Senior 2006 – Combinatoria 2

(F. V.)

Titolo nota

15/09/2006

Permutazioni

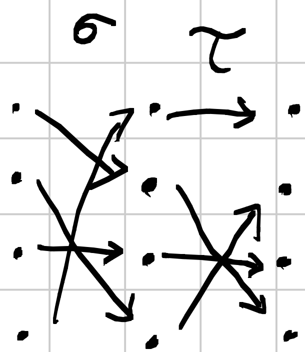
$$S_n = \{ \sigma : \{1, \dots n\} \to \{1, \dots n\} \text{ bigettive} \}$$

$$|S_n| = n! \qquad \sigma : \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix}$$

$$\sigma, \tau \in S_n \qquad \sigma \tau \qquad \sigma \circ \tau \, (i) = \sigma(\tau(i))$$

$$\sigma \tau \neq \tau \sigma$$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix} \qquad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$$



$$\tau \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 1 \end{pmatrix}$$

$$\sigma \tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}$$

$(a_1 \; \cdots \; a_k)$ k-ciclo

$\sigma \qquad (1 \;\; \sigma(1) \;\; \sigma^2(1) \cdots)(j \;\; \sigma(j) \;\; \sigma^2 \quad)$

$(a \; b)$ 2-ciclo Trasposizione

$$(a_1 \quad a_k) = (a_1\, a_2)(a_2\, a_3) \cdots \overset{(a_{i-1} a_{i})(a_i\, a_{i+1})}{\cdots} (a_{k-1}\, a_k)$$

$$\operatorname{sgn}(\sigma) \quad \pm \quad \cdot \prod_{1 \le i < j \le n} \frac{\sigma(i) - \sigma(j)}{i - j}$$

$i < j$ e $\sigma(i) > \sigma(j)$

(i, j) è un'inversione di σ

- Parità del numero di inversioni
- Parità del numero di trasposizioni in cui si scompone σ

$$\sigma \quad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

$$\frac{\sigma(1) - \sigma(2)}{1-2} \quad \frac{\sigma(1) - \sigma(3)}{1-3} \quad \frac{\sigma(2) - \sigma(3)}{2-3}$$

$$-1$$

$$\frac{\cancel{1-3}}{\cancel{1-2}} \quad \frac{\cancel{1-2}}{\cancel{1-3}} \quad \frac{\cancel{3-2}}{\cancel{2-3}}$$

$$\operatorname{sgn}(\sigma\,\tau) = \operatorname{sgn}(\sigma)\cdot\operatorname{sgn}(\tau)$$

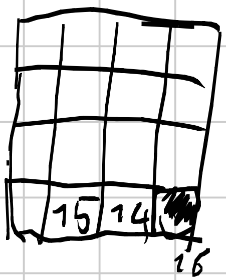
σ

$$P(x_1 \quad x_n) = \prod_{i<j} (x_i - x_j)$$

$$\sigma P(x_1 \quad x_n) = \prod_{i<j} \left(x_{\sigma(i)} - x_{\sigma(j)}\right)$$

$$\sigma P(x_1 \quad x_n) = \pm P(x_1 \quad x_n)$$

$$\operatorname{sgn}(k\text{-ciclo}) = (-1)^{k-1}$$



$\sigma \in S_n \qquad (a_1 \quad a_k)$

$$\sigma \, (a_1 \cdots a_k) \, \sigma^{-1} = (\sigma(a_1) \; \sigma(a_2) \quad \sigma(a_k))$$

$$\sigma(a_i) \xmapsto{\sigma^{-1}} a_i \xmapsto{(a_1 \cdots a_k)} a_{i+1} \xmapsto{\sigma} \sigma(a_{i+1})$$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix} \qquad (1 \; 3 \; 2)$$

