Senior 2006 - Combinatoria 1 (F.M.)

Titolo nota 13/09/2006

(1)

mr
ma
mo
mp

$13\ mr = 1\ ma$

$11\ ma = 1\ mo$

$7\ mo = 1\ mp$

Quanti prezzi positivi esistono $< 1\ mp$

$(0-6\ mo) \times (0-10\ ma) \times (0-12\ mr)$ 1000!

mp	mo	ma	mr
0	0	0	0

0	6	10	12

$7 \times 11 \times 13 = 1001$

$|A|\ |B|\ |C|$

$|A \times B \times C| = |A| \times |B| \times |C|$

Numero di divisori di un intero

$$N = \prod_{i=1}^{k} p_i^{a_i} \qquad \prod_{i=1}^{k} p_i^{b_i} \quad 0 \le b_i \le a_i$$

$$(0 - a_1) \times (0 - a_2) \times \ldots \times (0 - a_k)$$

$$\boxed{d(N) = \prod_{i=1}^{k} (a_i + 1)}$$

2. Una scimmia batte a caso su una tastiera con 26 tasti (A, B, ..., Z). Quanto vale la probabilità che scrive subito 'ABRACADABRA'?

$$1/26^{11}$$

→ quante sono le stringhe di 11 lettere da un alfabeto di 26

→ quante sono le funzioni da un insieme di 11 elementi ad un insieme di 26

$$\left|\{f : A \to B\}\right| = |B|^{|A|}$$

Quanti numeri naturali si esprimono in base 7 con fino a 4 cifre ?

$$7^4 = 2401$$

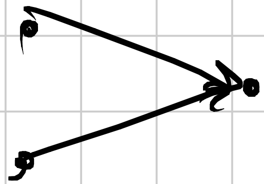
(3) Gara con 6 partecipanti. Quanti podi diversi sono possibili

$$6 \times 5 \times 4 = 120$$

$$6 \times 5 \times 4 = \left|\left\{ f : \{\mathrm{I}, \mathrm{II}, \mathrm{III}\} \longrightarrow \{a, b, c, d, e, f\},\ f \text{ iniettiva} \right\}\right|$$

f iniettiva se

$$i \neq j \Rightarrow f(i) \neq f(j)$$



vietato

$b = |B| \qquad a = |A|$

$$\left|\left\{ f : A \to B,\ f \text{ iniettiva} \right\}\right| = \underbrace{b(b-1)(b-2)\cdot \ldots \cdot (b-a+1)}_{a}$$

$$= \frac{b!}{(b-a)!}$$

Permutazioni

$$\left|\left\{ f : A \to A,\ f \ \underset{+\text{surgettive}}{\text{iniettiva}} \right\}\right| = |S_a|$$

$a = |A| \qquad |S_a| = a!$

Quante sono le classifiche ?

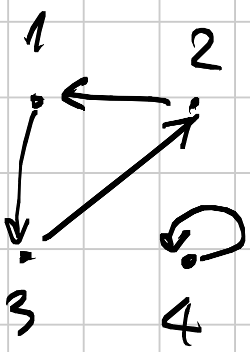
$$\left|\left\{ f : \{\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}, \mathrm{V}, \mathrm{VI}\} \to \{a, b, c, d, e, f\} \text{ invertibili} \right\}\right|$$

$$= 6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

$S_4 \ni \sigma$: $\boxed{3 \mid 1 \mid 2 \mid 4}$ (stringa)

$$\begin{aligned} 1 &\longrightarrow 3 \\ 2 &\longrightarrow 1 \\ 3 &\longrightarrow 2 \\ 4 &\longrightarrow 4 \end{aligned}$$

(funzione)



$(1, 3, 2)(4)$ (cicli)

$\boxed{0! = 1}$

$f : A \to A$ iniettiva $\Rightarrow$ surgettiva $\Rightarrow$ invertibile

solo SE A è finito.

