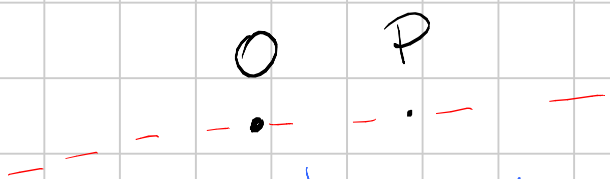
SENIOR 2006 - Geometria 3 (S.M.)

Titolo nota 14/09/2006

TRASFORMAZIONI del PIANO

1) Trasf. che conservano la distanza (ISOMETRIE)
2) Trasf. che conservano le similitudini tra triangoli. (SIMILITUDINI)

Omotetia: $k \in \mathbb{R}^*$



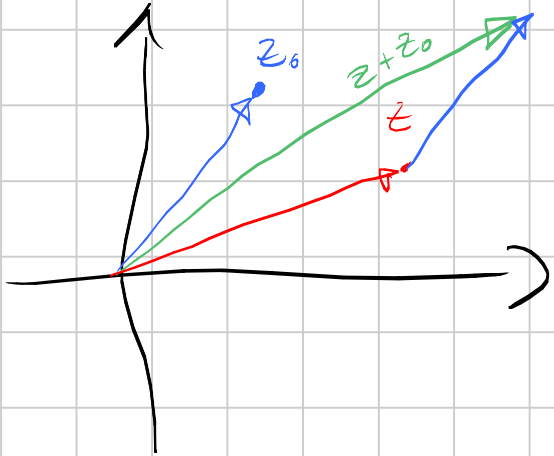
$k < 0$ $P' \in$ semir. da O che non contiene P

$k > 0$ P' t.c. P, P' sono sulla stessa semir. da O e $OP' = k \cdot OP$

1) ISOMETRIE

- Traslazione: fissato $z_0 \in \mathbb{C}$

$$T_{z_0}(z) = z + z_0$$

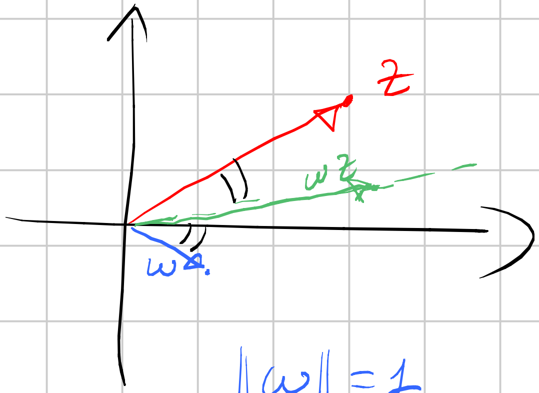


- Rotazione attorno a $0 \in \mathbb{C}$ di un angolo θ

$u = \rho(\cos\alpha + i \operatorname{sen}\alpha) \qquad v = r(\cos\beta + i \operatorname{sen}\beta)$

$$u \cdot v = \rho r \left(\cos(\alpha+\beta) + i \operatorname{sen}(\alpha+\beta)\right)$$

$\rho = 1 \iff \|u\| = 1 \implies u \cdot v = \rho\left(\cos(\alpha+\beta) + i \operatorname{sen}(\alpha+\beta)\right)$



