ALGEBRA 5

Titolo nota 05/09/2007

a, b, c LATI DI UN TRIANGOLO

$$a^3+b^3+c^3-3abc \geq 2\max\{|a-b|^3, |b-c|^3, |c-a|^3\}$$

$$2\sum_{cyc} a^2 b \geq \sum_{cyc} (ab^2+abc) \qquad a,b,c \in \mathbb{R}_{\geq 0}$$

$$|(a^2-b^2)ab+(b^2-c^2)bc+(c^2-a^2)ac| \leq M(a^2+b^2+c^2)^2$$

$$a, b, c \in \mathbb{R}$$

Hint: Scomporre !!!

(fattorizzare)

- $a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca) =$

$$= (a+b+c)\,\frac{1}{2}\cdot\left((a-b)^2+(b-c)^2+(c-a)^2\right)$$

$$(a+b+c)\left[(a-b)^2+(b-c)^2+(c-a)^2\right] \ge 4\max_{cyc}\left\{|a-b|^3\right\}$$

$a = x+y$

$b = y+z$

$c = z+x$

$$\overset{1}{\cancel{2}}\,\underbrace{(x+y+z)\left[(x-y)^2+(y-z)^2+(z-x)^2\right]}_{f(x,y,z)} \ge \overset{2}{\cancel{4}}\max_{cyc}\left\{|x-y|^3\right\}$$

$$(x,y,z) \to (x+k,\, y+k,\, z+k) \in \mathbb{R}^3_{\ge 0}$$

IL RHS NON CAMBIA

IL II° FATTORI DI LHS NON CAMBIA

CAMBIA SOLO $x+y+z$

$$f(x,y,z) \geq \max_{cyc} \{|x-y|^3\} = g(x,y,z)$$

$$g(x,y,z) = g(x+k,\ y+k,\ z+k)$$

$$k \geq 0$$

$$f(x+k,\ y+k,\ z+k) \geq f(x,y,z) \geq g(x,y,z) = g(x+k,\ y+k,\ z+k)$$

wlog $z = \min\{x,y,z\}$

$$f(x,y,z) \geq f(x-z,\ y-z,\ 0) \geq g(x-z,\ y-z,\ 0) \overset{\downarrow}{=} g(x,y,z)$$

se io dimostro $f(x,y,0) \geq g(x,y,0)$ ho finito

$$(x+y)\left[(x-y)^2+x^2+y^2\right] \geq 2\max\{|x-y|^3, |x|^3, |y|^3\}$$

$$\Uparrow \qquad x^3$$

$x \geq y$ $\Rightarrow \max\{|x-y|^3, |x|^3, |y|^3\} = |x|^3$

$$2(x+y)(x^2-xy+y^2) \geq 2x^3$$

$$2x^3+2y^3 \geq 2x^3$$

$y=0$

$$y^3 \geq 0 \quad \checkmark$$

$$(\ldots)\, x+y+z \geq x+y\,(\ldots)$$

$z=0$ $\quad$ $y=0$ $\quad$ $x=$ qualunque

$(\ldots)=0 \Rightarrow (x-y)^2+(y-z)^2+(z-x)^2=0 \Leftrightarrow x=y=z$

$(a=b\ ,\ c=0)$ $\quad (a=b=c)$

$$\sum_{cyc} a^2b-b^2a = (b-a)(c-b)(a-c)$$

$$\sum_{cyc} a^3b-b^3a = (a+b+c)(b-a)(c-b)(a-c)$$

$$\sum_{cyc} a^nb-b^na = (b-a)(c-b)(a-b)\left(\begin{array}{c}\text{polinom simmetrico}\\ \text{di grado } n-2\end{array}\right)$$

$n=4 \qquad a^2+b^2+c^2-ab-bc-ca$

$$\sum_{cyc} 2a^2b - b^2a - 3abc \geq 0$$

$$\left(\sum_{cyc} a^2b - b^2a\right) + \left(\sum a^2b - 3abc\right) =$$

$$= (\ldots) + \sum a^2b - a^3 + a^3 - \cdot abc =$$

$$= (\ldots\ldots) + \left(\sum_{cyc} a^3 - abc\right) - \left(\sum_{cyc} a^3 - a^2b\right)$$

$$\frac{2a^3 + b^3 - 3a^2b}{3}$$

$$\left(\sum_{cyc} \frac{2a^3+b^3-3a^2b}{3}\right)$$

$$\left(\frac{2a+b}{3}\right)(a-b)^2$$

$$= \left(\; \cdot \; \cdots \; \right) + \sum_{cyc} (a-b)^2 \left(\frac{a+b+c}{2} \right) - \sum_{cyc} (a-b)^2 \left(\frac{2a+b}{3} \right)$$

$$= \underbrace{(b-a)(c-b)(a-c)}_{= P} + \sum_{cyc} (a-b)^2 \left(\frac{b-a}{6} + \frac{c}{2} \right) =$$

$$= (b-a)(c-b)(a-c) + \underbrace{\frac{1}{6} \sum_{cyc} (a-b)^3}_{= \underbrace{\frac{1}{6} \cdot 3 (a-b)(b-c)(c-a)}_{= -\frac{1}{2} P}} + \sum_{cyc} \frac{c}{2} (a-b)^2$$