④ Quanti sottoinsiemi ha l'insieme $A = \{1, 2, \dots, 6\}$

$\emptyset$, $\{1\}$, $\{2\}$, ... $\{6\}$, $\{1,2\}$, $\{1,3\}$, ...

(64)

	1	2	3	4	5	6
$\emptyset$	·	·	·	·	·	·
$\{1\}$	x	·	·	·	·	·
$\{3,5\}$	·	·	x	·	x	·
$\{2,4,6\}$	·	x	·	x	·	x

stringhe di due simboli e lungh. 6

con i sottoinsiemi.

$$\mathcal{P}(A) = 2^{A} \qquad |\mathcal{P}(A)| = 2^{|A|}$$

Quanti sono i sottoinsiemi di 0/6 elementi = 1

" " " " " 1/5 " = 6

" " " " " 2/4 " = 15

insiemi di 2 elementi

1	2	3	4	5	6	
x	x					5
x		x				
x			x			
x				x		
x					x	
	x	x				4
	x		x			
	x			x		
	x				x	
		x	x			

$$5+4+3+2+1=15$$

$$n = |A|$$

$$(n-1)+(n-2)+\dots+2+1 = \frac{n(n-1)}{2}$$

3 elementi → 20

$\dot{x}_3 \; \dot{x}_1 \; \dot{x}_2$

x_1	x_2	x_3				4
x	x		x			
x	x			x		
x	x				x	
x		x	x			3
x		x		x		
x		x			x	
x			x	x		2
x			x		x	
x				x	x	1
	x	x	x			
	x	x		x		

$$4+3+2+1$$
$$+3+2+1$$
$$+2+1$$
$$+1$$

fisso x_1 $\rightarrow 6$

fisso x_2 $\rightarrow \times 5$

fisso x_3 $\dfrac{\rightarrow \times 4}{3!} = 20$

$$\frac{n(n-1)(n-2)}{3!} = \frac{n!}{3!\,(n-3)!} = \binom{n}{3}$$

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Coefficiente Binomiale

$$64 = \binom{6}{0} + \binom{6}{1} + \binom{6}{2} + \binom{6}{3} + \binom{6}{4} + \binom{6}{5} + \binom{6}{6}$$

$\rightarrow$ 1 6 15 20 15 6 1

5 C. B.

- # di sottoinsiemi
- coefficiente di un monomio nel bin. di N.
- somme di (somme di (s. di (... n. triangolari)));
- troppi altri ...

$$1+4+9+\dots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

(a) coeff. del binomio

$$(x+y)^n = (x+y)(x+y)\cdots(x+y)$$

k ... $n-k$

$$= \dots + \binom{n}{k} x^k y^{n-k} + \dots$$

(a') $\binom{n}{k}$ = Tartaglia

$$1$$
$$1 \quad 1$$
$$1 \quad 2 \quad 1$$
$$1 \quad \boxed{3} \quad 3 \quad \boxed{1}$$
$$1 \quad 4 \quad 6 \quad 4 \quad 1$$

$$\binom{4}{2} = 6$$

$$\frac{4!}{2!\,2!} = \frac{24}{4} = 6$$

(a'') $$\boxed{(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}}$$

Binomio di Newton

$$\to \sum_{k=0}^{n} \binom{n}{k} = \sum \binom{n}{k} 1^k\, 1^{n-k} = (1+1)^n = 2^n$$

$$\rightarrow \quad 0 = (1-1)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k = 0$$

$$1 - 6 + 15 - 20 + 15 - 6 + 1 = 0$$

$$\rightarrow \quad \sum_{k=0}^{[n/2]} \binom{n}{2k} = 2^{n-1} = \sum_{k=0}^{[n/2]} \binom{n}{2k+1}$$