$\|\omega\| = 1$

$\omega = \cos\theta + i \operatorname{sen}\theta$

$\omega \cdot z$ = ruotato di θ attorno all'origine

Rot. attorno a $z_0 \in \mathbb{C}$ di θ

$$z \longrightarrow z - z_0 \longrightarrow \omega(z-z_0) \rightsquigarrow$$

$$\rightsquigarrow \boxed{\omega(z-z_0)+z_0}$$

⊙ Riflessione

- Rispetto a $\{\operatorname{Im} z = 0\}$

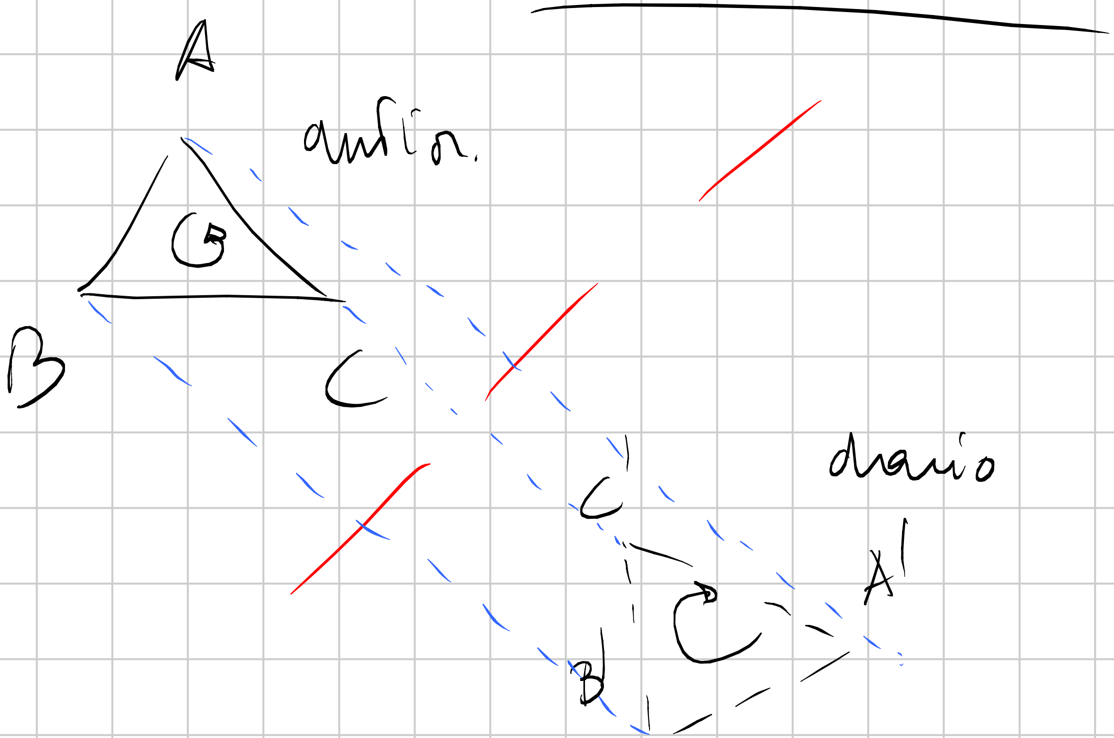
$$z \longmapsto \bar{z}$$

– Rifl. a una retta per z_0 inclinata di θ

$$z \rightsquigarrow z - z_0 \longrightarrow \bar{\omega}(z - z_0) \rightsquigarrow \omega(\bar{z} - \bar{z}_0)$$

$\omega = \cos\theta + i \operatorname{sen}\theta$

$$\rightsquigarrow \omega^2(\bar{z} - \bar{z}_0) \rightsquigarrow \boxed{\omega^2(\bar{z} - \bar{z}_0) + z_0}$$



2) SIMILITUDINI

Omotetia di centro $0 \in \mathbb{C}$
e fattore $k \in \mathbb{R}^*$

$$z \longmapsto k \cdot z$$

- centro $z_0 \in \mathbb{C}$

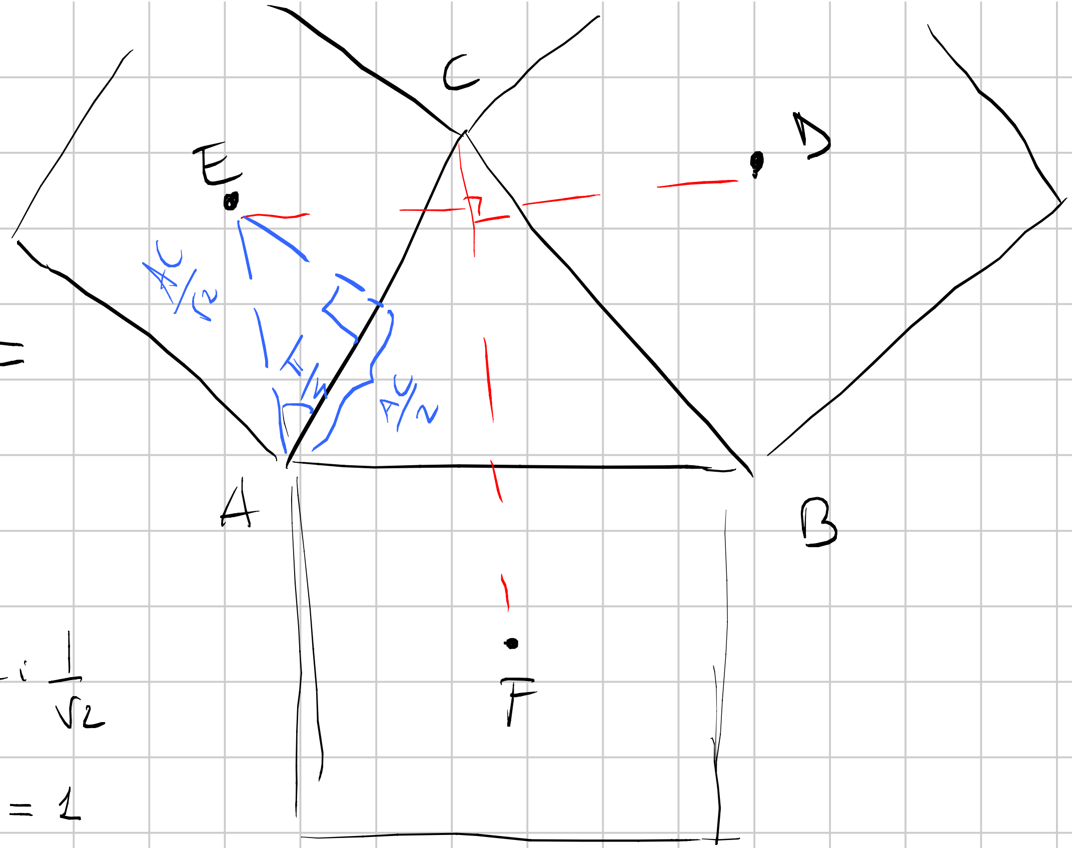
$$z \rightsquigarrow z - z_0 \rightsquigarrow k(z - z_0) \rightsquigarrow \boxed{k(z - z_0) + z_0}$$

$\cos(\theta + \pi) = -\cos(\theta)$
$\operatorname{sen}(\theta + \pi) = -\operatorname{sen}(\theta)$

3) AFFINITÀ

$$z \longrightarrow az + b\bar{z} + c \qquad a, b, c \in \mathbb{C}$$

Es: ABC Triangolo



$ED \equiv CF$ e $ED \perp CF$

a, b, c

$$e: Om_A^{\frac{1}{\sqrt{2}}}\left(Rot_A^{\frac{\pi}{4}}(c)\right)$$

$$\omega = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$$

$$\omega^8 = 1$$

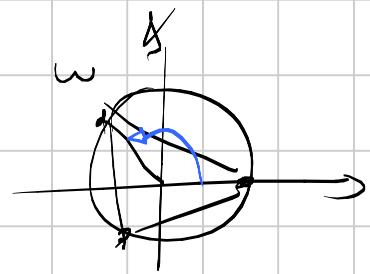
$$\frac{1}{\sqrt{2}}\left(\omega\,(c-a) + \cancel{a} - \cancel{a}\right) + a$$

