FUNZIONI GENERATRICI

a_n = n° richiesti per il parametro n

$$\boxed{\sum_{n=0}^{\infty} a_n t^n}$$

$$(1+t)^n = \sum_{k=0}^{n} \binom{n}{k} t^k$$

a_k = n° dei sottoinsiemi di k el. di $\{1, \dots, n\}$

F_n = n° numero di Fibonacci

$$F(t) = \sum_{n \geq 0} F_n t^n$$

$F_0 = 0 \quad F_1 = 1 \quad F_n = F_{n-1} + F_{n-2} \quad (n \geq 2$

$$\sum_{n \geq 2} F_n t^n = t \sum_{n \geq 2} F_{n-1} t^{n-1} + t^2 \sum_{n \geq 2} F_{n-2} t^{n-2}$$

$$F(t) - 1 - t = t\left(F(t) - 1\right) + t^2 F(t)$$

Risolvo: $\quad F(t) = \dfrac{1}{1-t-t^2} = \dfrac{1}{(1-\alpha t)(1-\beta t)} \quad (\alpha \neq \beta)$

$$= \frac{A}{1-\alpha t} + \frac{B}{1-\beta t} = A\left(\sum \alpha^n t^n\right) + B\left(\sum \beta^n t^n\right)$$

NUMERI DI CATALAN

Problema delle parentesi.

n variabili $x_1, \dots, x_n$

operazione fra le variabili

$n = 4$ $\quad x_1((x_2x_3)x_4) \quad x_1(x_2(x_3x_4))$

$(x_1x_2)(x_3x_4) \quad ((x_1x_2)x_3)x_4$

$(x_1(x_2x_3))x_4$

5 tipi di associazioni possibili

c_n = n° di modi di mettere le parentesi su n variabili

$$C(t) = \sum_{n \geq 1} c_n t^n \qquad c_0 = 0$$

$$c_n = c_1 c_{n-1} + c_2 c_{n-2} + \dots + c_{n-1} c_1$$

$$t + [C(t)]^2 = C(t)$$

$$C(t) = \frac{1 \pm \sqrt{1-4t}}{2}$$

i coefficienti devono essere positivi $\to$ segno giusto è $(-)$.

$$(1-4t)^{1/2} = \sum_{n \geq 0} \binom{1/2}{n} (-4t)^n$$

$$\binom{1/2}{n} = \frac{\frac{1}{2} \cdot \left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right) \cdots \left(\frac{1}{2}-(n-1)\right)}{n!}$$

$$\binom{1/2}{n}(-4)^n = -\frac{1 \cdot 3 \cdots (2n-3)}{n!} 2^n$$

$$= -\frac{1}{n} \frac{1 \cdot 3 \cdots (2n-3)}{(n-1)!} \frac{(n-1)!}{(n-1)!} 2^n = -\frac{1}{n} \frac{(2n-2)!}{(n-1)!\,(n-1)!} \, 2$$

$$= -\frac{2}{n} \binom{2n-2}{n-1}$$

$$c_n = \frac{1}{n} \binom{2n-2}{n-1}$$

$$\frac{1}{n+1}\binom{2n}{n} \in \mathbb{Z}$$

$$-\ \binom{2n}{n} + \binom{2n}{n-1} = \binom{2n+1}{n}$$

$$-\ \binom{2n}{n} = \frac{n+1}{n}\binom{2n}{n-1} \quad \left[= \frac{n+1}{n} \cdot \frac{2n \cdots 2n-n+2}{1 \cdot 2 \cdots n-1} \right]$$

$$x = \binom{2n}{n} \qquad x + \frac{n}{n+1}x = y$$

$$\frac{2n+1}{n+1}x = y \qquad (2n+1)x = y(n+1)$$

$$(n+1,\ 2n+1) = 1$$

$$n+1 \mid (2n+1)x \Rightarrow n+1 \mid x$$

FUNZIONI GENERATRICI ESPONENZIALI

$$F(t) = \sum_{n\geq 0} a_n \frac{t^n}{n!}$$

Permutazioni zig-zag

a_n n° permut. z-z con $\sigma(1)<\sigma(2)$

$\downarrow$

$2a_n$ = totale

$$\sigma(1)<\sigma(2)>\sigma(3)<\sigma(4)> \dots$$

$$\sigma(1)>\sigma(2)<\sigma(3)>\sigma(4)< \dots$$

perm. z-z su n+1 elementi.

