Senior 2006 - Algebra 2 (F.C. 4 S.D.M.)

Titolo nota 14/09/2006

DISUGUAGLIANZA DI RIARRANGIAMENTO

$$a_1 \geq a_2 \geq a_3 \cdots \geq a_n$$

$$b_1 \geq b_2 \geq b_3 \cdots \geq b_n$$

$$\sum_{i=1}^{n} a_i b_i \geq \sum_{i=1}^{n} a_i b_{\sigma(i)} \geq \sum_{i=1}^{n} a_i b_{n-i+1}$$

Supponiamo per assurdo max non sia quando le n-uple sono ordinate nello stesso modo $(\forall\, i, j \quad a_i \geq a_j \Rightarrow b_i \geq b_j)$

(i, j) t.c. $a_i > a_j \quad b_i \leq b_j$

$$\underbrace{\sum_{k=1}^{n} a_k b_k}_{} \leq \underbrace{\sum_{k \neq i,j} a_k b_k}_{} + a_i b_j + a_j b_i$$

$$a_i b_i + a_j b_j \leq a_i b_j + a_j b_i$$

$$\geq 0 \qquad \geq 0$$

$$0 \leq (b_j - b_i) a_i + a_j (b_i - b_j) = (b_j - b_i)(a_i - a_j)$$

$$x^2 + y^2 + z^2 \geq xy + yz + zx$$

$\{x, y, z\}$ $\{x, y, z\}$ $x \geq y \geq z$

$\{y, z, x\}$

$$x \cdot x + y \cdot y + z \cdot z \geq x \cdot y + y \cdot z + z \cdot x$$

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq a + b + c$$

$\{a^2, b^2, c^2\}$ $\left\{\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right\}$

$$a^2 \geq b^2 \geq c^2 \qquad \frac{1}{a} \leq \frac{1}{b} \leq \frac{1}{c}$$

⚠ NO!!!

$$a^2 \cdot \frac{1}{b} + b^2 \cdot \frac{1}{c} + c^2 \cdot \frac{1}{a} \geq a^2 \cdot \frac{1}{a} + b^2 \cdot \frac{1}{b} + c^2 \cdot \frac{1}{c} = a + b + c$$

Chebycheff

$$a_1 \geq a_2 \geq a_3 \cdots \geq a_n$$

$$b_1 \geq b_2 \geq b_3 \cdots \geq b_n$$

$$(a_1 + a_2 + a_3 \cdots + a_n)(b_1 + b_2 + b_3 + \cdots b_n) \leq n(a_1 b_1 + a_2 b_2 + \cdots a_n b_n)$$

$$a_1 b_1 + a_2 b_2 + a_3 b_3 - \quad - \quad a_n b_n \leq a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

$$a_1 b_2 + a_2 b_3 + a_3 b_4 + \cdots \quad a_n b_1 \leq a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

$$a_1 b_3 + a_2 b_4 + \cdots \quad \cdots + a_n b_2 \leq$$

$$\vdots \qquad \vdots \qquad \leq \qquad \vdots$$

$$a_1 b_n + a_2 b_1 + \cdots \quad - + a_n b_{n-1} \leq a_1 b_1 + a_2 b_2 \cdots + a_n b_n$$

$$a_1 \sum b_i + a_2 \sum b_i + \cdots \quad - + a_n \sum b_i \leq n (a_1 b_1 + a_2 b_2 + \cdots \quad \cdots + a_n b_n)$$

$$\sum a_i \sum b_i \leq n \sum a_i b_i$$

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \frac{a_3}{a_4} + \cdots + \frac{a_n}{a_1} \geq n$$

$$\{a_1, a_2, a_3, \ldots a_n\} \qquad \left\{\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \ldots, \frac{1}{a_n}\right\}$$

$$a_1 \frac{1}{a_2} + a_2 \frac{1}{a_3} \cdots + a_n \frac{1}{a_1} \geq a_1 \cdot \frac{1}{a_1} + a_2 \frac{1}{a_2} + \cdots + a_n \frac{1}{a_n}$$

