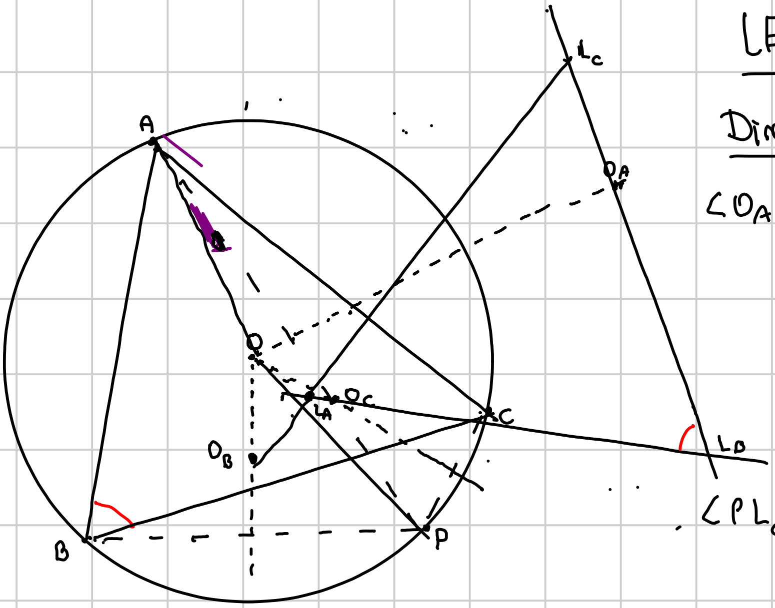


<u>Oss.</u> O_A, O_B, O_C allineati poichè sono sull'asse di OP

<u>Th.</u> OP tange la cfr circoscritta a $L_AL_BL_C$

<u>Sol.</u> [Idea] Disperata ma funziona: OP tange $\odot L_AL_BL_C$ in P, quindi uno dei passaggi sarà vedere la ciclicità $L_AL_BL_CP$.

<u>Remind</u> $\angle ABC = \angle(AB, BC)$ = quanto ruotare AB in senso antiorario per finire su BC. Ad esempio così vale ABCD ciclico $\iff$ $\angle ABC = \angle ADC$ e in generale sistema i problemi di conf. [e cicliche]



<u>LEMMA 1</u> P, O_A, O_C, L_B conciclici

<u>Dim.</u>

$$\angle O_APO_C = \angle O_AOO_C = \angle APC = \angle ABC = \angle O_AL_BO_C \quad \blacksquare$$

(O_AO_C asse OP; $OO_A \perp AP$, $OO_C \perp PC$; APBC ciclico; $O_AL_B \perp BC$, $O_CL_B \perp AB$)

<u>LEMMA 2</u> L_A, L_B, L_C, P conciclici

$$\angle PL_CL_A = \angle PL_CO_B \overset{\text{LEMMA 1}}{=} \angle PO_AO_B = \angle PO_AO_C \overset{\text{LEMMA 1}}{=} \angle PL_BO_C = \angle PL_BL_A \quad \blacksquare$$

(O_B, L_A, L_C allineati; O_A, O_B, O_C allineati; L_A, O_C, L_B allineati)

[Idea] Vogliamo per concludere $\angle OPL_A = \angle L_AL_CP$. Per giungere questi angoli...

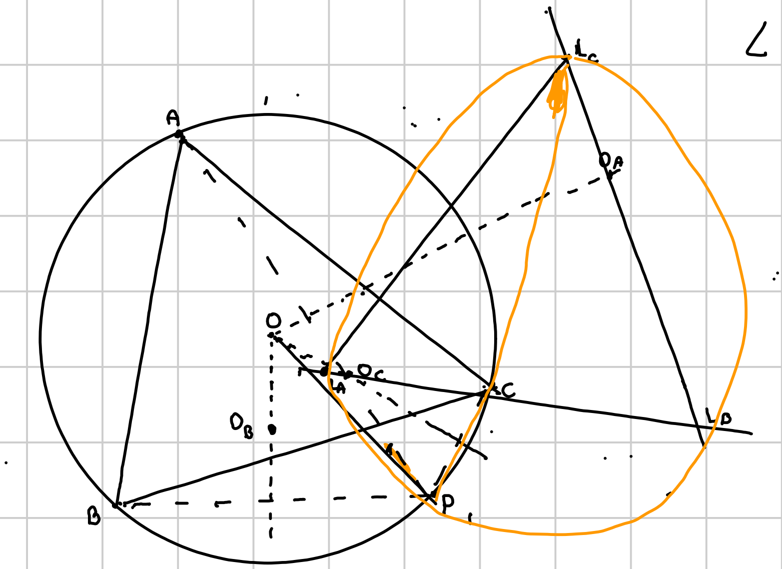
<u>LEMMA 3</u> [e cicliche] P, L_C, C allineati. Vogliamo $\angle PL_CL_A = \angle(PC, L_CL_A)$

Allora $\angle PL_CL_A = \angle PL_CO_B \overset{\text{LEMMA 1}}{=} \angle PO_AO_B = \frac{1}{2}\angle PO_AO = \angle PAO = \frac{\pi}{2} - \angle ACP$

(O_B, L_A, L_C all.; O_A, O_B, O_C asse di OP; Lavoro angoli)

$AC \perp L_AL_C \implies \angle(PC, L_AL_C)$

Conclusione



P, A, L_A allineati

$\angle OPL_A \overset{?}{=} \angle OPA = \angle OAP = \angle L_A L_C P$ ☒

<u>Idea</u>: Invertite in P...