$$(a-b) + (b-c) + (c-a) = 0$$

$$= \quad + \frac{3}{2} P + \frac{1}{2} \sum_{cyc} c\,(a-b)^3 \geq 0$$

$$\frac{1}{\cancel{2}} \sum_{cyc} c(a-b)^2 \geq -P \cdot \frac{3}{\cancel{2}}$$

$$\underbrace{\sum_{cyc} c(a-b)^2}\geq 3(a-b)(b-c)(c-a)$$

$$\overset{\text{AM-GM}}{\geq} 3\sqrt[3]{abc}\,\left|(a-b)(b-c)(c-a)\right|^{2/3} \overset{H}{\geq} 3\left|(a-b)(b-c)(c-a)\right| \geq$$

$$\geq 3(a-b)(b-c)(c-a)$$

$$abc \overset{H}{\geq} \left|(a-b)(b-c)(c-a)\right|$$

$a=1 \quad b=2 \quad c=0 \qquad 0 \geq 2$

$$(\cancel{x}+\cancel{y})(y+z)(z+x) \geq (x-y)(y-z)(z-x)$$

1)

$$\sum_{sym} x^2 y + 2xyz \geq -\left(\sum_{cyc} x^2 y - y^2 z\right)$$

$$|x+y| \geq |x-y|$$

$$\Big|\underset{\substack{\uparrow\\ S}}{(a+b+c)}\underset{\substack{\uparrow\\ x}}{(a-b)}\underset{\substack{\uparrow\\ y}}{(b-c)}\underset{\substack{\uparrow\\ z}}{(c-a)}\Big| \leq M \underset{\substack{\| \\ \frac{x^2+y^2+z^2+s^2}{3}}}{\left(a^2+b^2+c^2\right)^2}$$

WLOG x, y hanno lo stesso segno

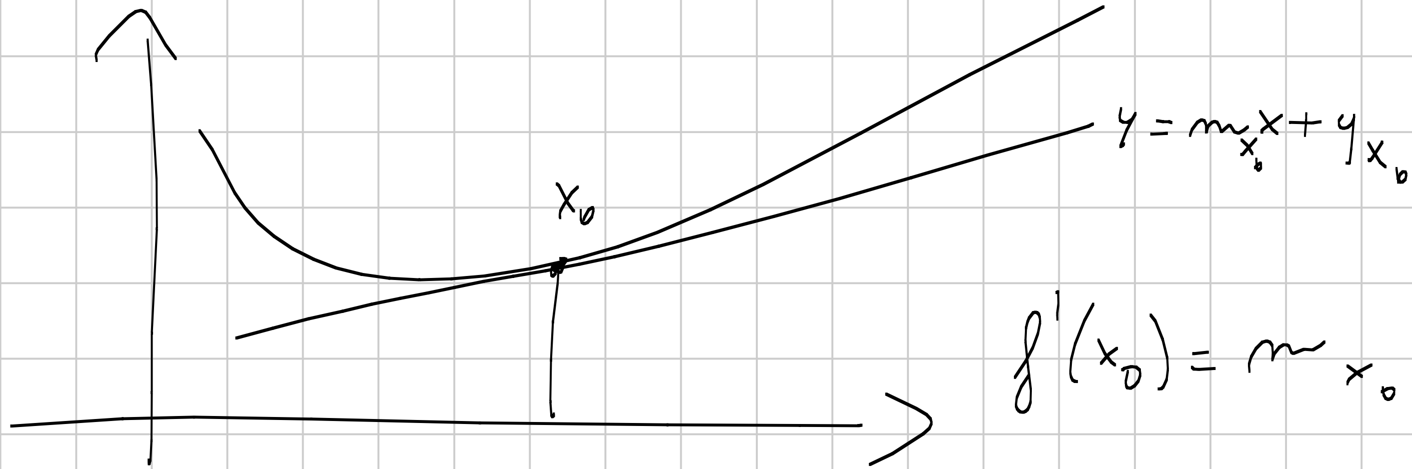
$x+y+z=0 \qquad |z| = |x+y|$

$$5xy(x+y) \le M\left(\frac{2(x^2+xy+y^2)+s^2}{3}\right)^2$$

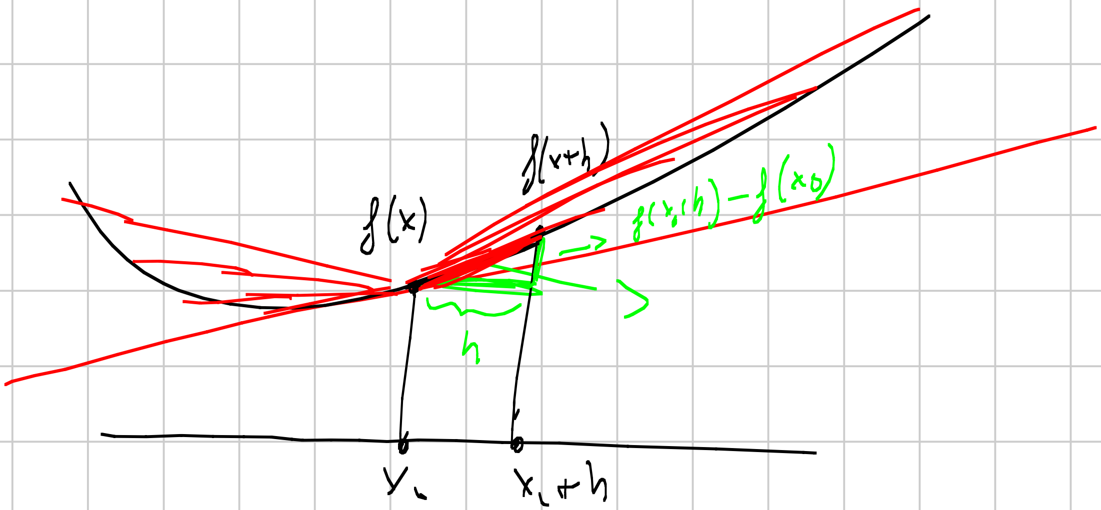
$$x = y = ks \qquad M \overset{?}{=} \frac{9}{4\sqrt{2}}$$