MEDIE

$$AM = \frac{x_1 + x_2 + \dots + x_n}{n}$$

$$GM = \sqrt[n]{x_1 x_2 x_3 \dots x_n}$$

$$AM \geq GM$$

$$\frac{x_1 + x_2}{2} \geq \sqrt{x_1 x_2}$$

$$x_1 - 2\sqrt{x_1 x_2} + x_2 \geq 0$$

$$\left(\sqrt{x_1} - \sqrt{x_2}\right)^2 \geq 0$$

SUPPONIAMO NEL MASSIMO DELLA MEDIA GEOMETRICA CI SIANO 2 ELEMENTI DIVERSI

$$\sqrt[n]{\left(\frac{x_1+x_2}{2}\right)^2 \cancel{x_3}\cancel{x_4}\cdots\cancel{x_n}} > \sqrt[n]{x_1 x_2 \cancel{x_3}\cdots\cancel{x_n}}$$

$$\left(\frac{x_1+x_2}{2}\right)^2 > x_1 x_2$$

$$x_1^2 - 2x_1x_2 + x_2^2 > 0$$

$$(x_1 - x_2)^2 > 0$$

NO!!

$$\boxed{x < 1}$$

1 $P_{2^n} \longrightarrow P_{2^{n+1}}$

$P_n \longrightarrow P_{n-1}$

$n=2$ GIÀ FATTO

Hp $\dfrac{x_1 + x_2 \dots + x_{2^n}}{2^n} \geq \sqrt[2^n]{x_1 x_2 \dots x_{2^n}}$

Th $\dfrac{x_1 + x_2 \dots + x_{2^{n+1}}}{2^{n+1}} \geq \sqrt[2^{n+1}]{x_1 x_2 \dots x_{2^{n+1}}}$

$$\frac{\overbrace{x_1+x_2 \dots + x_{2^m}} + \overbrace{x_{2^m+1} + \dots + x_{2^{m+1}}}}{2^{m+1}} \geq$$

$$\geq \frac{\sqrt[2^m]{x_1x_2 \dots x_{2^m}} + \sqrt[2^m]{x_{2^m+1} \dots x_{2^{m+1}}}}{2} \overset{?}{\geq} \sqrt[2^{m+1}]{x_1x_2 \dots x_{2^{m+1}}}$$

$P_n \rightarrow P_{n-1}$

I/p $\dfrac{x_1+x_2+\dots+x_n}{n} \geq \sqrt[n]{x_1x_2 \dots x_n}$

Th $\dfrac{x_1+x_2+\dots+x_{n-1}}{n-1} \geq \sqrt[n-1]{x_1x_2 \dots x_{n-1}}$

$$\frac{x_1 + x_2 \ldots + x_{n-1} + \left(\frac{x_1 + x_2 \ldots + x_{n-1}}{n-1}\right)}{n} \geq$$

$$\geq \sqrt[n]{x_1 x_2 x_3 \ldots x_{n-1}\left(\frac{x_1 + x_2 \ldots + x_{n-1}}{n-1}\right)}$$

$$\frac{x_1 + x_2 \ldots + x_{n-1}}{n-1} \geq \sqrt[n]{x_1 x_2 \ldots x_{n-1}}\,\sqrt[n]{\frac{x_1 + x_2 \ldots + x_{n-1}}{n-1}}$$

$$\boxed{\frac{1}{n} + \frac{1}{(n-1)n} = \frac{\cancel{n} - \cancel{1} + \cancel{1}}{\cancel{n}(n-1)} = \frac{1}{n-1}}$$

$$\left(\frac{x_1+x_2\ldots+x_{n-1}}{n-1}\right)^{\cancel{n}\; n-1} \geq x_1x_2\ldots x_{n-1}\cancel{\left(\frac{x_1+x_2\ldots+x_{n-1}}{n-1}\right)}$$

$$\frac{x_1+x_2+\ldots+x_{n-1}}{n-1} \geq \sqrt[n-1]{x_1x_2\ldots x_{n-1}}$$

$$a_1=\frac{x_1}{G} \quad a_2=\frac{x_1x_2}{G^2} \quad a_3=\frac{x_1x_2x_3}{G^3} \quad \ldots \quad a_n=\frac{x_1x_2\ldots x_n}{G^n}=1$$

$$\frac{a_2}{a_1}+\frac{a_3}{a_2}+\ldots+\frac{a_n}{a_{n-1}}+\frac{a_1}{a_n}\geq n$$