$$\sigma^i = \overbrace{\sigma \, \sigma \cdots \sigma}^{i \text{ volte}}$$

$$\sigma \, (1 \, 3 \, 2) \, \sigma^{-1} = (4 \; 1 \; 3)$$

$$\sigma^i \tau \sigma^{-i} = (\sigma^i(1) \; \sigma^i(2)) = (i+1 \;\; i+2)$$

$$\sigma = (1 \; 2 \cdots n) \qquad \tau = (1 \; 2)$$

$$\sigma^{-1} = \sigma^{n-1}$$

$(a\ b)$ $\quad a < b$

$$(a\ b) = (a\ \ a+1)(a+1\ \ a+2)\cdots(b-1\ \ b)(b-1\ \ b-2)\cdots$$

$(a+1\ \ a)$

$$(2\ 5) = (2\ 3)(3\ 4)(4\ 5)(3\ 4)(2\ 3)$$

$(1\ 2)\ ,\ (2\ 3)\ ,\ \quad (n-1\ \ n)$

$\sigma = (\overbrace{1 \quad\quad}\ n) \qquad \sigma^n = \mathrm{Id}$

$\tau = (a_1 \quad a_k) \qquad \tau^k = \mathrm{Id}$

σ min $i > 0$ $\sigma^i = Id$ ordine

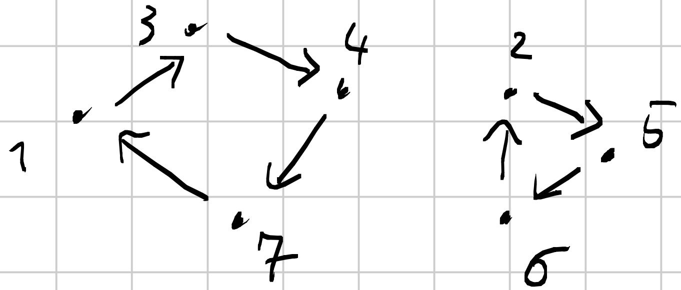
K-cicli ordine K

$$\sigma = (\ \ )^{K_1} (\ \ )^{K_2} (\ \ ) \cdots^{K_J}$$

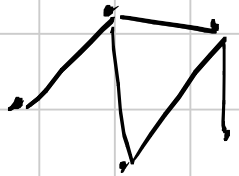
ordine di σ = $mcm(K_1, \dots K_J)$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 5 & 4 & 7 & 6 & 2 & 1 \end{pmatrix} = (1\ 3\ 4\ 7)(2\ 5\ 6)$$

$$\sigma^2 = (1\ 3\ 4\ 7)(2\ 5\ 6)(1\ 3\ 4\ 7)(2\ 5\ 6) = (1\ 3\ 4\ 7)^2 (2\ 5\ 6)^2$$

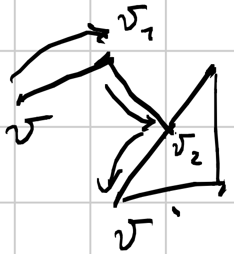


GRAFI



V vertici

E coppie di vertici



- connesso se $\forall\, v \quad v' \in V$

  $v \quad v_1 \quad v_2 \quad \dots \quad v'$

- <u>ciclo</u> se $v = v'$

- albero: connesso senza cicli

- ordine di un vertice v : numero di archi che
  (grado) escono da quel vertice $\quad d(v)$

$$\boxed{\sum_{v \in V} d(v) = 2E}$$

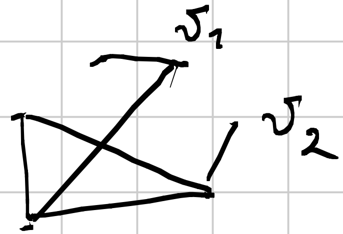
E : # lati

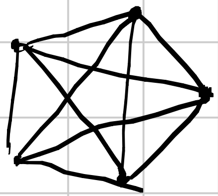
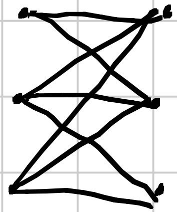
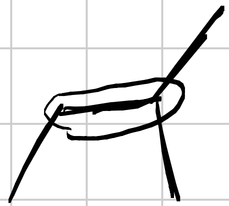
$$\sum_{\substack{d(v) \\ \text{pari}}} d(v) + \sum_{\substack{d(v) \\ \text{dispari}}} d(v) = 2E$$

