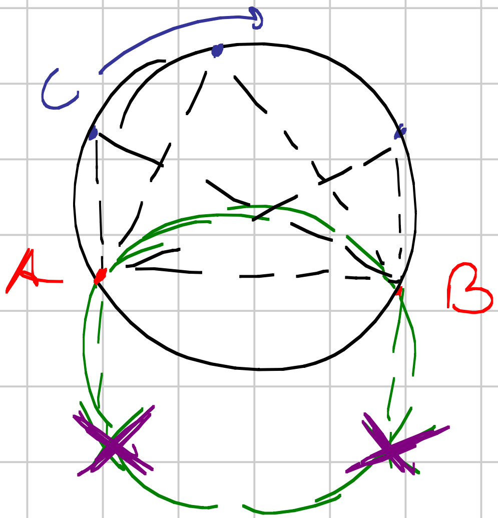
GEOMETRIA 2

Titolo nota 05/09/2007

1. Coordinate cartesiane
2. Vettori
3. Numeri Complessi

I - Luoghi geometrici

$\mathcal{L} = \{ P \text{ t.c. } \text{"} \ldots P \ldots \text{"} \}$



$P : (x, y) \qquad f(x, y) = 0$

$\mathcal{L} = \{ P \mid PA = PB \}$ A, B fissati

asse del segmento AB

$P : (x, y)$

$A : (a_1, a_2)$

$B : (b_1, b_2)$

$$\sqrt{(x-a_1)^2 + (y-a_2)^2} = \sqrt{(x-b_1)^2 + (y-b_2)^2}$$

$$(x-a_1)^2 + (y-a_2)^2 = (x-b_1)^2 + (y-b_2)^2$$

$$2x(a_1 - b_1) + 2y(a_2 - b_2) - a_1^2 - a_2^2 + b_1^2 + b_2^2 = 0$$

<u>Es</u>: A, B punti fissati $r \parallel AB$. Descrivere, al variare di C su r, il luogo degli ortocentri di ABC.

- **Un problema invarianti per affinità**

Trasformazione del piano (affine)

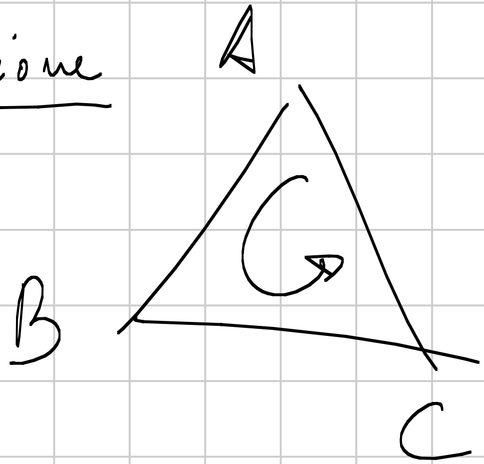
- manda rette in rette
- è bigettiva
  (⇒ conserva parallelismo e concorrenza)
- conserva il rapporto tra segmenti allineati
  (⇒ anche tra segmenti paralleli)
- conserva i rapporti tra aree.

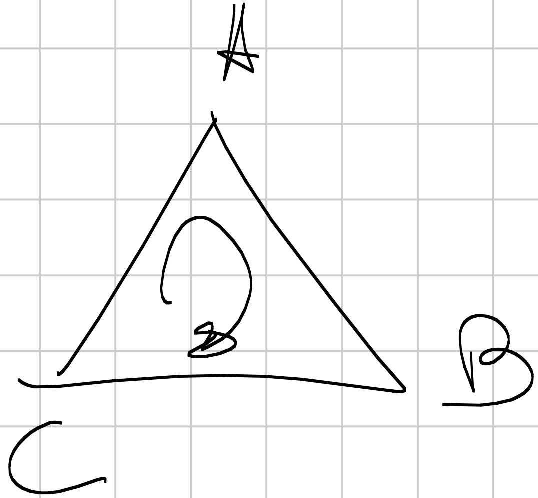
$$(x, y) \longrightarrow (ax + by + c,\ dx + ey + f)$$

con $ae - bd \neq 0$ (determinante della trasf.)

$$\begin{cases} x' = ax + by + c \\ y' = dx + ey + f \end{cases} \qquad ae - bd \neq 0$$