(b) $$\binom{n}{k} = \binom{n}{n-k}$$

$$\binom{n}{0} = \binom{n}{n} = 1$$

$$\binom{n}{1} = \binom{n}{n-1} = n$$

$$\binom{n}{2} = \frac{n(n-1)}{2}$$

(c) $$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

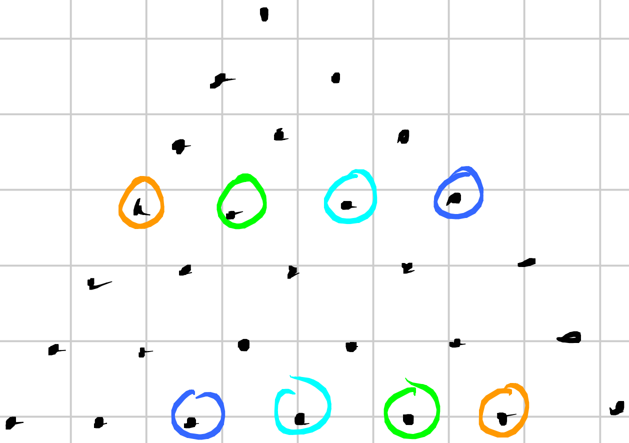
1 6 15 20 15 6 1

$$a \le k \le b$$

$$\sum_{n=a}^{b}\binom{n}{k}=\binom{b+1}{k+1}-\binom{a}{k+1}$$

$$\sum_{k=0\vee l-m}^{l\wedge n}\binom{n}{k}\binom{m}{l-k}=\binom{n+m}{l}$$

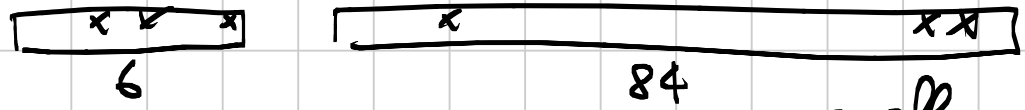
(d)



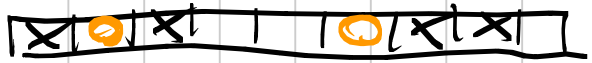
$$P(\text{fare } 6)=\frac{1}{\binom{90}{6}}\simeq\frac{1}{600\text{ mln}}$$

$$\binom{90}{6} = \frac{90!}{84!\,6!} = \frac{\cancel{90}\cdot 89\cdot \cancel{88}\cdot 87\cdot \cancel{86}\cdot 85}{\cancel{6}\cdot\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1}$$

$$p(\text{fare } 3) = \frac{\binom{6}{3}\binom{84}{3}}{\binom{90}{6}} \cong \frac{1}{326}.$$



(e) $\binom{9}{5}\binom{5}{2} = \frac{9!}{5!\,4!}\cdot\frac{5!}{3!\,2!} = \frac{9!}{4!\,3!\,2!}$ coeff. trinomiale

$\Longrightarrow$ 

$$\binom{9}{2}\binom{7}{4} = \frac{9!}{2!\,3!\,4!}$$

$$(x+y+z)^n = \sum_{\substack{a,b,c=0\\ a+b+c=n}}^{n} \frac{n!}{a!\,b!\,c!}\, x^a y^b z^c$$

Coefficiente multinomiale $\frac{n!}{a_1! \cdot a_2! \cdot \ldots \cdot a_k!}$ t.c. $\sum_1^k a_i = n$

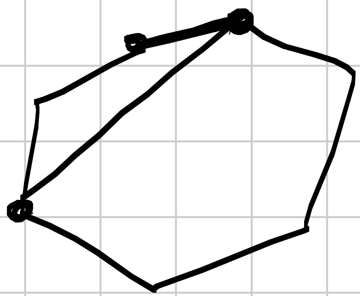
$$\left(\sum_1^k x_i\right)^n = \sum \ldots$$

Quanti sono gli anagrammi di 'ABRACADABRA'?