Albero $\quad n$ vertici $\Longleftrightarrow$ $n-1$ lati

diametro (grafo connesso) : max distanza tra vertici

$\hookrightarrow$ min lunghezze dei cammini tra i vertici



Circuito euleriano

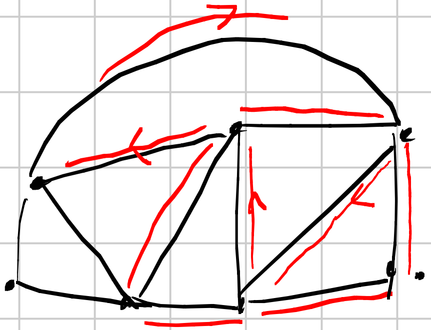
percorrere il grafo passando una e una sola volta
per ogni lato

Circuiti hamiltoniani
per ogni vertice

Circuito euleriano se e solo se:
- tutti i vertici hanno ordine pari
- esattamente 2 vertici hanno ordine

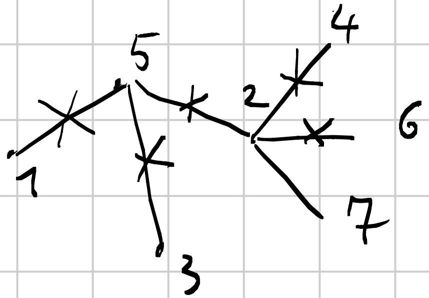
dispari

tutti i vertici hanno ordine pari:



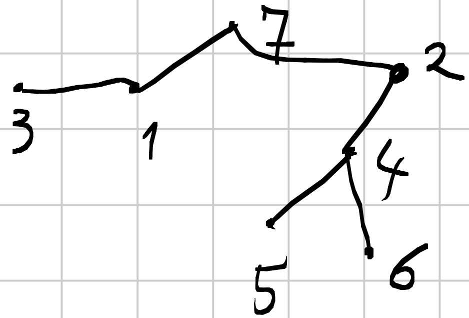
Quanti alberi etichettati con n vertici?