<u>Orientazione</u>





$$\begin{cases} x' = x \\ y' = -y \end{cases} \qquad \det = -1$$

<u>Proprietà</u> (importantissima): Comunque scelgo 3 punti non allineati esiste $\overline{F}$ affinità A-C

$F(A) = (0,0) \qquad F(B) = (1,0) \qquad F(C) = (0,1)$

<u>Dim</u>: $(a_1, a_2) \qquad (b_1, b_2) \qquad (c_1, c_2)$

$$\begin{cases} 0 = a a_1 + b a_2 + c \\ 0 = d a_1 + e a_2 + f \\ 1 = a b_1 + b b_2 + c \\ 0 = d b_1 + e b_2 + f \\ 0 = a c_1 + b c_2 + c \\ 1 = d c_1 + e c_2 + f \end{cases}$$

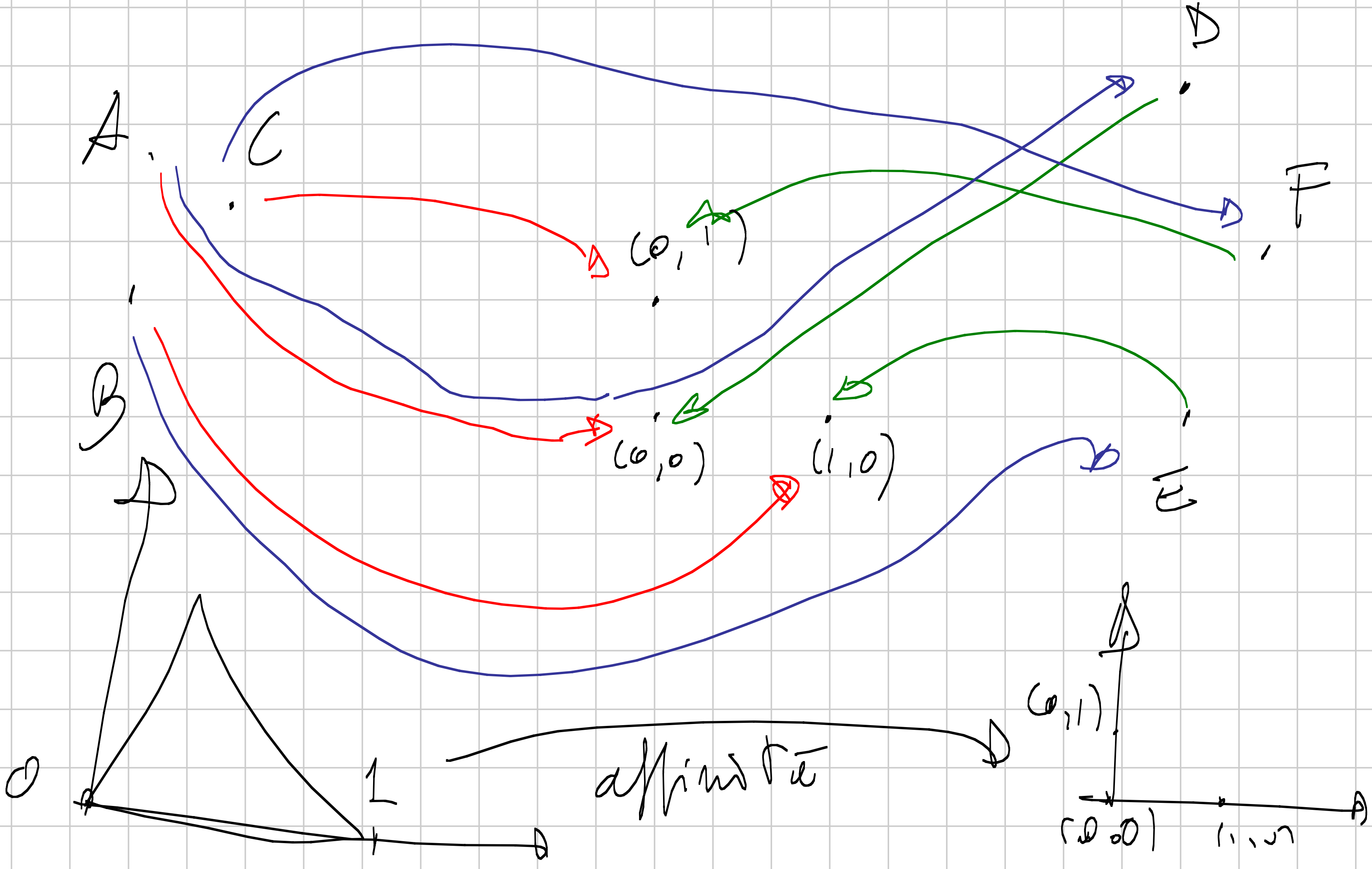
che si risolve in modo unico. $\square$

<u>Cor</u>: Se A, B, C non sono all, D, E, F non sono all

$\exists$ un'unica aff. T t.c.