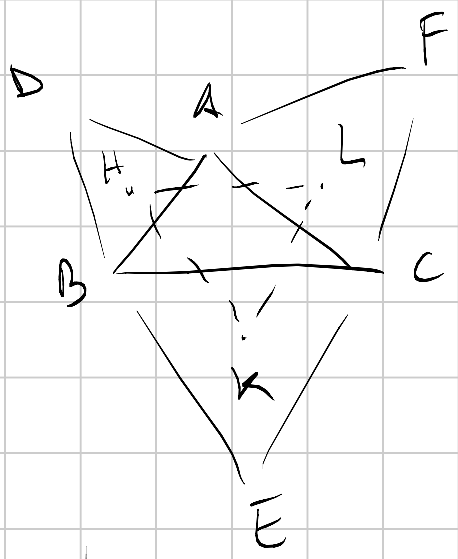
$$\frac{1}{\sqrt{2}}\left(\omega\,(c-a)\right) + a$$

$$\frac{e-d}{c-f} = \pm i$$

Es: ABC triangolo ABD, BCE, CAF equilateri

H, K, L baricentri $\Longrightarrow$ HKL equilatero.





ω rad 3^a di 1.

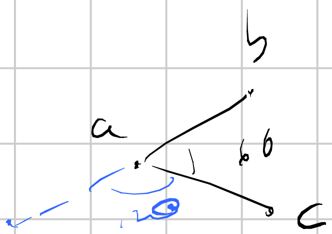
a, b, c fanno un tri. equil.

$\Longleftrightarrow$

$$a + \omega b + \omega^2 c = 0$$

o

$$a + \omega c + \omega^2 b = 0$$



$$\omega(c-a) = -(b-a) \Longleftarrow \boxed{1 + \omega = -\omega^2}$$

ABD equi. (1) $a + \omega b + \omega^2 d = 0$

AFC " (2) $a + \omega f + \omega^2 c = 0$

BEC " (3) $b + \omega c + \omega^2 e = 0$

$$h = \frac{a+b+d}{3}$$

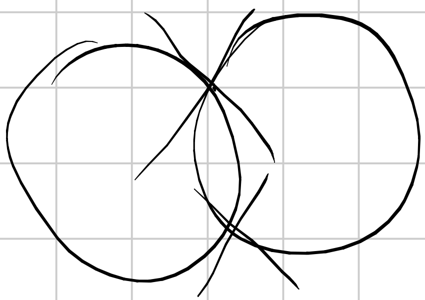
$$k = \frac{a+f+c}{3}$$

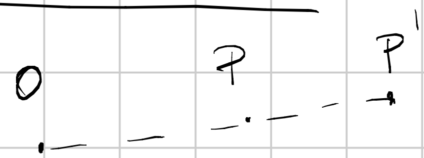
$$l = \frac{b+c+e}{3}$$

$$h + \omega l + \omega^2 k = 0 \quad (4)$$

$$(1) + (2) + (3) = A \cdot (4)$$

Inversione



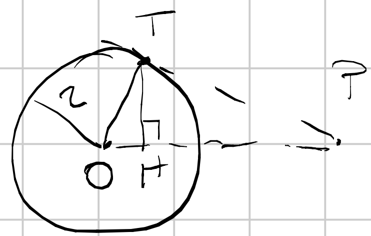


P' ∈ semiretta OP

t.c.

$OP \cdot OP' = r^2$

$r \in \mathbb{R}^+$



$OH \cdot PO = r^2$

H è P'!

Centro in $0 \in \mathbb{C}$

1) $\arg z = \arg z'$

2) $|z| \cdot |z'| = r^2$

$$\cancel{z' = \frac{r^2}{z}}$$ No

$$z' = \frac{r^2}{\bar{z}}$$ Sì

Centro in z_0

$$\boxed{z' = \frac{r^2}{\bar{z} - \bar{z}_0} + z_0}$$

Ex 2:

A, B

$Rot_A^\alpha \quad Rot_B^\beta \quad Om_A^k \quad Om_B^h$

$$Rot_A^h(z) = \omega(z-a)+a$$

$$+\; Om_A^k(z) = k(z-a)+a$$

$$= k\omega(z-a)+a$$

$$\underbrace{Rot_A^\alpha\, Om_A^k}_{v}\; \underbrace{Rot_B^\beta\, Om_B^h}_{u} \overset{?}{=} \underbrace{Rot_B^\beta\, Om_B^h}_{u}\; \underbrace{Rot_A^\alpha\, Om_A^k}_{v}$$

$$= v(z-a)+a$$

$$v(u(z-b)+b-a)+a \overset{?}{=} u(v(z-a)+a-b)+b$$

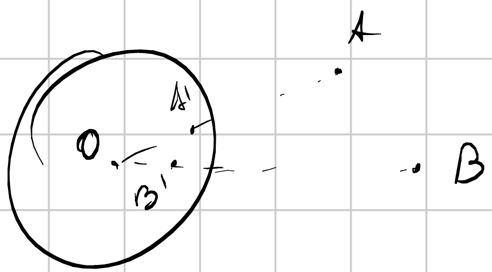
$$\cancel{vuz} - vub + vb - va + a \overset{?}{=} \cancel{uvz} - uva + ua - ub + b$$

$$uv(a-b) - v(a-b) - u(a-b) + (a-b) \overset{?}{=} 0$$

$$(a-b)(uv - v - u + 1) = (a-b)(u-1)(v-1) \overset{?}{=} 0$$

- $a = b$
- $u = 1$
- $v = 1$

Lunghezze sotto l'inversione



$$A'B' = r^2 \frac{AB}{OA \cdot OB}$$

Inversione è inversa di se stessa.