n $\qquad$ n^{n-2}

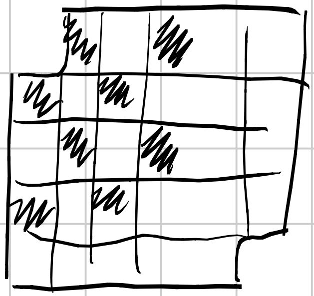


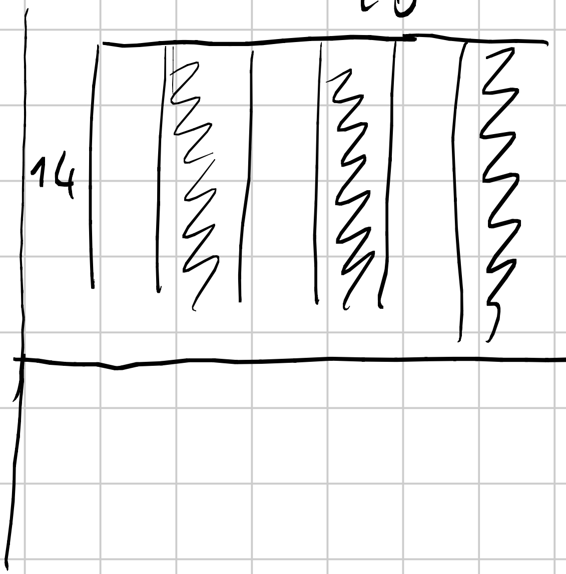
$\longleftrightarrow$ 5 5 2 2 2

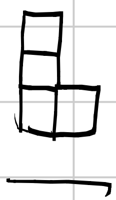
1 7 4 4 2

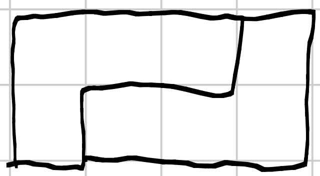


INVARIANTI E COLORAZIONI









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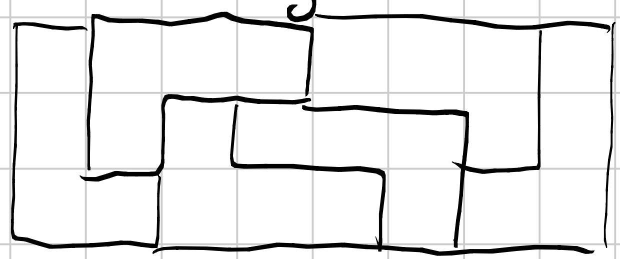

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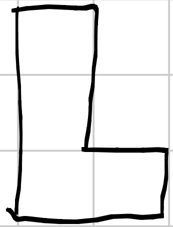
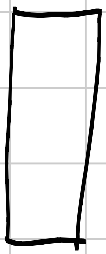
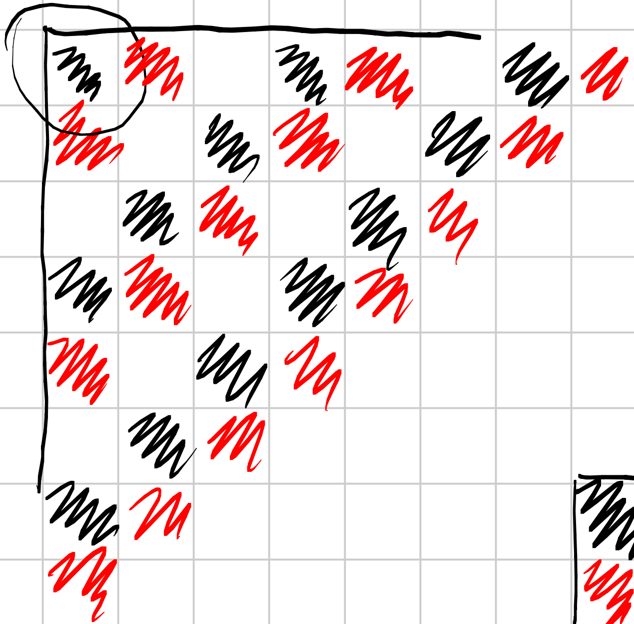