$T(A)=D \quad T(B)=E \quad T(C)=F$

Dim: Ottengo H_1, H_2 $\quad H_2^{-1}\circ H_1$



Isometrie:

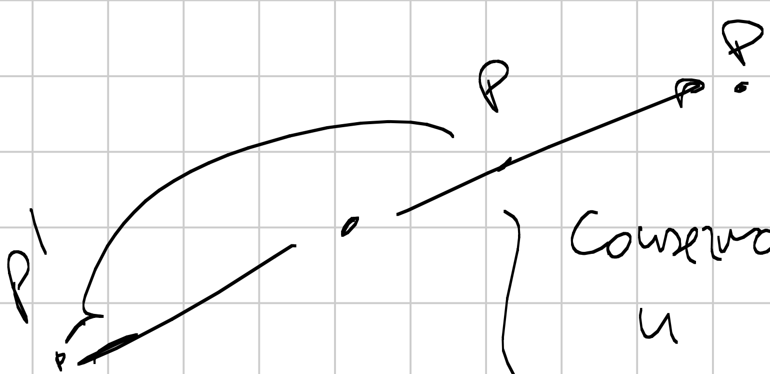
- Rotazione
- Traslazione
- Simmetria assiale
- loro composizioni

Conservano le distanze

⇒ Conservano gli angoli
conservano i rapporti
+ Tutto quello che facevano le affinità.

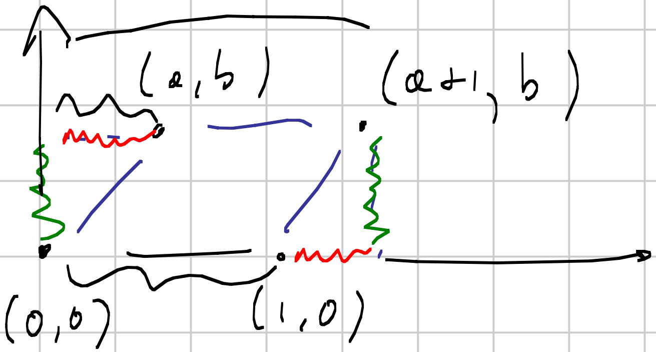
Similitudini

- Isometrie
- Omotetie
- loro composizioni



Conservano gli angoli
e i rapporti
+ Tutto quello che c.
. – – –

2 - Vettori



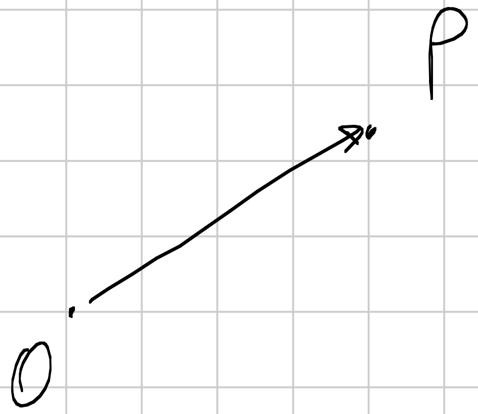
$$F: \begin{cases} x' = \alpha x + \beta y \\ y' = \gamma x + \delta y \end{cases}$$

$$\alpha\delta - \gamma\beta \neq 0$$

$F(0,0) = (0,0)$ $\quad$ $F(1,0) = (\alpha, \gamma)$

$F(a,b) = (\alpha a + \beta b, \gamma a + \delta b)$ $\quad$ $F(a+1,b) = (\alpha(a+1) + b\beta, \gamma(a+1) + \delta b)$