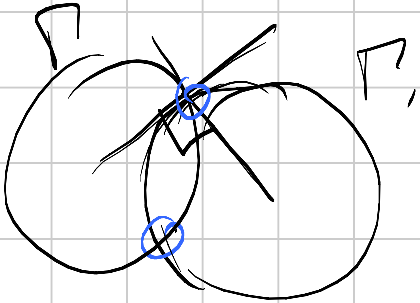
1) rette per O $\longleftrightarrow$ rette per O

2) circ. per O $\longrightarrow$ rette non per O

3) rette non per O $\longrightarrow$ circ. per O

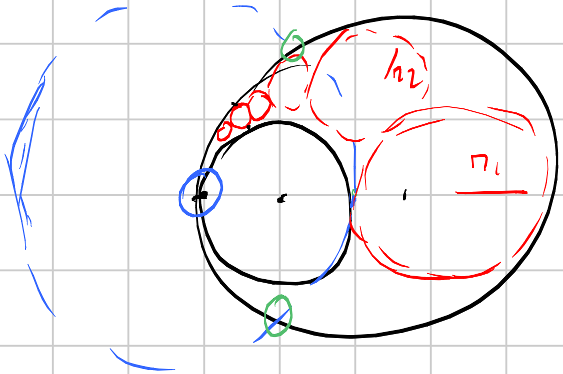
4) circ non per O $\longrightarrow$ circ non per O

Circ. ortogonali

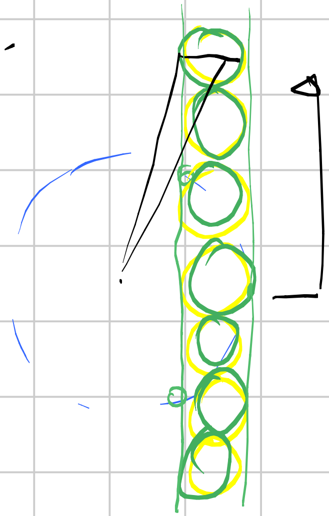


Inv. di Γ_1 risp a Γ è Γ_1.

Es

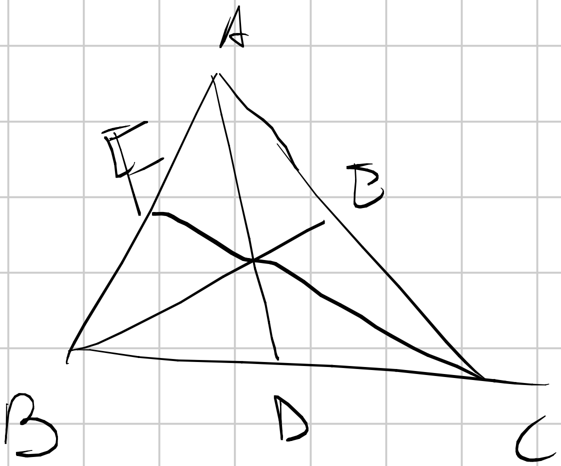


$r_n = ?$



Geometria Sintetica

1) T° di CEVA

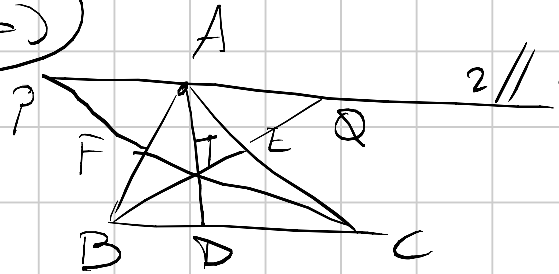


AD, BE, CF concorrono

$\Longleftrightarrow$

$$\frac{AE}{EC} \cdot \frac{CD}{DB} \cdot \frac{BF}{FA} = 1 \quad (*)$$

Dim: ($\Longrightarrow$)



$AFP \simeq FBC$ $\quad \dfrac{AF}{FB} = \dfrac{PA}{BC}$

$AEQ \simeq BEC$ $\quad \dfrac{AE}{EC} = \dfrac{AQ}{BC}$

PQT, BTC omotetico di centro T

$\Rightarrow$ D e A omot di " "

$$\Rightarrow \frac{PA}{AQ} = \frac{DC}{DB}$$

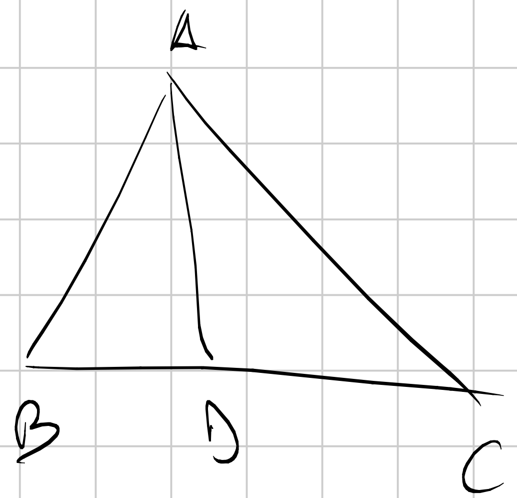
$$\frac{AB}{EC} \cdot \frac{CD}{DB} \cdot \frac{BF}{FA} = \frac{\cancel{AQ}}{\cancel{BC}} \cdot \frac{\cancel{PA}}{\cancel{AQ}} \cdot \frac{\cancel{BC}}{\cancel{PA}} = 1$$