$$\frac{x_2}{G}+\frac{x_3}{G}+\ldots+\frac{x_1}{G}\geq n$$

$$\frac{x_1 + x_2 \ldots + x_n}{G} \geq n$$

$$\frac{x_1 + x_2 \ldots + x_n}{n} \geq G$$

$$(a+3b)(b+4c)(c+2a) \geq 60abc \qquad c \geq b \geq a \geq 0$$

$$\boxed{\frac{a+3b}{2} \geq \sqrt{3ab}}$$

$$\boxed{\frac{a+b+b+b}{4} \geq \sqrt[4]{ab^3}}$$

$$(a+3b)(b+4c)(c+2a) \geq 2\sqrt{3ab} \cdot 2 \cdot 2\sqrt{bc} \cdot 2 \cdot \sqrt{2ac}$$

$$(a+b+b+b)(b+c+c+c+c)(c+a+a) \geq 4\sqrt[4]{ab^3} \cdot 5\sqrt[5]{bc^4} \cdot 3\sqrt[3]{ca^2}$$

$$| \vee \longrightarrow ?$$

$$60abc$$

$$a^{\frac{1}{4}+\frac{2}{3}} \, b^{\frac{1}{5}+\frac{3}{4}} \, c^{\frac{4}{5}+\frac{1}{3}} \geq abc$$

$$c^8 \geq a^5 b^3$$

ALTRE MEDIE

$$QM = \sqrt{\frac{x_1^2 + x_2^2 \dots + x_n^2}{n}}$$

$$HM = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} \dots + \frac{1}{x_n}}$$

$$pM = \left(\frac{x_1^p + x_2^p + \dots + x_n^p}{n}\right)^{\frac{1}{p}}$$

$$\boxed{p \geq q} \qquad \boxed{pM \geq qM} \qquad \emptyset M = GM$$

$$\sqrt[3]{2(a+b)\left(\frac{1}{a}+\frac{1}{b}\right)} \geq \sqrt[3]{\frac{a}{b}}+\sqrt[3]{\frac{b}{a}}$$

$$\frac{\sqrt[3]{2(a+b)^2}}{\cancel{\sqrt[3]{ab}}} \geq \frac{\sqrt[3]{a^2}+\sqrt[3]{b^2}}{\cancel{\sqrt[3]{ab}}}$$

$$2(a+b)^2 \geq \left(\sqrt[3]{a^2}+\sqrt[3]{b^2}\right)^3$$

$$\left(\frac{a+b}{2}\right)^2 \geq \left(\frac{\sqrt[3]{a^2}+\sqrt[3]{b^2}}{2}\right)^3$$

$$\frac{a+b}{2} \geq \left(\frac{a^{\frac{2}{3}}+b^{\frac{2}{3}}}{2}\right)^{3/2}$$

ok

Disuguaglianza di Cauchy-Schwarz

$$\left(a_1^2+a_2^2+\dots a_n^2\right)\left(b_1^2+b_2^2+\dots b_n^2\right) \geq \left(a_1b_1+a_2b_2+\dots+a_nb_n\right)^2$$

$$P(x) = (a_1\overset{\overset{0}{\shortparallel}}{x}-b_1)^2+(a_2\overset{\overset{0}{\shortparallel}}{x}-b_2)^2+\dots+(a_n\overset{\overset{0}{\shortparallel}}{x}-b_n)^2$$

$$=\left(a_1^2+a_2^2+\dots a_n^2\right)x^2-2\left(a_1b_1+a_2b_2\dots+a_nb_n\right)x+\left(b_1^2+b_2^2+\dots b_n^2\right)$$

$$\frac{\Delta}{4}\leq 0 \qquad \left(a_1b_1+a_2b_2+\dots a_nb_n\right)^2-\left(a_1^2+a_2^2+\dots a_n^2\right)\left(b_1^2+\dots b_n^2\right)\leq 0$$

$$\underbrace{(a_1^2 + a_2^2 + \cdots a_n^2)}_{\shortparallel \atop A^2}\underbrace{(b_1^2 + b_2^2 + \cdots + b_n^2)}_{\shortparallel \atop B^2} \geq (a_1b_1 + a_2b_2 + \cdots a_nb_n)^2$$

$$(AB)^2 \geq (a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2$$

$$|AB| \geq |a_1b_1 + a_2b_2 + \cdots a_nb_n|$$

$$1 \geq \left| \underbrace{\frac{a_1}{A} \cdot \frac{b_1}{B}} + \frac{a_2}{A}\frac{b_2}{B} + \cdots + \frac{a_n}{A} \cdot \frac{b_n}{B} \right|$$