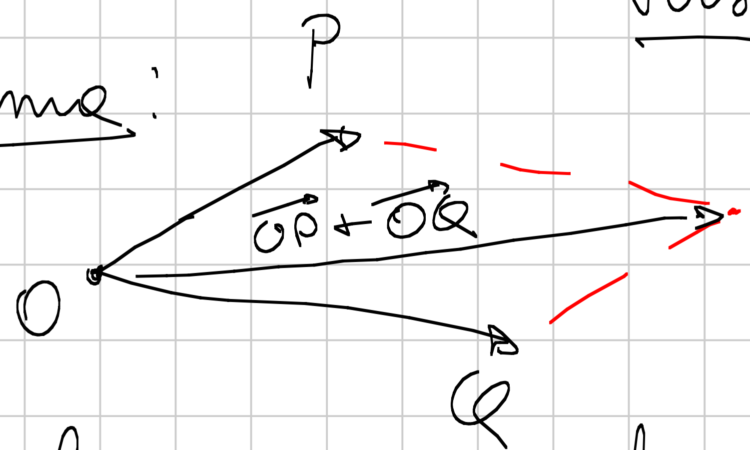
$$\big((\alpha) + (\alpha a + \beta b), (\gamma) + (\gamma a + \delta b)\big)$$? ok !!



$\vec{OP}$, $\vec{P}$ vettore dall'origine a P

- $\|\vec{OP}\| = |\vec{OP}| = d(O,P)$
  norma (modulo)
- retta per O, P   direzione del vettore
- verso di "percorrenza" del vettore

Somma:



Regola del parallelogramma.

Moltiplicazione per scalare: $\lambda \in \mathbb{R}$ $\vec{OP}$ vettore

$\lambda \cdot \overrightarrow{OP}$

Se $\lambda = 0$ $\lambda \cdot \overrightarrow{OP} = \overrightarrow{OO} = O$ (zero)

Se $\lambda > 0$ $(\lambda \cdot \overrightarrow{OP})$ ha norma $\lambda \cdot \|\overrightarrow{OP}\|$
ha direz. di $\overrightarrow{OP}$ e ha verso di $\overrightarrow{OP}$

Se $\lambda < 0$ $(\lambda \cdot \overrightarrow{OP})$ ha norma $|\lambda| \, \|\overrightarrow{OP}\|$
ha direz. di $\overrightarrow{OP}$ e ha verso opposto a $\overrightarrow{OP}$

F affinità $F(O) = O$

$F(\overrightarrow{OP}) = \overrightarrow{O F(P)}$

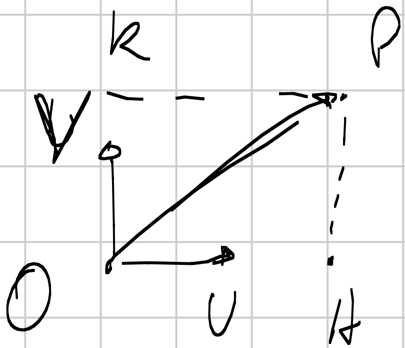
Proprietà ovvie

$\overrightarrow{OP} + \overrightarrow{OQ} = \overrightarrow{OQ} + \overrightarrow{OP}$

$\overrightarrow{OP} - \overrightarrow{OP} = O$

$\lambda(\overrightarrow{OP} + \overrightarrow{OQ}) = \lambda \overrightarrow{OP} + \lambda \overrightarrow{OQ}$

ecc. ..



$$\frac{OH}{OU} = x$$

$$\frac{OK}{OV} = y$$

Segmenti

I vettori non si dividono!!

In questo modo

$$\overrightarrow{OP} + \overrightarrow{OQ} = \overrightarrow{OR}$$

$P : (x, y)$

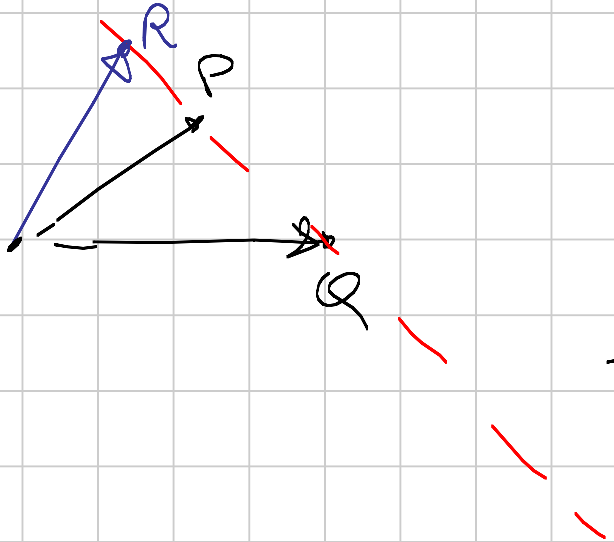
$Q : (z, w)$ $\Rightarrow R : (x + z, y + w)$