Moltiplicatori di Lagrange

$y = f(x)$



$$f'(x_0) = m_{x_0}$$



$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{(x+h) - x} = f'(x)$$

$$f(x) = x^n \qquad f'(x) = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} =$$

$$= \lim_{h\to 0} \frac{\cancel{x^n} + \cancel{h} x^{n-1} \cdot \binom{n}{1} + h^{\cancel{2}} x^{n-2} \binom{n}{2} + \dots + h^{n}\cancel{-x^{n}}}{\cancel{h}} =$$

$$= \lim_{h\to 0} n x^{n-1} + \underbrace{h\left(- - - - \right)}_{\to 0} = n x^{n-1}$$

$$(f+g)' = f' + g' \qquad (fg)' = f'g + g'f$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2} \qquad f(g(x))' = f'(g(x)) \cdot g'(x)$$

$$f(f^{-1}(x)) = x$$

$$f'(f^{-1}(x))\, f^{-1\,'}(x) = 1 \qquad f^{-1\,'}(x) = \frac{1}{f'(f^{-1}(x))}$$

$f(x)$	$f'(x)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
x^{a}	$a x^{a-1}$
$\ln\lvert x\rvert$	$\frac{1}{x}$
$\operatorname{tg}(x)$	$1 + \operatorname{tg}^2 x$
e^x	e^x

$\forall a \in \mathbb{R}$

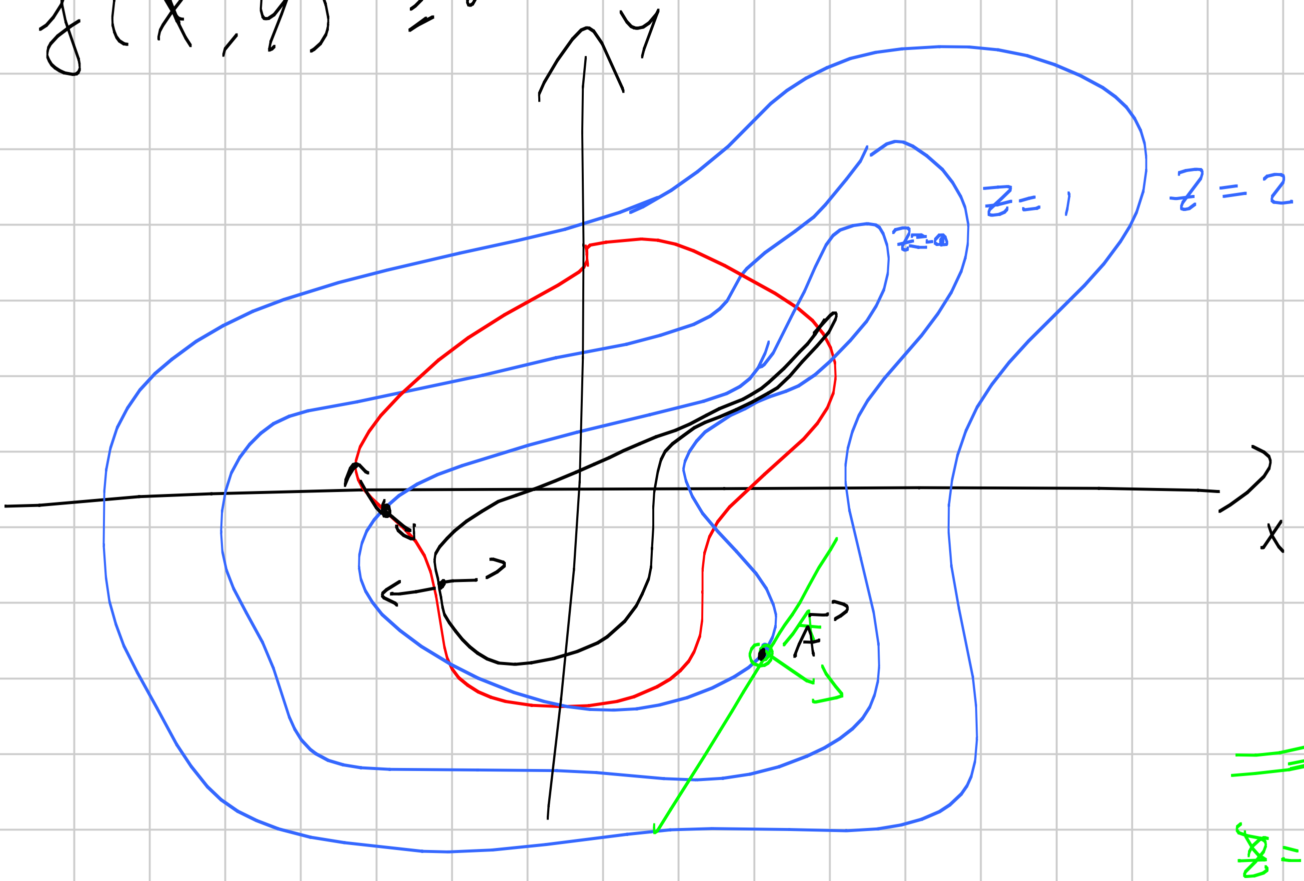
$\operatorname{arctg} x \quad \Big| \quad \frac{1}{1+x^2}$