$$\frac{11!}{5! \cdot 2! \cdot 2! \cdot 1! \cdot 1!}$$

5 × A
2 × B
2 × R
1 × C
1 × D

⑥ (a) Diagonali di un poligono



n lati

$$\binom{n}{2} - n = \frac{n(n-3)}{2}$$

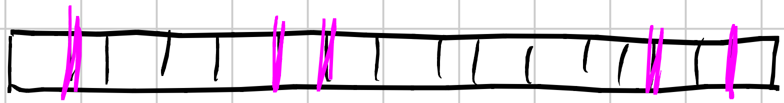
(b) Partizioni di un intero

$$n = \sum_{i=1}^{k} x_i \qquad x_i \geq 1$$

$x_1\ x_2\ x_3 \quad k=3$

$n = 4 = 2+1+1$
$1+2+1$
$1+1+2$

$\binom{3}{2} = 3$

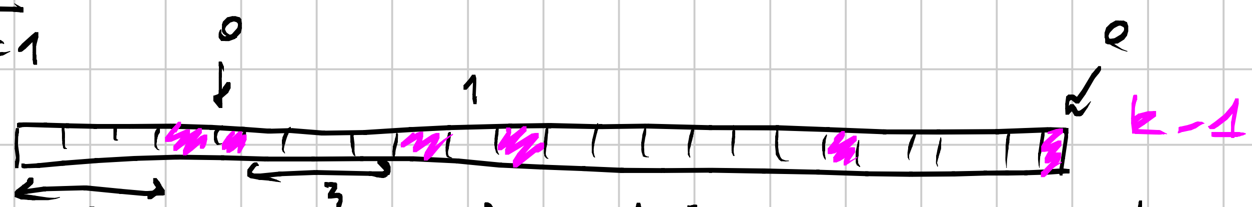


n

$\binom{n-1}{k-1}$

$$n = \sum_{i=1}^{k} x_i \qquad x_i \geq 0$$

$\binom{8}{2} = 28$



n spazi residui

$n+k-1$

$\binom{n+k-1}{k-1}$

(c) Prob. di prezzo che si paga con esattamente 6 monete?

$6 = x_r + x_a + x_b \qquad x_i \geq 0$

(7) Double Counting

(a) scacchiera n righe e m colonne $\quad m = 17n$

su ogni colonna ci sono 3 pedoni

su ogni riga c'è un numero diverso e $\leq n$ di pedoni

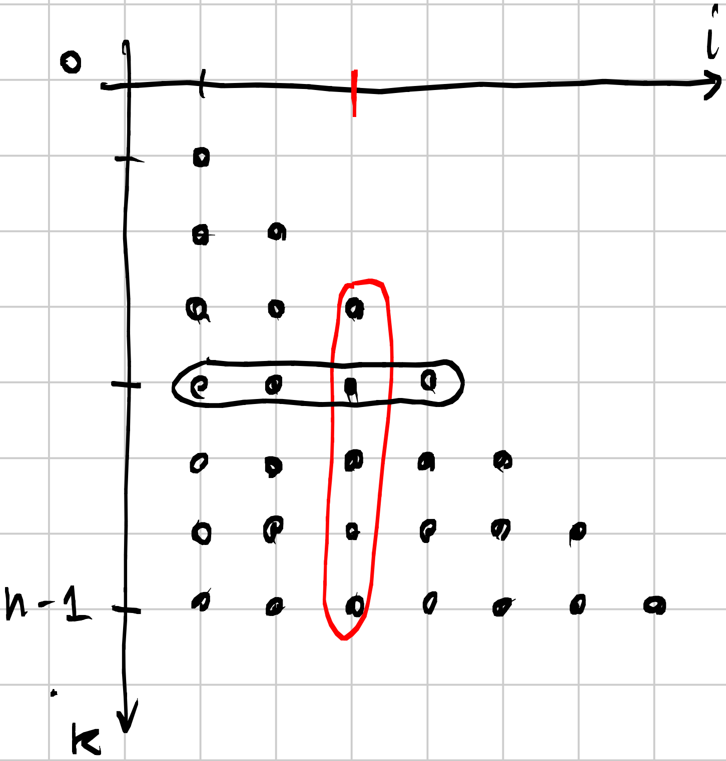
$n = ?$

$$N = 3m = 51n = \sum_{i=1}^{n} r_i = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$102n = n^2 + n \iff n = 101$$