$$\lambda \cdot \overrightarrow{OP} = \overrightarrow{OR}$$

$P : (x, y)$

$\Downarrow$ $R : (\lambda x, \lambda y)$

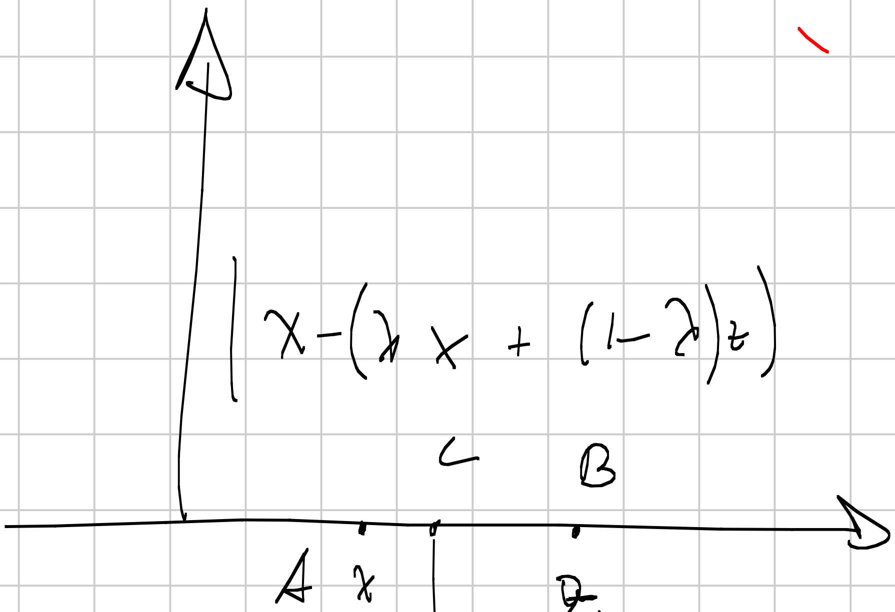
Combinazione lineare



$$\underset{(x,y)}{\lambda\cdot\overrightarrow{OP}}+(1-\lambda)\cdot\underset{(z,w)}{\overrightarrow{OQ}}=\overrightarrow{OR}$$

$$(\lambda x,\lambda y)+\left((1-\lambda)z,(1-\lambda)w\right)=$$

$$=\left(\lambda x+(1-\lambda)z,\ \lambda y+(1-\lambda)w\right)$$



$$|x-(\lambda x+(1-\lambda)z)| = |(1-\lambda)x-(1-\lambda)z| =$$

$$=|1-\lambda|\cdot|x-z|$$

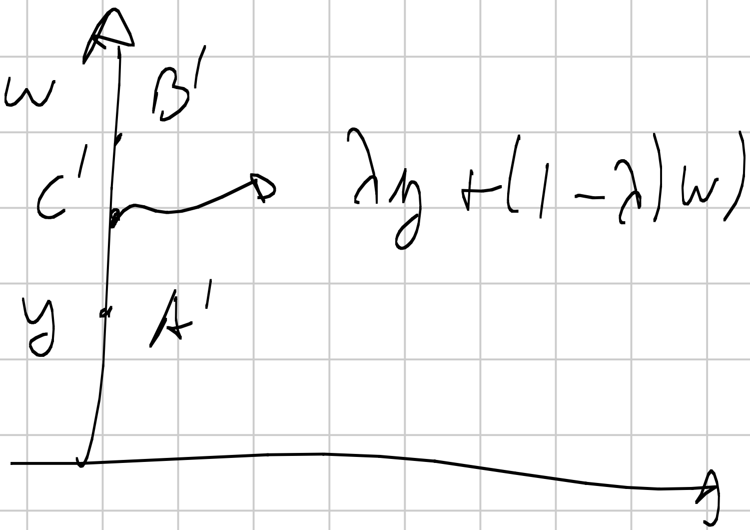
$$|z-(\lambda x+(1-\lambda)z)|=|\lambda|\,|x-z|$$