$\Leftarrow$ Vale (*) $T = BE \cap CF$ $AT \cap BC = D'$

$$\frac{\cancel{AB}}{\cancel{EC}} \cdot \frac{CD'}{D'B} \cdot \frac{\cancel{BF}}{\cancel{FA}} = 1 = \frac{\cancel{AB}}{\cancel{EC}} \cdot \frac{CD}{DB} \cdot \frac{\cancel{BF}}{\cancel{FA}}$$

$\Rightarrow D = D'$ $\Rightarrow$ AD, BE, CF concorrono $\boxtimes$

Dim 2 :



$$\frac{BD}{DC} = \frac{S'_{ABD}}{S'_{DCA}}$$

$$\frac{a}{b} = \frac{c}{d} = \frac{a-c}{b-d}$$

Dim 3 :

$$\begin{cases} x' = ax + by + c \\ y' = dx + ey + f \end{cases}$$

$$ae - bd \neq 0$$

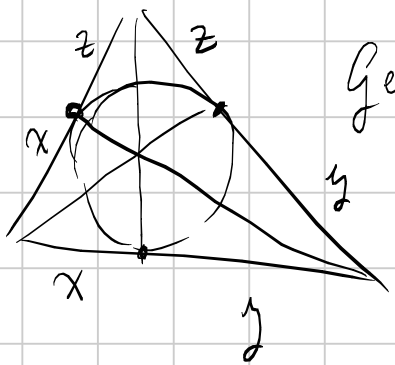
1. rette in rette
2. rapporti fra seg. allineati
3. rapporti fra aree.
4. Parallelismo.

UN'AFFINITA' MANDA 3 punti NON ALLINEATI in 3 punti NON ALLINEATI QUALSIASI.

Cerco a, b, c, d, e, f t.c. l'affinità mande il mio

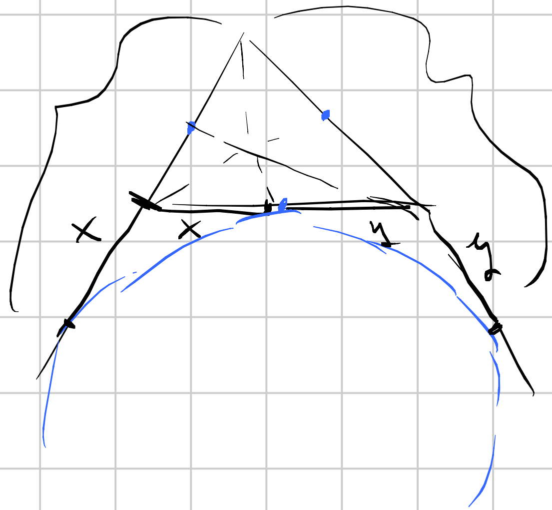
Triangolo in $(0,0)$, $(1,0)$, $(0,1)$.

Es:

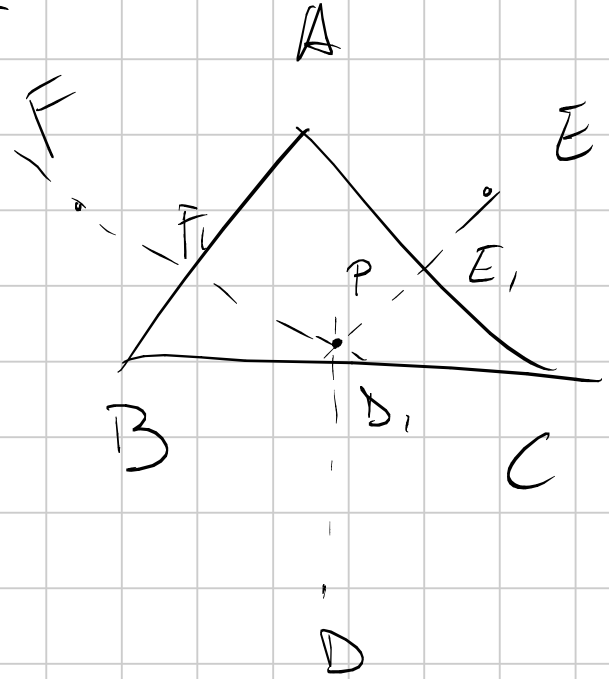


Gergonne

Es:



Teo di CARNOT



D_1 " di D su BC

E_1 proiez. di E su AC

F_1 " di F su AB

DD_1, EE_1, FF_1 concorrono

$\Updownarrow$

$$CD^2 + BF^2 + AE^2 = DB^2 + FA^2 + EC^2$$

Teo di MENELAO

ABC Triangolo

$D \in BC$

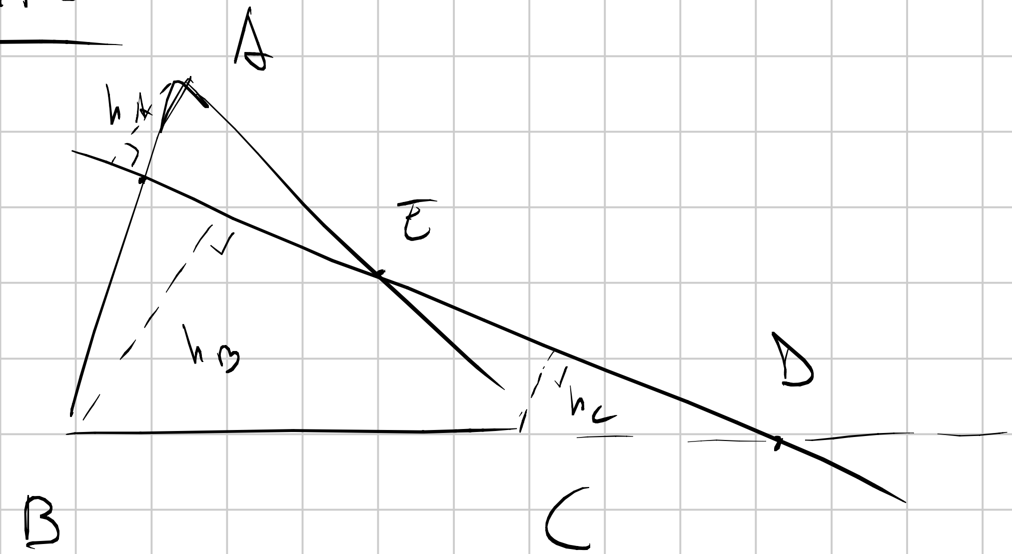
$E \in AC$

$F \in AB$

sono allineati

$\Updownarrow$

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1$$

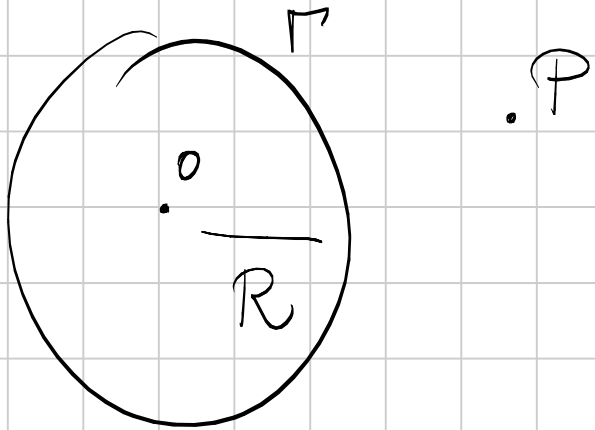


Dim: $\Rightarrow$

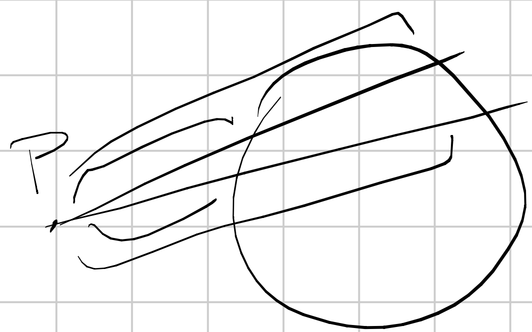
$$\frac{BD}{DC} = \frac{h_B}{h_C}$$

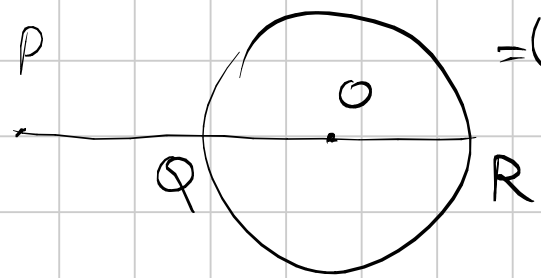
$$\frac{AE}{EC} = \frac{h_A}{h_C} \qquad \frac{AF}{FB} = \frac{h_A}{h_B}$$

Potenza di 1 punto risp. a una circonf.



$$pow_\Gamma(P) = R^2 - PO^2$$





$$PQ \cdot PR =$$
$$= (PO - OQ)(PO + OR) =$$
$$= PO^2 - R^2$$

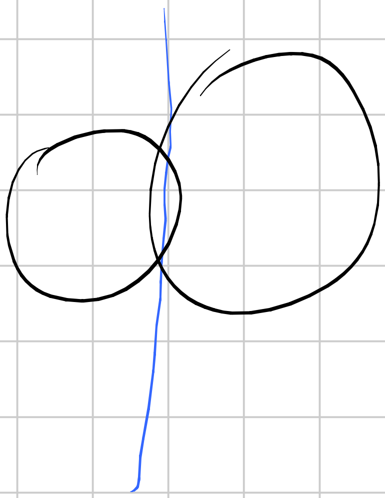
$pow_\Gamma(P) < 0$ P esterno

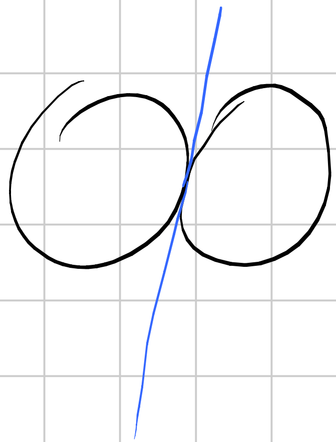
$pow_\Gamma(P) > 0$ P interno

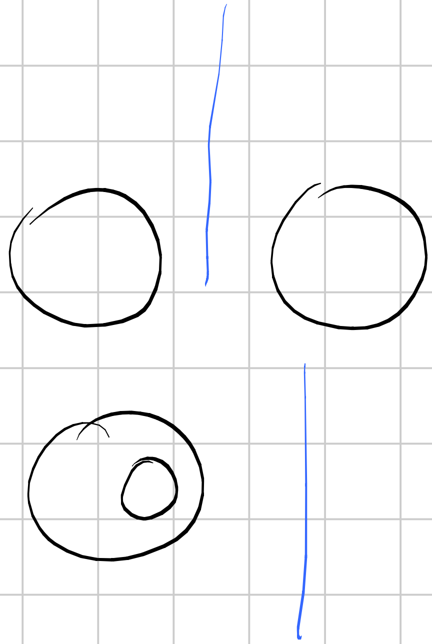
$pow_\Gamma(P) \leq R^2$ $\qquad$ $pow_\Gamma(P) = R^2 - PO^2$