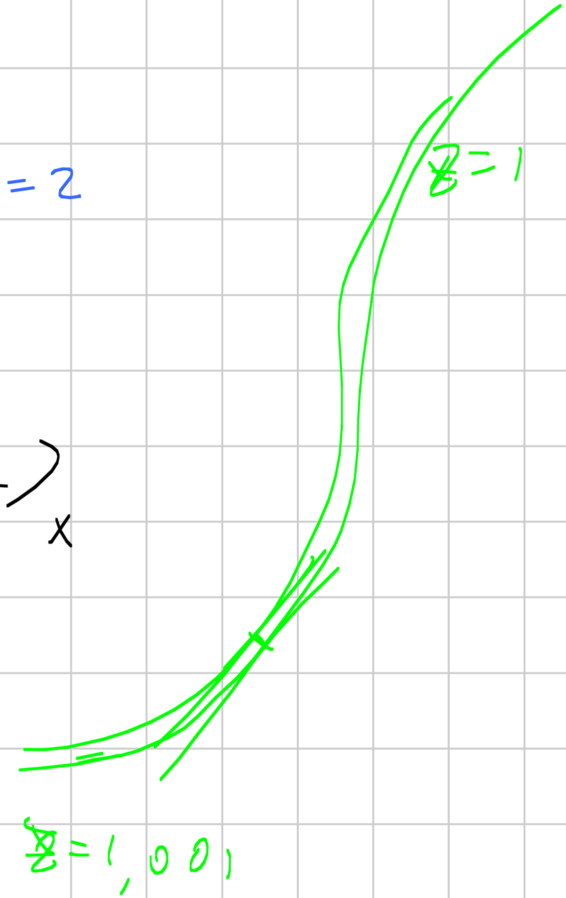
$$g(x_1, x_2, x_3, \dots, x_n) = 0$$

$$g(x, y) = x^2 + y^2 - 1 + \operatorname{tg} x$$

$$\begin{cases} g(x_1, \dots, x_n) = 0 \\ g(x_1, x_2, x_3, \dots, x_n) \geq 0 \end{cases}$$

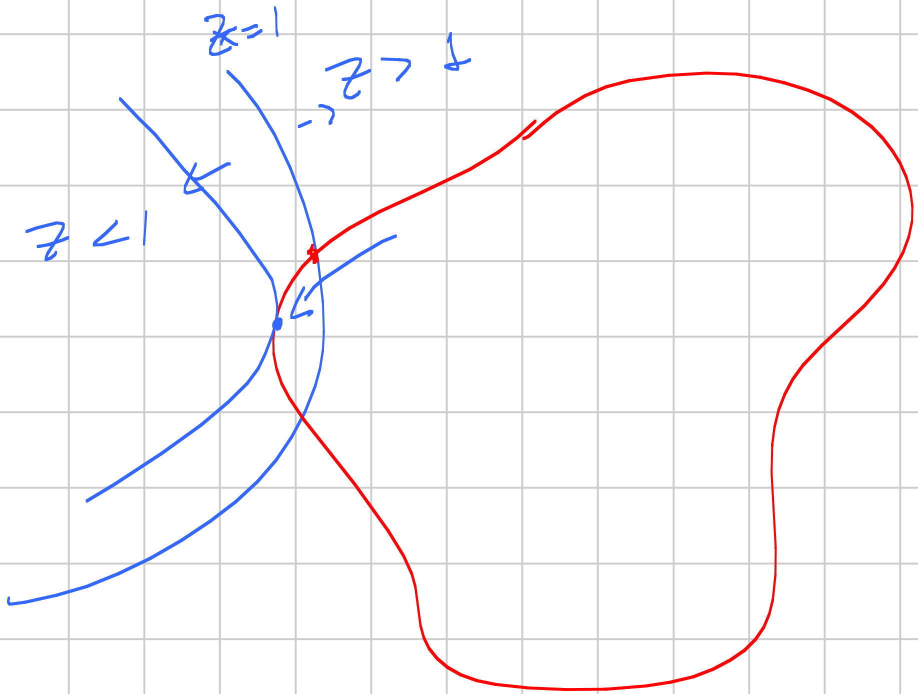
$$\begin{cases} g(x,y) = 0 \\ f(x,y) \geq 0 \end{cases}$$





$$\nabla f(x_0, y_0) = \vec{A}$$

$$\nabla f(x_0, y_0) = k \nabla g(x_0, y_0)$$



$f(x_1, - - - - , x_n)$ $\qquad$ $f: \mathbb{R}^n \to \mathbb{R}$

$$Df := \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix} \qquad \frac{\partial f}{\partial x_1} = f'(x_1)\Big|_{x_2, \dots, x_n}$$

Prendo x_1 come variabile e $x_2, \dots, x_n$ come costanti e derivo rispetto a x_1

$$x_1^2 + x_2^2 + \dots + x_n^2 = f(x_1, x_2, \dots, x_n)$$

$$\frac{\partial f}{\partial x_i} f(x_1, x_2, \dots, x_n) = 2x_i$$

$$\nabla f(x_1, \dots, x_n) = 2 \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Quando possiamo usare i moltiplicatori di Lagrange?