(b)
$$\sum_{k=0}^{n-1} \frac{k}{2^k} = \sum_{k=0}^{n-1} \frac{1}{2^k} \sum_{i=1}^{k} 1 = \sum_{k=0}^{n-1} \sum_{i=1}^{k} \frac{1}{2^k} = \sum_{i=1}^{n-1} \underbrace{\sum_{k=i}^{n-1} \frac{1}{2^k}}$$

$$= \sum_i \frac{1}{2^i} \left(1 + \frac{1}{2} + \dots + \frac{1}{2^{n-i+1}}\right) = \sum_i \frac{1}{2^i} \frac{1 - \frac{1}{2^{n-1}}}{1 - \frac{1}{2}} = 2 - \frac{n+1}{2^{n-1}}$$



$$\sum_{k=1}^{n-1}\sum_{i=1}^{k} f(i,k)$$

$$\sum_{i=1}^{n-1}\sum_{k=i}^{n-1}$$

(c) Più grande potenza di p che divide $n!$

primo

$$n! = 1 \cdot 2 \cdot 3 \cdot \ldots \ldots \cdot n$$

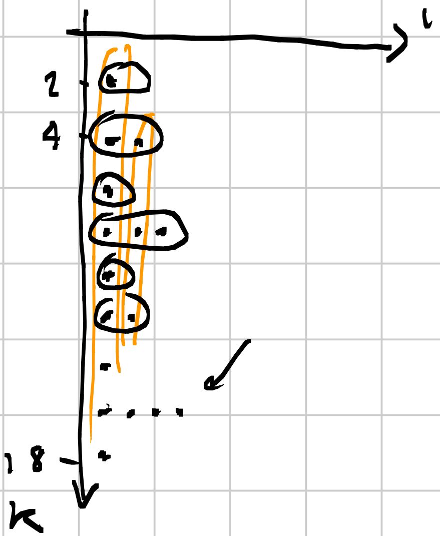
$$\ldots\, p \,\ldots\, 2p \,\ldots\, 3p \,\ldots\, p^2 \,\ldots\, (p+1)p \,\ldots\, p^3$$

$\forall k \in \mathbb{N}^+$ sia $a(k)$ t.c. $p^{a(k)} \,\|\, k$

$p = 3 \qquad k = 144 \qquad a(k) = 2$

$$a(n!) = \sum_{k=1}^{n} a(k) = \sum_{k=1}^{n} \sum_{i=1}^{a(k)} 1 = \sum_{i=1}^{\infty} \sum_{\substack{k:\, i \le a(k) \\ 1 \le k \le n}} 1$$

$p=2$



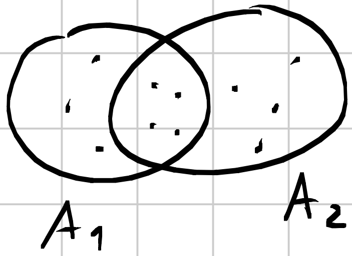
$$a(k) \ge i \iff p^i \mid k$$

$$= \sum_{i=1}^{\infty} \underbrace{\sum_{\substack{1 \le k \le n: \\ k:\, p^i \mid k}} 1} = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor$$

8 Principio di inclusione - esclusione

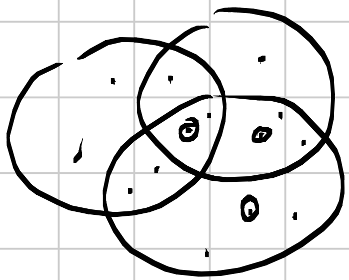
$A_1, A_2, \dots, A_n$ insiemi finiti

$n = 2$



$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

$n = 3$



$$|A_1 \cup A_2 \cup A_3| = |(A_1 \cup A_2) \cup A_3|$$

$$= |A_1 \cup A_2| + |A_3| - |(A_1 \cup A_2) \cap A_3|$$

$$= |A_1| + |A_2| - |A_1 \cap A_2| + |A_3| - |(A_1 \cap A_3) \cup (A_2 \cap A_3)|$$