il n° n+1 sta in una certa posizione

a sinistra ci sono k numeri

a destra " n−k numeri

$$2a_{n+1} = \sum_{k=0}^{n} \binom{n}{k} a_k a_{n-k} \qquad n > 0$$

$$a_0 = 1 \quad a_1 = a_2 = 1$$

$$\sum_n 2a_{n+1} \frac{t^n}{n!} = 1 + \sum_{n\geq 0} \left(\sum_{k=0}^{n} \frac{a_k}{k!} \frac{a_{n-k}}{(n-k)!} \right) t^n$$

$$A(t) = \sum a_n \frac{t^n}{n!} \qquad \frac{1}{k!(n-k)!} = \frac{1}{n!}\binom{n}{k}$$

$$2A'(t) = 1 + [A(t)]^2$$

$$\frac{d}{dt} a_{n+1} \frac{t^{n+1}}{(n+1)!} = a_{n+1} \frac{t^n}{n!}$$

$$2 \frac{A'(t)}{[A(t)]^2 + 1} = 1 \qquad 2 \frac{d}{dt} \operatorname{arctg} A(t) = 1$$

$$\cancel{2}\ \operatorname{arctg} A(t) = t + \text{cost}.$$

$$A(t) = \operatorname{tg}\left(\frac{t}{2} + \frac{\pi}{4}\right)$$

PARTIZIONI di n $\qquad P_n$

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k > 0$$

$$\lambda_1 + \lambda_2 + \dots + \lambda_k = n$$

$P_{n,k}$ = n° di partizioni di n in esattamente k addendi

$$P_n = P_{n,1} + \dots + P_{n,n}$$

$$P_{n,k} = \underbrace{P_{n-k,1} + P_{n-k,2} + \dots}_{} + P_{n-k,k}$$

(1 in ogni scatola e le altre in $1, 2, \dots, k$ scatole)

$$P_{n,k} = P_{n-1,k-1} + P_{n-k,k}$$

$$F(t,u) = \sum_{n,k} P_{n+k,k}\, t^n u^k$$

$$P_{n+k,k}\, t^n u^k = P_{n+\underbrace{k-1}, \underbrace{k-1}}\, t^n u^k + P_{n\cdot\ ,k}\, t^n u^k$$

$$\underline{F(t},u) = \underline{u\, F(t},u) + F(t,tu)$$

$$F(t,u) = \frac{1}{1-u} F(t,tu) = \frac{1}{1-u}\cdot\frac{1}{1-tu} F(t,t^2u)$$

$$= \frac{1}{1-u}\cdot\frac{1}{1-tu}\cdots\frac{1}{1-t^j u}\cdot F(t,t^{j+1}u) \qquad = 1 + \text{termini di grado elevato}$$

$$\downarrow$$

$$= \prod_{k\geq 0} \frac{1}{1-t^k u} \qquad 1 \quad (t<1 \quad k\to\infty \quad F(t,0)\to 1)$$

$$u = t \qquad F(t,t) = \sum_{n,k} P_{n+k,k}\, t^{n+k}$$

$$n+k=m \qquad \sum_m \sum_{k=0}^{m} P_{m,k}\, t^m = \sum_m P_m t^m$$

$$F(t) = \sum P_n t^n = \prod_{k \geq 1} \frac{1}{1-t^k} = \frac{1}{1-t} \cdot \frac{1}{1-t^2} \cdot \frac{1}{1-t^3} \cdots$$