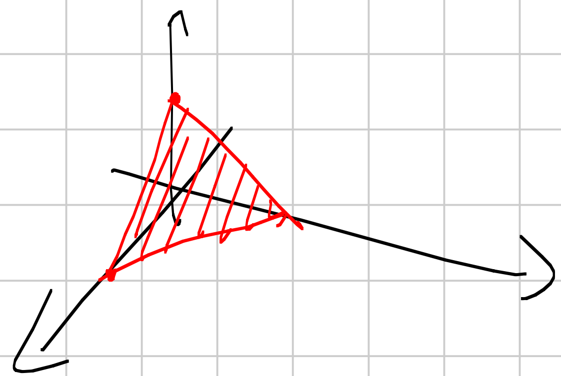
Quando l'insieme in cui variano $x_1, x_2, \dots, x_n$ è chiuso e limitato.

$$\{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 = 1\} = A$$

$\exists \; M$ t.c. $|x_i| < M \quad \forall \; x_i \in A$

$[0, 1) \qquad 1 - \frac{1}{n} \in [0, 1)$

$x + y + z = 1 \quad , x, y, z \geq 0$



$\{(x, y, z) \mid x^2 + y^2 + z^2 < 1\}$ non è chiuso

$0 \le a, b, c \le 1 \qquad a+b+c=1$

$$a^2+b^2+c^2 \le a^2b+b^2c+c^2a+1$$

$$g(a,b,c) = a+b+c-1$$

$$f(a,b,c) = a^2+b^2+c^2-a^2b-b^2c-c^2a-1$$

$$\nabla f = \begin{pmatrix} 2a-2ab-c^2 \\ 2b-2bc-a^2 \\ 2c-2ac-b^2 \end{pmatrix} \qquad \nabla g = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{cases} a+b+c=1 \\ 2a-2ab-c^2=k \\ 2b-2bc-a^2=k \\ 2c-2ac-b^2=k \end{cases}$$

← mi dà solo i punti INTERNI (controllare il bordo)

$$2(a+b+c)-(a+b+c)^2=3k$$

$k=\frac{1}{3}$

$$2(a-b)+2b(c-a)+(a-c)(a+c)=0$$

$$2(a-b)+(a+c-2b)(a-c)=0$$

$$\begin{cases} 2a-2ab-c^2=\frac{1}{3} \\ 2b-2bc-a^2=\frac{1}{3} \\ 2c-2ac-b^2=\frac{1}{3} \end{cases}$$

$2a=2a(a+b+c)$

$$\text{cyc}\left\{ 2a^2+2ac-c^2=\frac{1}{3} \right.$$

Funzione lineare $f(x) = ax + b$

max, min è agli estremi

$x_1 + x_2 + \dots + x_n - x_1x_2 - x_2x_3 - \dots - x_nx_1 \leq \lfloor \frac{n}{2} \rfloor$

$0 \leq x_1, x_2, \dots, x_n \leq 1$

$(x_1, \dots, x_n) = (1, 0, 1, 0, \dots, 1, 0)$

Supp. per assurdo che il max è ottenuto da una n-upla $(x_1, \ldots, x_n)$ t.c. $\exists\, x_i \neq 0,\ x_i \neq 1$

Dire che $\exists$ il sup $\in \mathbb{R}$ prima di usare roba del genere

$$\sum x_i^3 = 0 \quad , \quad x_i \in [-1, 1]$$

$$\Rightarrow \quad \sum x_i \le \frac{n}{3}$$

$$\underbrace{\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, \frac{1}{2}}_{8}, -1, \underbrace{\frac{1}{2}, \ldots \frac{1}{2}}_{8}, -1$$

$$\sum x_i^3 = \Big(\overbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} \ldots}^{8}\Big) - 1 + \Big(\overbrace{\frac{1}{8} + \frac{1}{8} \ldots \frac{1}{8}}^{8}\Big) - 1 \ldots = 0$$

$$\sum x_i = \Big(\underbrace{\frac{1}{2} + \frac{1}{2} \ldots \ldots + \frac{1}{2} - 1}_{3}\Big) + \Big(\underbrace{\ldots\ldots}_{3}\Big) + (\ldots) = \frac{n}{3}$$

noto che l'uguaglianza è in un caso particolare dove $x_i = \{\pm\frac{1}{2}, \pm 1\}$

$$\sum x_i = \sum (x_i + k x_i^3)$$

$x_i + k x_i^3 \in [\quad\quad]$ $\quad\quad$ $x + kx^3 = a(2x-1)^2 q(x) + r(x)$

$x_i \in [-1, 1]$

$k = -\frac{3}{4}$ $\quad\quad$ $x_i - \frac{4}{3} x_i^3 \le \frac{1}{3}$ $\quad\quad$ $\forall\, x \in [-1, 1]$

$$\sum x_i - \frac{4}{3} x_i^3 \le \frac{n}{3}$$

$$x_1 - \frac{4}{3}x_1^3 - \frac{1}{3} = -\frac{4}{3}\left(\cancel{x} - \frac{1}{2}\right)^2 (x-1) \le 0$$