$$= |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_1 \cap A_3| + |A_1 \cap A_2 \cap A_3|$$

$$\underbrace{\left|\bigcup_{i=1}^{n} A_i\right|}_{\substack{\shortparallel \\ A}} = \sum_{i=1}^{n} |A_i| - \sum_{\substack{1\le i<j\le n \\ \{i,j\}\subset\{1,2,\dots,n\}}} |A_i\cap A_j| + \sum_{1\le i<j<k\le n} |A_i\cap A_j\cap A_k| - \dots \pm \left|\bigcap_{i=1}^{n} A_i\right|$$

Come si dimostra? $\qquad |H| = \sum_{a\in H} 1$

$$\sum_{a\in A} 1 \overset{?}{=} \sum_{i=1}^{n} \sum_{a\in A_i} 1 - \sum_{i,j} \sum_{a\in A_i\cap A_j} 1 + \dots$$

$$= \sum_{a\in A} \underbrace{\sum_{i:A_i\ni a}}_{k(a)} 1 - \sum_{a\in A} \sum_{i,j:A_i\cap A_j\ni a} 1 + \dots$$

$$= \sum_{a\in A} \Bigg[\underbrace{\binom{k(a)}{1} - \binom{k(a)}{2} + \binom{k(a)}{3} - \dots \pm \binom{k(a)}{0}}_{1} \Bigg]$$

$$\overbrace{\underbrace{1-\binom{k(a)}{1}+\binom{k(a)}{2}-\dots}_{0}}^{-1}$$

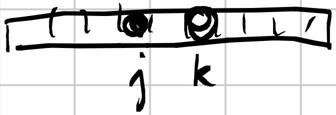
9 (a) Contare le permutazioni di S_n che non hanno punti fissi

$\sigma(k)=k$

$2\,4\,1\,3 \quad (1,2,4,3)$

$1\,4\,2\,3 \quad (1)(2,4,3)$

$$n! - n\cdot(n-1)! + \binom{n}{2}(n-2)! - \binom{n}{3}(n-3)! + \dots$$



$$A_i : \{\sigma \in S_n \mid \sigma(i)=i\}$$

$$|S_n| - \left|\bigcup_{i=1}^{n} A_i\right| = n! - \left|\bigcup_{i=1}^{n} A_i\right| = n! - \sum|A_i| + \sum|A_i\cap A_j| - \dots$$

$$= \emptyset$$

$$|A_i| = (n-1)! \qquad A_i \cap A_j = \{\sigma \in S_n \mid \sigma(i)=i, \sigma(j)=j\}$$

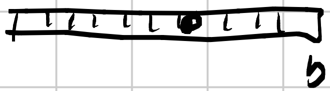
$$|A_i \cap A_j| = (n-2)!$$

$$\circledast = n! - n(n-1)! + \binom{n}{2}(n-2)! - \dots$$

$$= \sum_{k=0}^{n} \binom{n}{k}(n-k)!(-1)^k = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

(b) $f : A \to B$ surjective $\quad a = |A| \quad b = |B| \quad b \leq a$

$$b^a - b(b-1)^a + \binom{b}{2}(b-2)^a - \binom{b}{3}(b-3)^a + \dots$$



$$E_i = \{ f : A \to B \ : \ f^{-1}(i) = \emptyset \} \qquad i \in B$$

$$b^a - \left| \bigcup_{i=1}^{n} E_i \right| \overset{i\text{-}e}{=} \sum_{k=0}^{b} \binom{b}{k} (b-k)^a (-1)^k$$

$$|E_i| = (b-1)^a$$

$$|E_i \cap E_j| = (b-2)^a$$

$$E_i \cap E_j = \{ f : A \to B : f^{-1}(i) = \emptyset, \ f^{-1}(j) = \emptyset \}$$

. . .

Esercizi C1: 1, 5, 7, 10