$$\frac{\left(\frac{a_1}{A}\right)^2 + \left(\frac{b_1}{B}\right)^2}{2} \geq \sqrt{\left(\frac{a_1}{A}\right)^2 \cdot \left(\frac{b_1}{B}\right)^2}$$

$$\underbrace{\underbrace{\frac{b_1^2}{B^2}+\frac{b_2^2}{B^2}+\frac{b_3^2}{B^2}+\dots\frac{b_n^2}{B^2}}_{\frac{b_1^2+b_2^2+\dots b_n^2}{B^2}=1}+\underbrace{\frac{a_1^2}{A^2}+\frac{a_2^2}{A^2}+\frac{a_3^2}{A^2}+\dots\frac{a_n^2}{A^2}}_{\frac{a_1^2+a_2^2+\dots a_n^2}{A^2}=1}}_{2} \geq RHS$$

$$1=\frac{1+1}{2} \geq \left|\frac{a_1b_1}{AB}+\frac{a_2b_2}{AB}+\dots\frac{a_nb_n}{AB}\right|$$

$$(x^2+y^2+z^2)^2 \geqslant (xy+yz+zx)^2$$

$$(x^2+y^2+z^2)(x^2+y^2+z^2) \geqslant (xy+yz+zx)^2$$

a_1	a_2	a_3	b_1	b_2	b_3
$\parallel$	$\parallel$	$\parallel$	$\parallel$	$\parallel$	$\parallel$
x	y	z	y	z	x

$$(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2) \geqslant (a_1b_1+a_2b_2+a_3b_3)^2$$

$$a_1 x - b_1 = 0 \qquad x = \frac{b_1}{a_1}$$

$$a_2 x - b_2 = 0 \qquad x = \frac{b_2}{a_2}$$

$$a_3 x - b_3 = 0$$

$$\vdots \qquad \vdots$$

CASO DI UGUAGLIANZA

$$\frac{b_1}{a_1} = \frac{b_2}{a_2} = \frac{b_3}{a_3} = \cdots = \frac{b_n}{a_n}$$

$$a_i = \lambda b_i \qquad \forall 1 \le i \le n$$

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq a + b + c$$

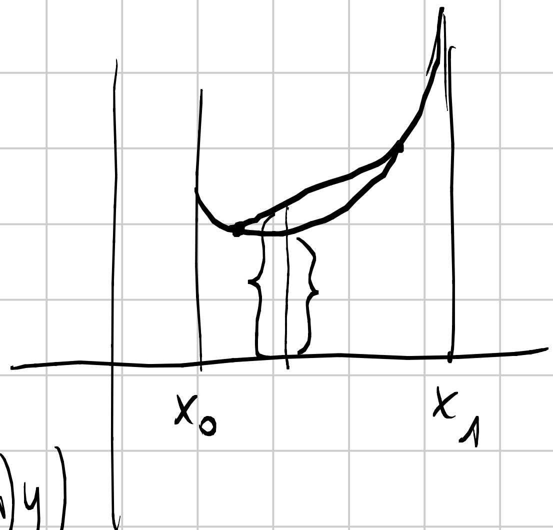
$$(a + b + c)\left(\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a}\right) \geq (a + b + c)^2$$

$$x_1 = \sqrt{a} \qquad x_2 = \sqrt{b} \qquad x_3 = \sqrt{c} \qquad y_1 = \frac{c}{\sqrt{a}} \qquad y_2 = \frac{a}{\sqrt{b}} \qquad y_3 = \frac{b}{\sqrt{c}}$$