$$x_i \in [-1, 1] \qquad x_i = \sin(\alpha_i)$$

$$\sum \sin^3(\alpha_i) = 0 \qquad \sin(3\alpha) = 4\sin^3(\alpha) - 3\sin(\alpha)$$

$$\sum \sin(3\alpha) - 3\sin\alpha = 0$$

$$\sum x_i = \sum \sin\alpha_i = \frac{\sum \overset{\le 1}{\sin 3\alpha}}{3} \le \frac{n}{3}$$

$$\sin 3\alpha = 1 \qquad \alpha = 30^\circ, 270^\circ \qquad \boxed{\sin\alpha = \begin{cases} \frac{1}{2} \\ -1 \end{cases}}$$

$$P_1(x) = x^2 - 2$$

$$P_{n+1}(x) = P_1(P_n(x))$$

$P(x) = x$ ha 2^n soluzioni reali e distinte

$$P_1(2\cos\alpha) = 2\cos 2\alpha$$

$x^2 - 2$ $\quad$ $2x^2 - 1$

$$P_n(2\cos\alpha) = 2\cos(2^n\alpha)$$

$P_n(x) = x$

$2\cos(\alpha) = 2\cos(2^n \alpha)$

$2^n \alpha = \alpha + 2\pi k$

$2^n \alpha = -\alpha + 2\pi k$

$\alpha = \frac{2\pi k}{2^n - 1} \leftarrow \frac{2^n - 1}{2}$

$\alpha = \frac{2\pi k}{2^n + 1} \leftarrow \frac{2^n + 1}{2}$

$\cos\left(\frac{2\pi k}{2^n - 1}\right) = \cos\left(\frac{2\pi (2^n - 1 - k)}{2^n - 1}\right)$

$$a = \cos\alpha \leftarrow -1 \le a \le 1$$

$$a = \frac{b + \frac{1}{b}}{2} \leftarrow \quad |a| \ge 1$$

$$\cosh(x) = \frac{e^{x} + e^{-x}}{2}$$

$$\cos(x) = \cosh(ix)$$

Funzionali balorde

$f: \mathbb{N} \to \mathbb{N}$ t.c.

$$f(m+f(n)) = f(f(m)) + f(n)$$

Guess: $f(n) \stackrel{?}{=} n$ e $\in \mathbb{N}$

$m=0$ (1) $f(f(n)) = f(\cancel{f(0)}) + f(n)$ $\to$ $\boxed{x \in f(\mathbb{N}) \quad f(x) = \cancel{a} + x}$

$$f(f(m)+f(n)) = f(ff(m))) + f(n) = f(f(m)) + f(n) + f(f_0)$$
$$= f(m) + f(n) + 2f(f(0))$$

$$= f(ff(n))) + f(m)$$

(1) $m = 0 \Rightarrow f(0) = 0$

$$f(\overbrace{f(m) + f(n)}) = \overbrace{f(m) + f(n)}$$

$x \in f(\mathbb{N}) + f(\mathbb{N})$

$f(x) = x$

$$f(m + f(n)) = f(f(m)) + f(n) = f(m) + f(n)$$

se $\underline{m + f(n) \in \overline{f(\mathbb{N})}} \Rightarrow f(m + f(n)) = m + f(n)$

$$\boxed{m + f(n) = f(a)}$$

$$\cancel{f(n)+f(n)} = f(m + f(n)) = f(f(a)) = f(a) = m + \cancel{f(n)}$$

$$f(m) = m \quad \Leftarrow \quad m \in f(\mathbb{N}) - f(\mathbb{N})$$

$$A = \{k \mid f(k) = k\} = d\mathbb{N} = \{dk \mid k \in \mathbb{N}\}$$

$$f(m + f(n)) = f(m) + f(n)$$

$$(1) \quad f(m + kd) = f(m) + kd$$

$0 - f(0), f(1), \ldots, f(d-1)$

$f(1+kd)$ $\qquad$ $f(1+f(kd)) = f(1) + f(kd)$

$f(1+kd) = f(1) + kd$

$$\begin{cases} f(0) = 0 \\ f(1) = k_1 d \\ f(2) = k_2 d \\ \vdots \\ f(d-1) = k_{d-1} d \end{cases}$$

$x \in I_n \Leftrightarrow f(x) = x$

$f(x) = x \Leftrightarrow x \in I_n$

IMO 1996/3

Polinomiali

- Guardare le radici
- Guardare il grado con delle stime
- Provare a sostituire e trovare i coefficienti

$$P(-x) = P(x-1) \qquad P(x) \in \mathbb{R}[x]$$

$P(x) \equiv c$

$P(x) = A(x-1)^{2n}$

$P(x) \equiv B(x^2+x+1)^{\alpha}$

$P(\alpha) = 0 \qquad \alpha \in \mathbb{R}$

α radice $\quad -\alpha - 1$ radice $\quad \alpha$

$$P(x) = Q\left(\left(x + \frac{1}{2}\right)^2 \right)$$

$$P(-x) = P(x-1) \qquad Q(x) = P\left(x - \frac{1}{2}\right)$$

$$Q\left(x + \frac{1}{2}\right) = P(x)$$

$$\underline{Q\left(-x + \frac{1}{2}\right) = Q\left(x - \frac{1}{2}\right)} \qquad -\left(x - \frac{1}{2}\right) = y$$

$$\underline{Q(y) = Q(-y)} \qquad \underline{Q \text{ è } \underline{\text{pari}}} \qquad Q(x) = R(x^2)$$