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots$$

$$\frac{1}{1-t^2} = 1 + t^2 + t^4 + \cdots$$

$$\frac{1}{1-t^3} = 1 + t^3 + t^6 + \cdots$$

termini di grado n $\qquad t^{x_1 + 2x_2 + 3x_3} \cdots$

$$x_1 + 2x_2 + 3x_3 + \cdots = n$$

PARTIZIONI IN PARTI DISTINTE

$$\| \quad \underline{(1+t)(1+t^2)(1+t^3)} \cdots$$

PARTIZIONI CON PARTI DISPARI

$$\frac{1}{1-t} \cdot \frac{1}{1-t^3} \cdot \frac{1}{1-t^5} \cdots$$

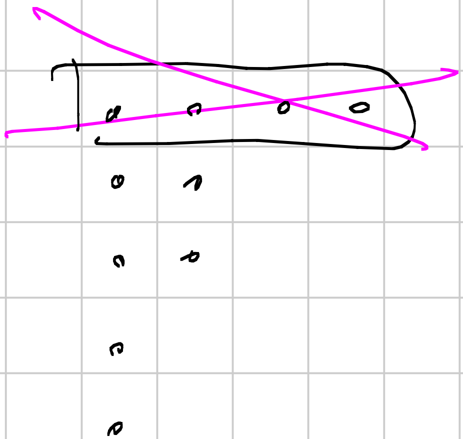
$$1+t = \frac{1-t^2}{1-t} \qquad 1+t^2 = \frac{1-t^4}{1-t^2} \qquad 1+t^3 = \frac{1-t^6}{1-t^3}$$

IL NUMERO DI PARTIZIONI DI n CON NESSUNA PARTE $> k$ È $P_{n+k,k}$.

($n+k$ in esattamente k parti)

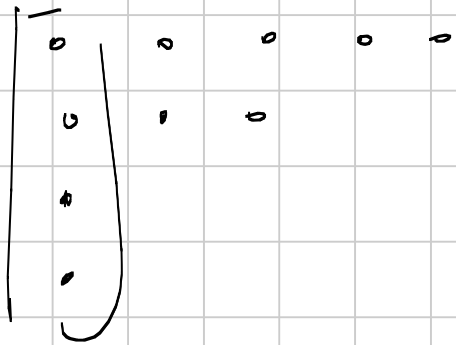
$n = 6 \qquad k = 4$

nessuna parte > 4



$P_{10,4}$

$10 = 5 + 3 + 1 + 1$



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Numeri di Bell B_n

(Partizioni di un insieme di n elementi)

FORMULA RICORSIVA

$$\boxed{B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k}$$

$|X| = n$

$Y = X \cup \{x\}$

$Y = Y_1 \cup \dots \cup Y_m$, $x \in Y_1$

x sta in un certo sottoinsieme con $1 \leq n°el. \leq n+1$

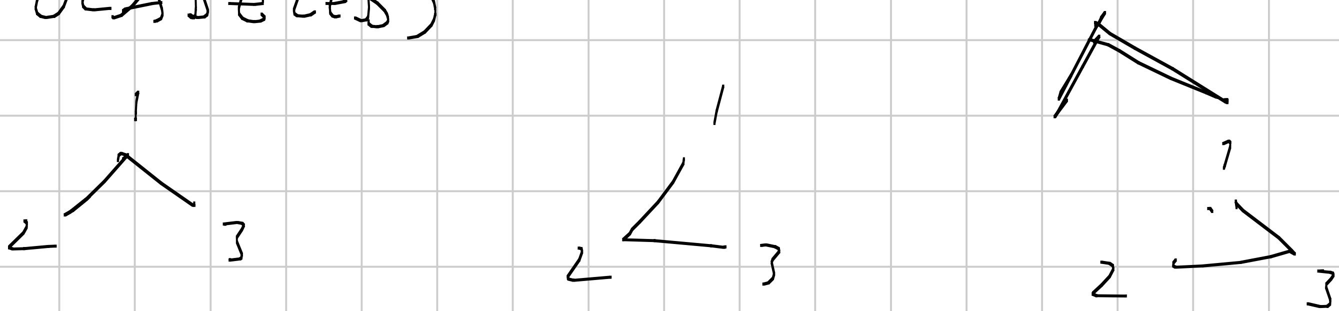
$n+1-k$

con $0 \leq k \leq n$

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{n-k} B_k$$

$$F(t) = \sum B_n \frac{t^n}{n!} = e^{e^t - 1}$$

ALBERI CON n VERTICI
↓
(LABELED)