Γ, Γ_1 circ. $\qquad \mathcal{L} = \{P / pow_\Gamma(P) = pow_{\Gamma_1}(P)\}$

$=$ asse radicale







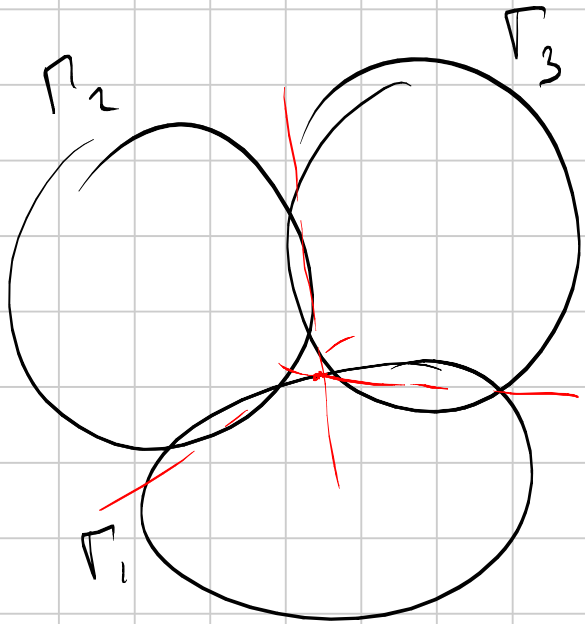
$$R^2 - (x-x_0)^2 - (y-y_0)^2 = p(x,y)$$

$$p(x,y) = \mathrm{pow}_\Gamma(x,y) \qquad \{p(x,y)=0\} = \Gamma$$

$$\Gamma : \quad \underbrace{x^2+y^2+2\alpha x+2\beta y+\gamma}_{=\,q(x,y)} = 0 \qquad P=(x_P, y_P)$$

$$\mathrm{pow}_\Gamma(P) = -q(x_P, y_P)$$

Asse radicale $\perp$ congiungente i centri.



$\forall$ terne di circ. con i centri non allineati, gli assi radicali concorrono.

$r_{1,2} \cap r_{2,3} = P$ t.c. $pow_{\Gamma_1}(P) = pow_{\Gamma_2}(P)$ xché $P \in r_{12}$

$pow_{\Gamma_2}(P) = pow_{\Gamma_3}(P)$ xché $P \in r_{23}$

$\Rightarrow pow_{\Gamma_1}(P) = pow_{\Gamma_3}(P) \Rightarrow P \in r_{13}$

$\Rightarrow r_{13} \cap r_{12} \cap r_{23} = P \Rightarrow$ concorrono

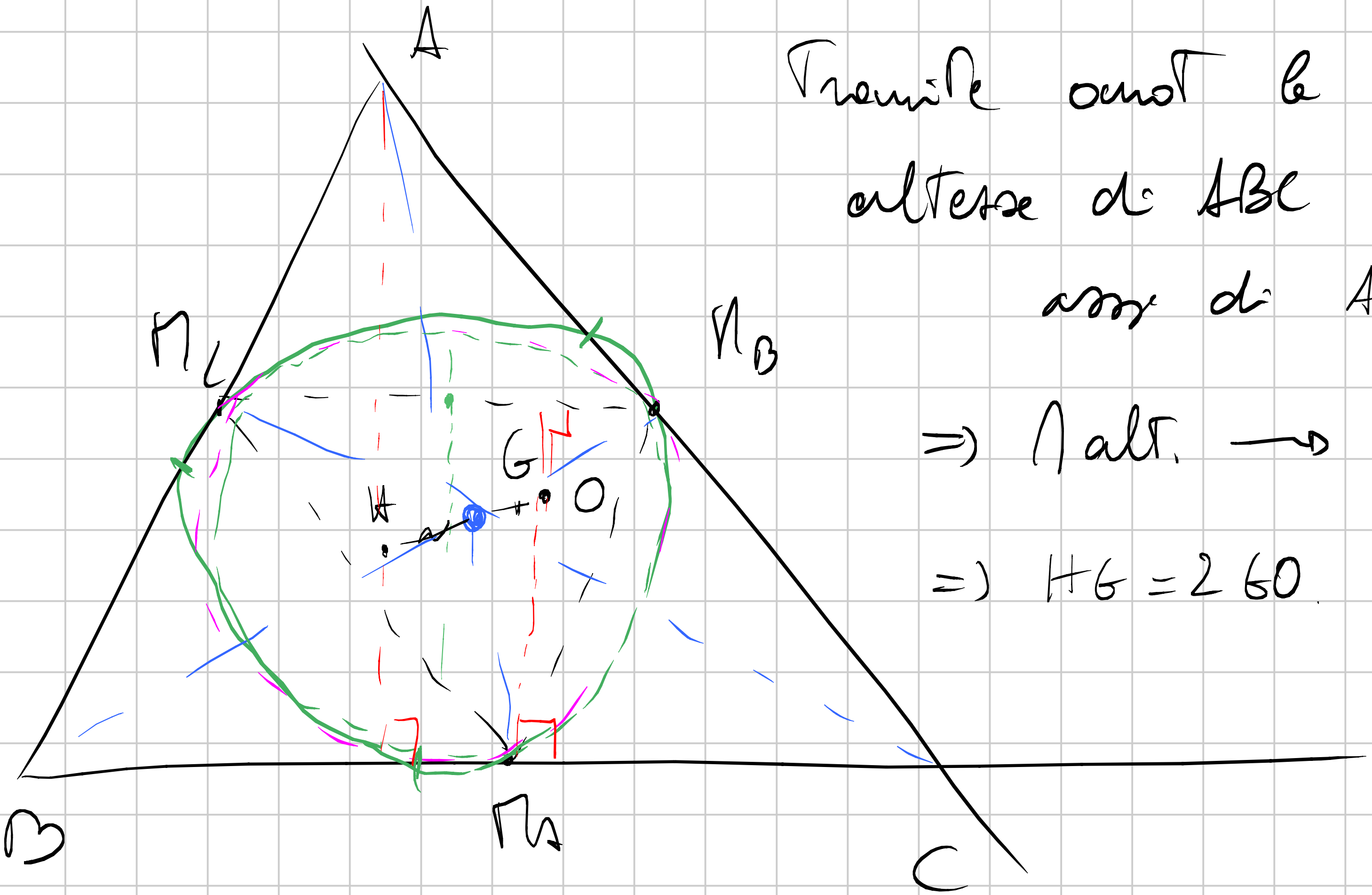
Es: ABC triangolo O, I circoc., incentro