$f \in \mathbb{R}[x]$ $\qquad a \quad 2a \qquad 3ac + 2a^2 = 0$

se $\forall$ a, b, c t.c. $ab + bc + ca = 0$ $\qquad c = -\frac{2}{3}a$

$$2 f(a+b+c) = f(a-b) + f(b-c) + f(c-a)$$

$f(0) = 0$ $\qquad 2ac + a^2 = 0$

$$2 f(2a + c) = f(c - a) + f(a - c) \qquad a(2c+a) = 0$$

$$\cancel{2} f\left(\frac{3}{2}a\right) = \cancel{f\left(\frac{3}{2}a\right)} + f\left(-\frac{3}{2}a\right)$$

$f(x)$ è pari

$xy + yz + zx = 0$

$f(x) = x^2 \qquad 2(x+y+z)^2 \overset{?}{=} (x-y)^2 + (y-z)^2 + (z-x)^2$

$2x^2 + 2y^2 + 2z^2 + 0 = 2x^2 + 2y^2 + 2z^2 - 0$

$+0 = -0$

$f(x) = x^4 \quad \checkmark$

$f(x) \sim x^6 \quad \Rightarrow \quad 2f\left(\frac{7}{3}a\right) = f(a) + f\left(\frac{5}{3}a\right) + f\left(\frac{8}{3}a\right)$

$f(x) = Ax^n + \text{———} - -$

$A\left(a^n\left(\left(\frac{8}{3}\right)^n + \left(\frac{5}{3}\right)^n + 1 - 2\left(\frac{7}{3}\right)^n\right)\right) - a^{n-1}\left(\ldots \right) = 0$

$$A\left(\left(\frac{?}{3}\right)^n + \left(\frac{5}{3}\right)^n + 1 - 2\left(\frac{7}{3}\right)^n\right) - \frac{1}{a}\left(- - - - \right) = 0$$

$$\left(\frac{8}{7}\right)^{6} > 2 \qquad \left(1+\frac{1}{7}\right)^{6} > 1+\frac{6}{7}$$

$$\left(\frac{8}{3}\right)^{6} > 2\left(\frac{7}{3}\right)^{6}$$

$$8^2 + 5^2 + 3^2 - 2\cdot 7^2$$

$$64 + 25 + 9 - 98 = 0$$

$$4096 + 625 + 81 - 2\cdot 2401$$

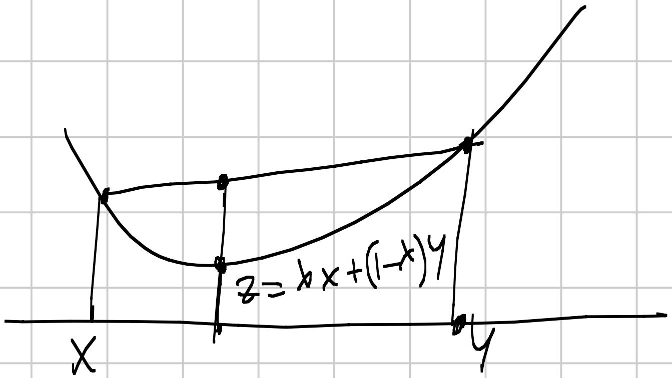
$$4696 + 106 - 4802 = 0$$

$$p(x) = ax^2 + bx^4$$

IMO 2004/2

KARAMATA INEQUALITY

Sia $f(x)$ una funzione convessa



$$f(z) \le \lambda f(x) + (1-\lambda) f(y)$$

$\lambda \in [0,1]$

$$f(\lambda x + (1-\lambda) y) \le \lambda f(x) + (1-\lambda) f(y)$$

$f''(x) \ge 0 \quad \forall x \in [a,b] \implies f(x)$ e' convessa in $[a,b]$

$$\frac{f(x_1) + f(x_2) + \dots + f(x_n)}{n} \ge f\left(\frac{x_1 + x_2 + \dots + x_n}{n}\right)$$

$$x_1 \geq x_2 \geq x_3 \geq x_4 - \cdots \geq x_n$$

$$y_1 \geq y_2 \geq y_3 - \cdots \geq y_n$$

$$x_1 \geq y_1$$

$$x_1 + x_2 \geq y_1 + y_2$$

$$\vdots$$

$$x_1 + x_2 + x_3 + \cdots + x_{\cancel{k}} \geq y_1 + y_2 + \cdots y_k \quad \forall 1 \leq k \leq n$$

$$x_1 + x_2 + x_3 + \cdots + x_n = y_1 + y_2 + \cdots + y_n$$

$$\underbrace{(x_1, x_2, \cdots, x_n)}_{X} \succ \underbrace{(y_1, y_2, \cdots, y_n)}_{Y}$$

Se f è convessa e $X \succ Y$

$$\Rightarrow \quad f(x_1) + f(x_2) + \dots + f(x_n) \geq f(y_1) + f(y_2) + \dots + f(y_n)$$