$n \times m$

$8 \mid nm$

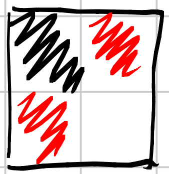
$n, m \geq 2$

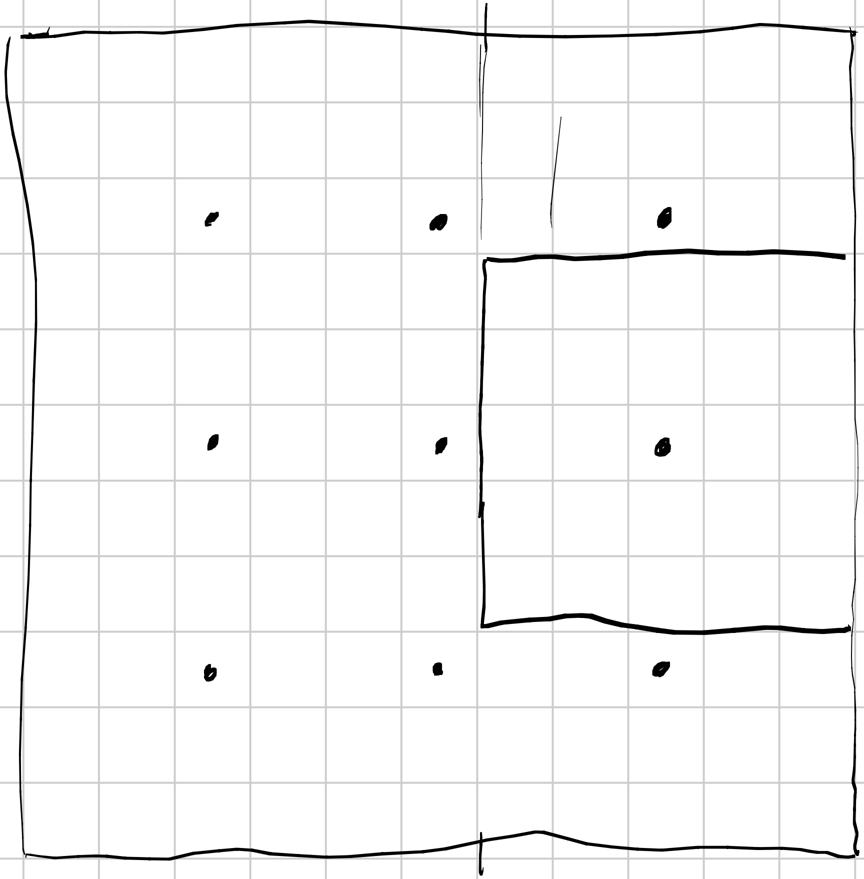




11×11

0	1
2	2
1	1





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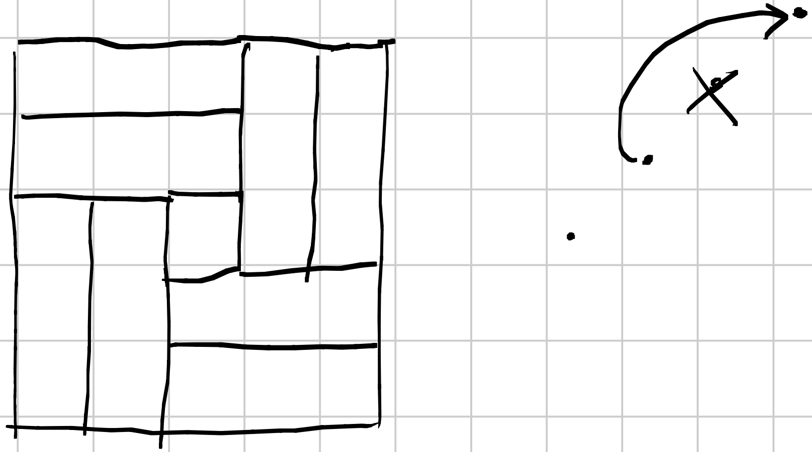
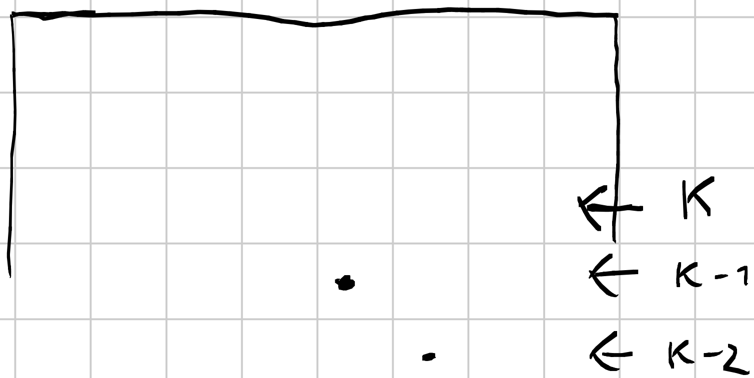
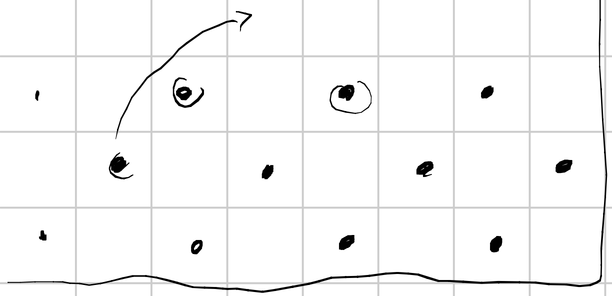