vertice v_i $\to$ variabile t_i

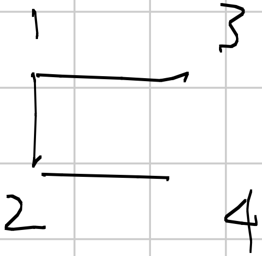
grafo $\xrightarrow{\alpha}$ monomio $t_1^{d_1} \cdots t_n^{d_n}$

n vertici

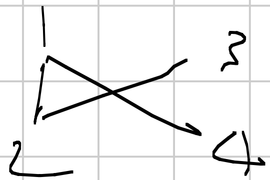
d_i = grado di v_i

- Si ottengono solo monomi di grado $= 2n-2$
- " " " " divisibili per $t_1 \cdots t_n$

è una funzione iniettiva?



$$F(t_1,\dots,t_n) = \sum_{d_1,\dots,d_n} T(n; d_1,\dots,d_n)\, t_1^{d_1} \cdots t_n^{d_n}$$



$$T(n; d_1, \dots, d_n) = T(n-1; d_1 - 1, d_2, \dots, d_{n-1}) + T(n-1; d_1, d_2 - 1, \dots, d_{n-1}) + \dots + T(n-1; d_1, \dots, d_{n-2}, d_{n-1} - 1)$$

$$\boxed{d_1 \geq d_2 \geq \dots \geq d_n = 1}$$

$$T(3; 2, 1, 1) = \underbrace{T(2; 1, 1)}_{= 1} + \underbrace{T(2; 2, 0)}_{= 0}$$

$\boxed{\text{Tesi}}$

$$T(n; d_1, \dots, d_n) = \binom{n-2}{d_1 - 1, \dots, d_{n-1}}$$

$$= \frac{(n-2)!}{(d_1 - 1)! \dots (d_{n-1})!} \qquad \times \qquad (d_1 - 1) + \dots + (d_n - 1) = n - 2$$

$$\underbrace{t_1 \dots t_n \sum T(n; d_1, \dots, d_n)\, t_1^{d_1 - 1} \dots t_n^{d_n - 1}}_{= (t_1 + \dots + t_n)^{n-2}}$$

$t_i = 1$ $\quad n^{n-2}$

$$S(t) = \sum_{k=0}^{n} s_k t^k = \prod_{i=1}^{n} (1 + x_i t)$$

s_k ↓ simmetriche elementari

$$s_1 = x_1 + \dots + x_n$$

$$s_2 = \sum_{i<j} x_i x_j$$

$$P(t) = \sum_{r \geq 1} p_r t^{r-1} = \sum_{r \geq 1} \sum_{i=1}^{n} x_i^r t^{r-1}$$

$$p_r = x_1^r + \dots + x_n^r$$

$$= \sum_{i=1}^{n} \sum_{r \geq 1} x_i^r t^{r-1} = \sum_{i=1}^{n} \frac{x_i}{1 - x_i t} =$$