1. JENSEN
2. MUIRHEAD (BUNCHING)
3. SHUR

JENSEN

$f(x)$ CONVESSA



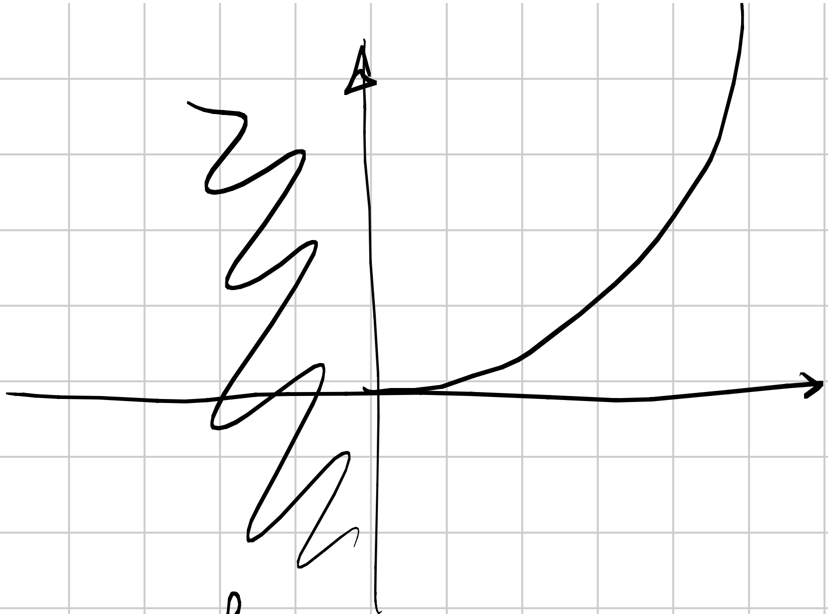
$$\lambda f(x) + (1-\lambda) f(y) \geq f(\lambda x + (1-\lambda) y)$$

$$0 \leq \lambda \leq 1$$

$$\frac{f(x_1) + f(x_2) + \dots + f(x_n)}{n} \geq f\left(\frac{x_1 + x_2 + x_3 \dots + x_n}{n}\right)$$

$f(x)$ CONVESSA

$$y = x^{\frac{p}{q}} \qquad p > q$$



$$\frac{x_1^{\frac{p}{q}} + x_2^{\frac{p}{q}} \ldots + x_n^{\frac{p}{q}}}{n} \geq \left(\frac{x_1 + \ldots + x_n}{n}\right)^{\frac{p}{q}}$$

$$x_i = y_i^{q} \longrightarrow \left(\frac{y_1^{p} + \ldots y_n^{p}}{n}\right)^{\frac{1}{p}} \geq \left(\frac{y_1^{q} + \ldots + y_n^{q}}{n}\right)^{\frac{\cancel{p}}{q}}$$

$$\sum_{sym} x^2y = x^2y + xy^2 + y^2z + yz^2 + x^2z + xz^2$$

$$\sum_{sym} xy = 2xy + 2yz + 2zx$$

$$\sum_{sym} xyz = 6xyz$$

$$\alpha_1 \geq \beta_1$$

$$\sum_{sym} x^{\alpha_1} y^{\alpha_2} z^{\alpha_3} \geq \sum_{sym} x^{\beta_1} y^{\beta_2} z^{\beta_3}$$

$$\alpha_1 + \alpha_2 \geq \beta_1 + \beta_2$$

$$\alpha_1 + \alpha_2 + \alpha_3 = \beta_1 + \beta_2 + \beta_3$$

$$\alpha_1 \geq \alpha_2 \geq \alpha_3 \qquad \beta_1 \geq \beta_2 \geq \beta_3$$

$$\boxed{\frac{1}{a^3+b^3+1}+\frac{1}{b^3+c^3+1}+\frac{1}{a^3+c^3+1}\leq 1 \qquad abc=1}$$

$$\frac{1}{a^3+b^3+abc}+\frac{1}{b^3+c^3+abc}+\frac{1}{a^3+c^3+abc}\leq\frac{1}{abc}$$

$$\sum_{sym}a^6b^3\geq\sum_{sym}a^5b^2c^2$$

SCHUR

$$\longrightarrow \sum_{sym} x^3 + \sum_{sym} xyz \geq 2\sum_{sym} x^2y$$

$$x^3+y^3+z^3+3xyz \geq x^2y+xy^2+y^2z+yz^2+xz^2+x^2z$$

$$\underbrace{x^3-x^2y-x^2z+xyz}+y^3-y^2x-y^2z+xyz+z^3-z^2x-z^2y+xyz \geq 0$$

$$x(x-y)(x-z)+y(y-x)(y-z)+z(z-x)(z-y) \geq 0$$

$x \geq y \geq z \geq 0$ $\quad + \quad - \quad +$

$$|x\cancel{(x-y)}(x-z)| \geq |y\cancel{(y-x)}(y-z)|$$

$x \geq y$

$x-z \geq y-z$

$$(ab+bc+ca)\left(\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\right)\geq\frac{9}{4}$$

$$4\sum a^5b-\sum a^4b^2-3\sum a^3b^3+\sum x^4yz-2\sum x^3y^2z+\sum x^2y^2z^2\geq 0$$