Dim. I° passo. $n = 1 \Rightarrow$ ovvio

$n = 2$ $\quad x_1 + x_2 = y_1 + y_2$

$x_1 \geq y_1$ $\quad\quad f(x_1) + f(x_2) \geq f(y_1) + f(y_2)$

$x_1 \geq x_2 \quad y_1 \geq y_2$

$$x_2 \qquad y_2 \qquad y_1 \qquad x_1$$

$$y_1 = \lambda x_1 + (1-\lambda) x_2$$

$$y_2 = \lambda x_2 + (1-\lambda) x_1$$

$$f(y_1) = f(\lambda x_1 + (1-\lambda) x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$$

$$f(y_2) \; f(\lambda x_2 + (1-\lambda) x_1) \leq (1-\lambda) f(x_1) + \lambda f(x_2)$$

$$f(y_2) + f(y_1) \leq f(x_1) + f(x_2)$$

Passo induttivo: $P(n) \cup P(2) \Rightarrow P(n+1)$

Lemma. se $(x_1, x_2, \dots, x_{n+1}) \succ (y_1, y_2, \dots y_{n+1})$

$\Rightarrow \exists \; k$ t.c. $x_k \geq y_{k+1} \geq x_{k+1}$

$\Rightarrow (x_1, x_2, \dots, x_k + x_{k+1} - y_{k+1}, x_{k+2}, \dots x_{n+1}) \succ_1$

$\succ (y_1, y_2, \dots, y_{k-1}, y_{k+1}, \dots, y_{n+1})$

Supponiamo per assurdo che $\forall k \; x_k < y_{k+1}$

oppure $x_{k+1} > y_{k+1}$

Sicuramente $x_1 \geq y_1$. Voglio dimostrare che

allora $x_i \geq y_i \;\; \forall i$

Passo base $i = 1$ ce l'ho!

Passo induttivo

$x_k < y_{k+1} \; \vee \; \boxed{x_{k+1} > y_{k+1}}$

$x_k \geq y_k$

$y_k \geq y_{k+1}$

$x_k \geq y_k \geq y_{k+1}$

$$x_1 + x_2 + x_{\ldots} \quad + x_n = y_1 + y_2 + y_3 - \quad + y_n$$

$x_1 = y_1$
$x_2 = y_2 \quad -$

$\exists \; k \text{ s.t. } x_k \geq y_{k+1} \geq x_{k+1}$

$$x_1 \geq x_2 \geq x_3 - - \geq x_{k-1} \geq x_k + x_{k+1} - y_{k+1} \geq x_{k+2} \geq - -$$

$x_{k+1} - y_{k+1} \leq 0$ $\quad$ $x_k - y_{k+1} \geq 0$

$$y_1 \geq y_2 \geq y_3 - - - - - - \geq y_{k-1} \geq y_{k+1} \geq - - \geq y_{n+1}$$

$$x_1 \geq y_1$$

$$x_1 + x_2 \geq y_1 + y_2$$

$$\vdots$$

$$x_1 + \cdots x_{k-1} \geq y_1 + y_2 \cdots + y_{k-1}$$

$$x_1 + x_2 \cdots x_{k-1} + x_k + x_{k+1} - y_{k+1} \geq y_1 + \cdots + y_k$$

$$\vdots$$

$$f(x_1) + \cdots + f(x_{k-1}) + f(x_k + x_{k+1} - y_{k+1}) + f(x_{k+1}) \cdots + f(x_n) \geq$$

$$f(y_1) + f(y_2) \cdots + f(y_{k-1}) + f(y_{k+1}) + \cdots + f(y_{n+1})$$

$$f(x_k + x_{k+1} - y_{k+1}) \le f(x_k) + f(x_{k+1}) - f(y_{k+1})$$

$$f(x_k + x_{k+1} - y_{k+1}) + f(y_{k+1}) \le f(x_k) + f(x_{k+1})$$

Vera perché ipotesi induttiva per $n=2$

x_{k+1} $\quad$ $x_k + x_{k+1} - y_{k+1}$ $\quad$ y_k $\quad$ x_k

Disuguaglianza di popoviciu

$$f(a)+f(b)+f(c)+3f\left(\frac{a+b+c}{3}\right)\geq 2f\left(\frac{a+b}{2}\right)+2f\left(\frac{b+c}{2}\right)+2f\left(\frac{c+a}{2}\right)$$

$$a \geq b \geq \frac{a+b+c}{3} \geq c$$

$$a \geq \frac{a+b+c}{3} \geq b \geq c$$

$$f(x) = \cancel{x}e^{2x} \qquad f''(x) = 4e^{2x}$$

e^x, $\frac{1}{x}$, x^{2n} sono convesse

$$a^2+b^2+c^2+3\sqrt[3]{a^2b^2c^2} \geq 2(ab+bc+ca)$$

1. Dire se $\exists$ $f : \mathbb{Q} \to \{-1, 1\}$ t.c.

se $x \neq y \in \mathbb{Q}$ $\quad x+y=1 \lor xy=1 \lor x+y=0$

$$\Rightarrow \quad f(x)f(y) = -1$$

2. $f : \mathbb{R} \to \mathbb{R}$ dire tutte le f tali che

$$f(x - f(y)) = f(f(y)) + x f(y) + f(x) - 1$$

3. $f : \mathbb{R} \to \mathbb{R}$

$$f(f(x) + y) = f(f(x) - y) + 4 f(x) y$$

4. Sia $n \geq 3$ Siano $t_1, t_2, \ldots, t_n \in \mathbb{R}_{>0}$ t.c.

$$n^2 + 1 > (t_1 + t_2 + \cdots + t_n)\left(\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \cdots + \frac{1}{t_n}\right)$$

Dimostra che t_i, t_j, t_k sono lati di un triangolo $\forall\ i, j, k$ t.c. $1 \leq i < j < k \leq n$