$$\Rightarrow \quad OI^2 = R^2 - 2rR$$

Retta di Eulero

ABC tr. O, G, H allineati in ordine e $2OG = GH$.

Dim: 1) omot. di centro G e fattore $-\frac{1}{2}$ manda A, B, C nei pt. medi dei lati.



Tramite omot le altezze di ABC diventano assi di ABC

$\Rightarrow \cap$ alt. $\longrightarrow \cap$ assi

$\Rightarrow HG = 2GO$

Oss: circo ABC $\leadsto$ circo di $M_A M_B M_C$ O, H, G allineati

$\quad\;\;\, \| \qquad\qquad\qquad\qquad\qquad\quad \|$

$\quad\;\; O \qquad\qquad\qquad\qquad\qquad\;\; N \qquad\quad OG = 2GN$

La circ. circ a $M_A M_B M_C$ passa per:

- i piedi delle altezze
- i punti medi di AH, BH, CH.

Si dice circ. di FEUERBACH.

Teo di TOLOMEO

$ABCD$ punti non tutti allineati

$\Rightarrow AB \cdot CD + BC \cdot DA \geq AC \cdot BD$

Vale $=$ $\Leftrightarrow$ $ABCD$ ciclico

Dim:

inv. risp. a cfr blu

$$ABC \longrightarrow A'B'C'$$

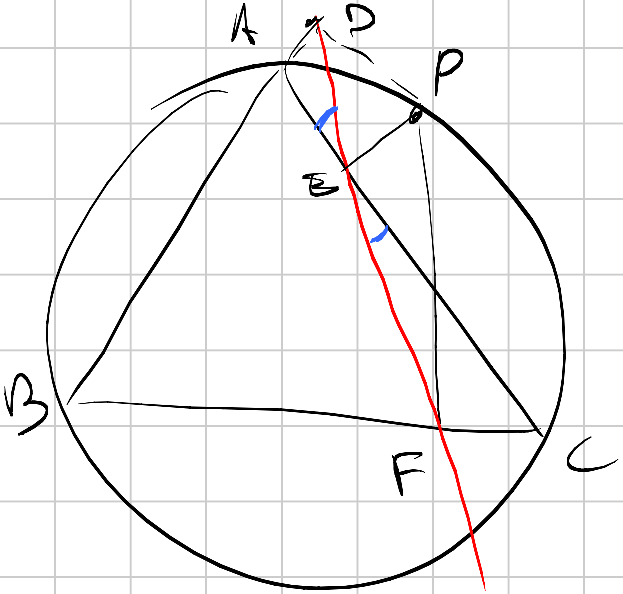
$$(\circledast)\quad A'B' + B'C' \geq A'C'$$

$$\frac{\cancel{R^2}\, AB}{DA \cdot DB} + \frac{BC\, \cancel{R^2}}{DB \cdot DC} \geq \frac{AC\, \cancel{R^2}}{DA \cdot DC}$$

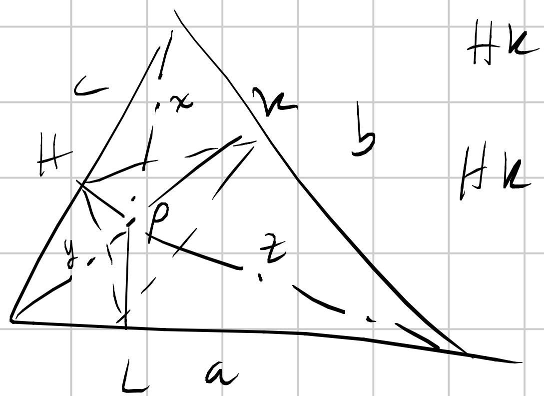
$$AB \cdot DC + BC \cdot DA \geq AC \cdot DB$$

Vale $=$ $\Longleftrightarrow$ $A'B'C'$ all. $\Longleftrightarrow$ ABC conciclici con D.

Teo di Simson



Le proiez. di P sui lati di ABC sono allineate $\iff$ P $\in$ circ. circ. di ABC.



$HK = \dfrac{ax}{2R}$

$HK + KL \geq HL$