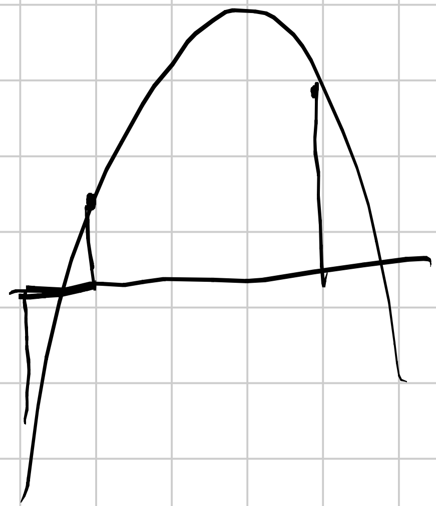
$$\underbrace{3\left(\sum a^5b-\sum a^3b^3\right)}_{+}+\underbrace{\sum a^5b-\sum a^4b^2}_{+}+\underbrace{xyz(\text{SHUR})}_{+}\geq 0$$

(1)

$$a^2+b^2+c^2 \le a^2b+b^2c+c^2a+1 \qquad 0\le a,b,c\le 1$$

$$a^2(b-1)+ac^2+b^2c+1-b^2-c^2\ge 0$$

$$f(a)=x_1a^2+x_2a+x_3$$



$$xyz = x + y + z \qquad x, y, z > 0$$

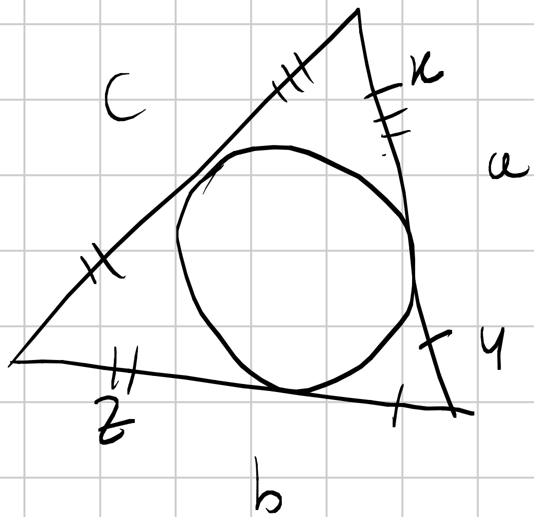
$$\frac{1}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+y^2}} + \frac{1}{\sqrt{1+z^2}} \le \frac{3}{2}$$

$$x, y, z \qquad \operatorname{tg}\alpha \quad \operatorname{tg}\beta \quad \operatorname{tg}\gamma \qquad \alpha+\beta+\gamma = \pi$$

$$\cos\alpha + \cos\beta + \cos\gamma \le \frac{3}{2}$$

a, b, c lati di un triangolo

$$2ab + 2bc + 2ca > a^2 + b^2 + c^2$$



$a = x + y$

$b = z + y$

$c = x + z$

$a + b - c = 2y$

$$2\left[(x+y)(x+z) + (x+z)(z+y) + (y+z)(y+x)\right] > (x+y)^2 + (y+z)^2 + (z+x)^2$$

$$2\left[\cancel{x^2} + \cancel{xy + xz + yz} + \cancel{z^2} + zx + zy + xy + \cancel{y^2} + xy + yz + zx\right] >$$

$$> \cancel{2\left(x^2 + y^2 + z^2\right)} + \cancel{2\left(xy + yz + zx\right)}$$

$$4\left(xy + yz + zx\right) > 0$$

Sul libretto 1, 3, 5

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x} \geq x+y+z \qquad xyz=1$$

$x, y, z > 0$

POSSIBILI PER IL TEST FINALE

$\boxed{\text{WUBERBEACH}}$

NON POSSIBILI PER IL TEST FINALE

Sul libretto 4, 9, 10 ⟶ ISTRUTTIVI

$$\sqrt{x}+\sqrt{y}+\sqrt{z} \geq xy+yz+zx \qquad x+y+z=3 \qquad x, y, z > 0$$

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq \frac{a+b}{b+c}+\frac{c+b}{b+a}+1 \qquad a, b, c > 0$$

⟶ DISTRUTTIVI

CIAO! :)