13

8

← 5

← 3

← 2

12

PARTIZIONI (di un intero)

$$n = a_1 + a_2 + \cdots + a_k$$

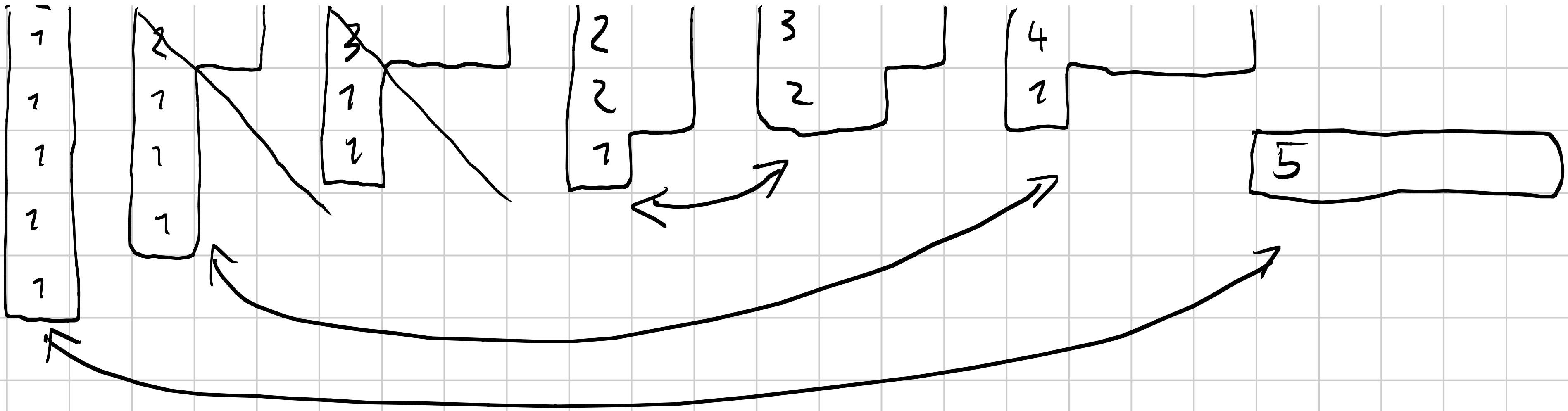
5

- $1+1+1+1+1$
- $2+1+1+1$
- $2+2+1$
- $3+1+1$
- $3+2$
- $4+1$
- 5

3

$\underline{n}$

$\#$ Partizioni in $\leq n$ interi $=$ $\#$ Partizioni in interi $\leq n$



$\#$ Partizioni di n in addendi dispari $=$ $\#$ Partizioni di n in addendi distinti $\leftarrow$

$$\rightarrow \sum_{k=0}^{n}\binom{n}{k}^2=\binom{2n}{n} \qquad (1+x)^n=\sum_{k=0}^{n}\binom{n}{k}x^k$$

$$c_k=\sum_{i=0}^{k} a_i\, b_{k-i} \qquad \sum_{k\ge 0} a_k x^k \cdot \sum_{k\ge 0} b_k x^k = \sum_{k\ge 0} c_k x^k$$

$$(1+x)^{2n} = (1+x)^n (1+x)^n$$

coeff. di x^n

$$\binom{2n}{n} = \sum_{i=0}^{n} \binom{n}{i} \cdot \binom{n}{n-i} = \sum_{i=0}^{n} \binom{n}{i}^2$$