$$= \sum_{i=1}^{n} \frac{d}{dt} \log \frac{1}{1 - x_i t}$$

$$\log(1 - x_i t) \underset{=}{\longrightarrow} \log \frac{1}{1 - x_i t}$$

$$\underline{P}(-t) = \frac{d}{dt} \log \prod (1 + x_i t)^{-1}$$

$$= \frac{d}{dt} \log \frac{1}{S(t)} = - \frac{S'(t)}{S(t)}$$

$$P(-t)\, S(t) = -S'(t)$$

termine di grado $m = n-1$

$$\pm p_r t^{r-1} \qquad s_k t^k$$

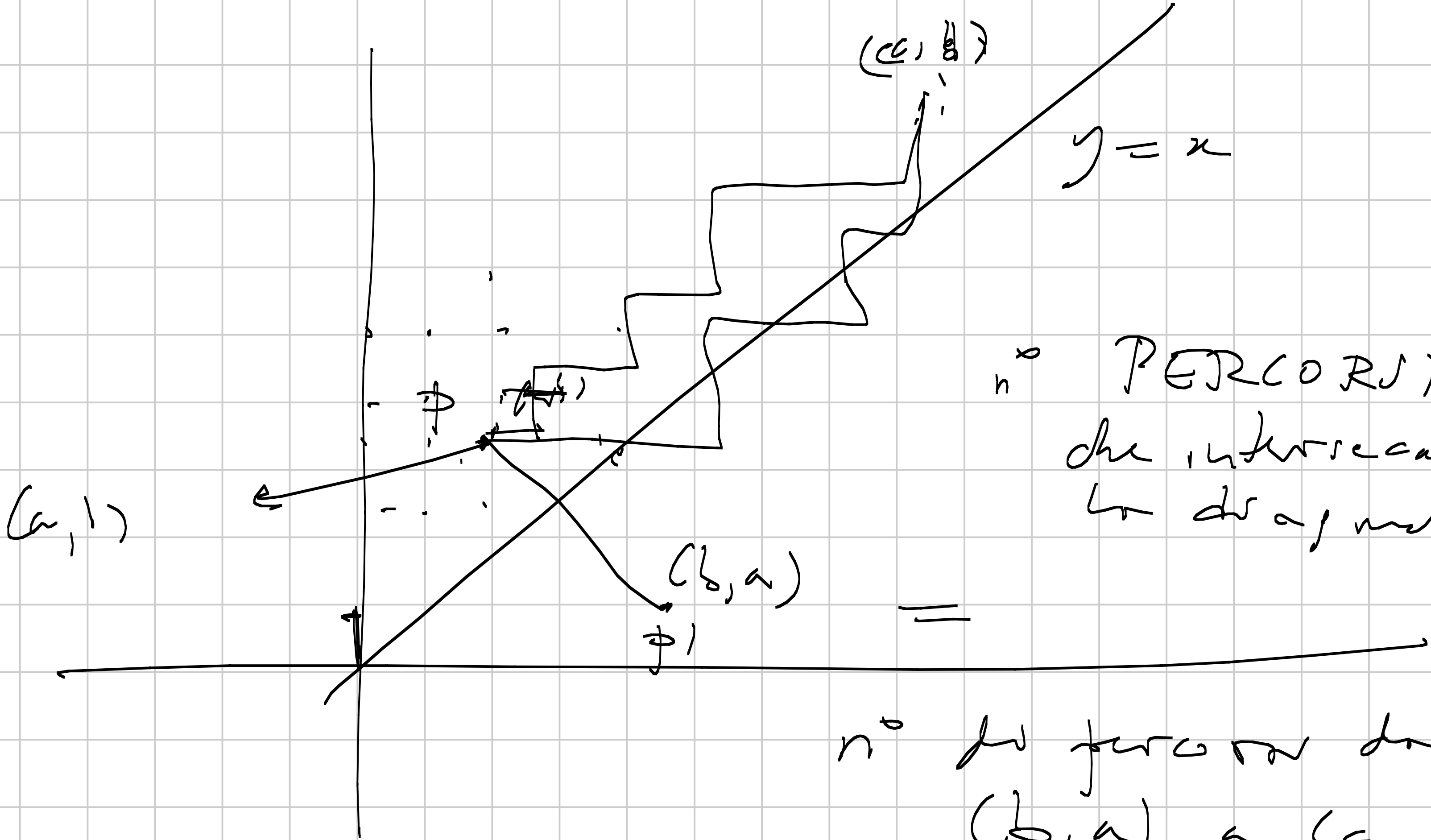
$$r + k = n$$

$$\sum_{r=1}^{n} (-1)^{r-1} p_r s_{n-r} = n s_n$$

$$= p_0 s_n$$

$$\frac{d}{dt} s_n t^n = n s_n t^{n-1}$$

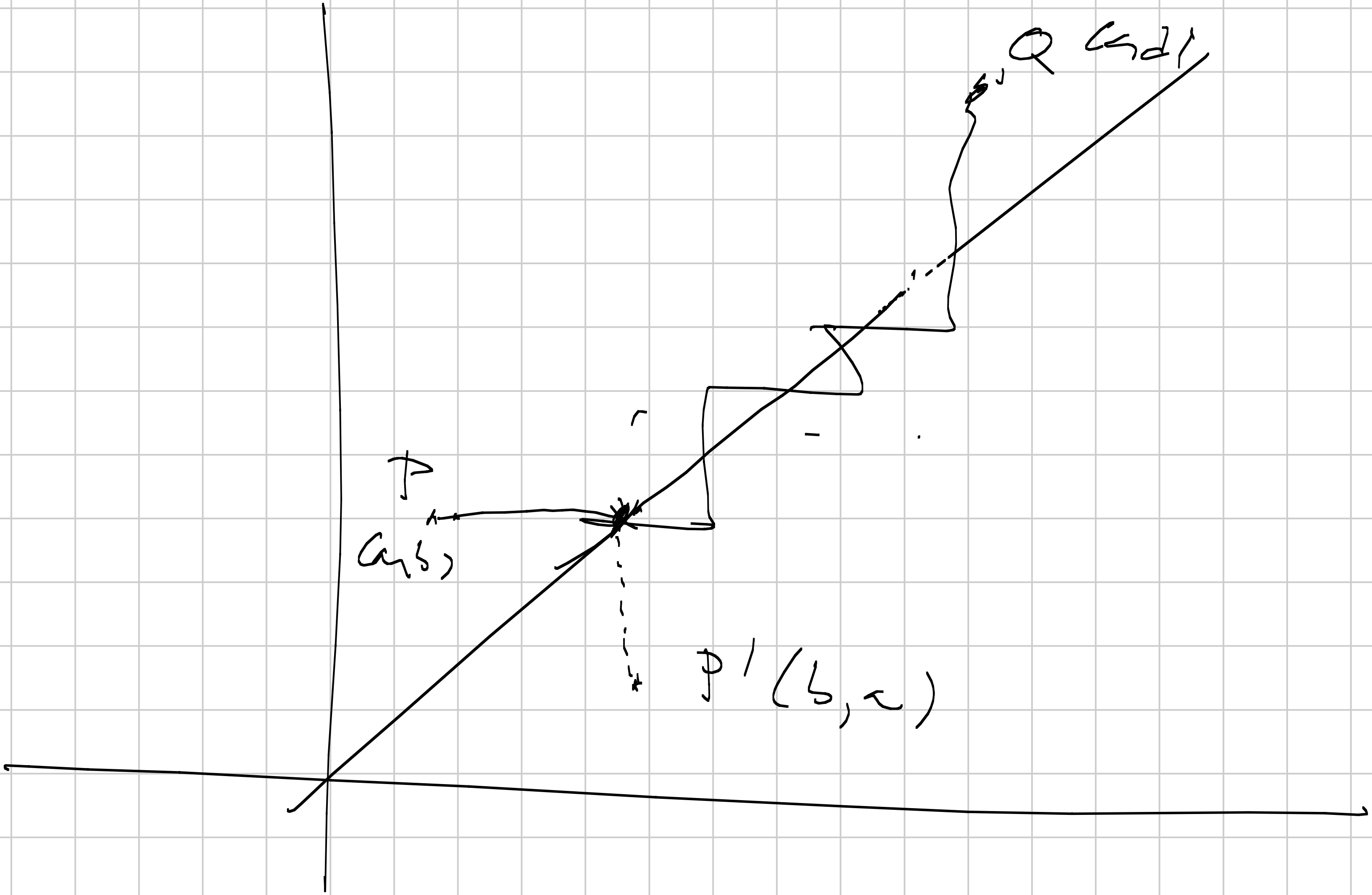
$$\boxed{p_0 s_n - p_1 s_{n-1} + p_2 s_{n-2} - \cdots = 0}$$



n° PERCORSI che intersecano la diagonale

=

n° dei percorsi da (b,a) a (c,d)



Q (c,d)
P (a,b)
P' (b,a)