$(\lambda x+(1-\lambda)z)$

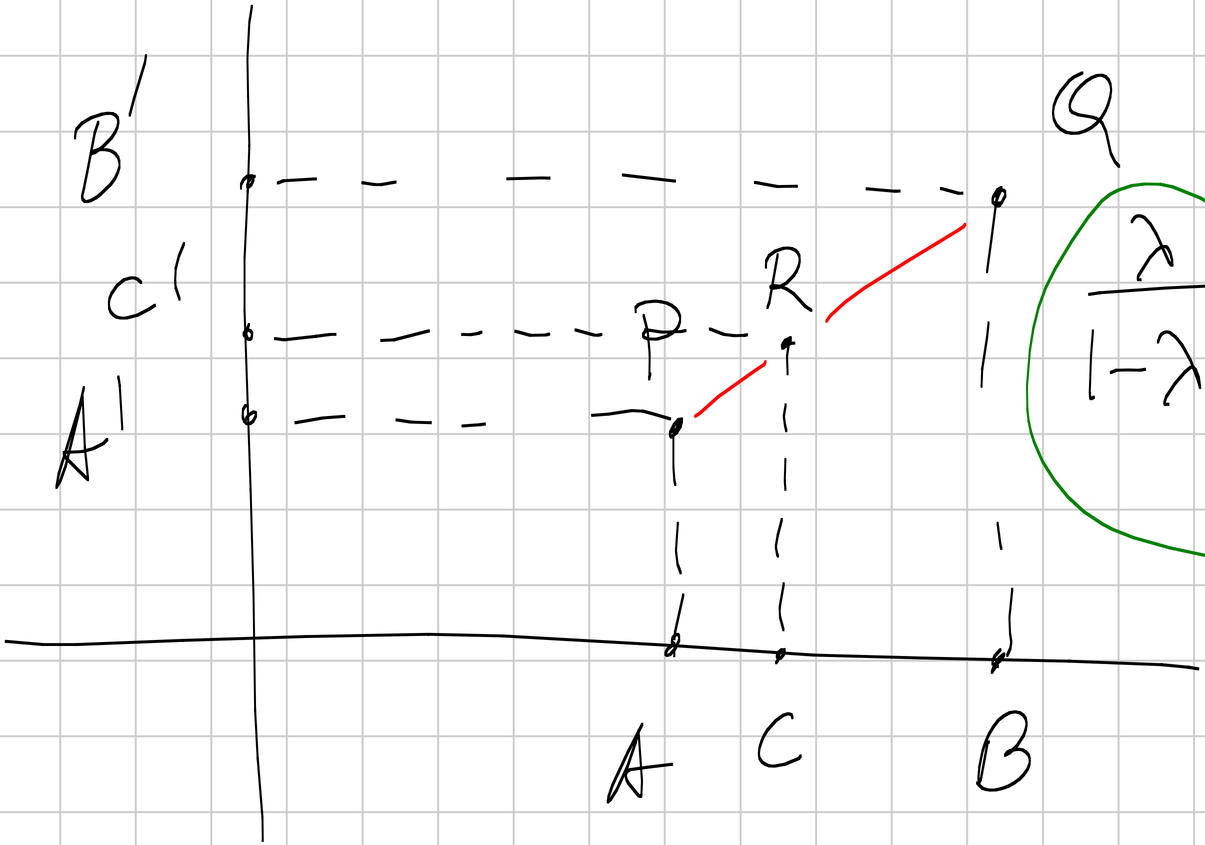
Divide il segmento tra x e z in rapporto

$$\frac{BC}{CA} = \frac{\lambda}{1-\lambda}$$

positivo con C tra A e B
negativo altrimenti



$$\frac{B'C'}{C'A'} = \frac{\lambda}{1-\lambda}$$



$$\frac{\lambda}{1-\lambda} = \frac{BC}{CA} = \frac{B'C'}{C'A'} = \frac{QR}{RP}$$

$\Rightarrow$ Abbiamo capito chi è $\overrightarrow{OR}$

Dati $\overrightarrow{OP}$, $\overrightarrow{OQ}$ trovare $\overrightarrow{OR}$ t.c. $\dfrac{PR}{RQ} = k$

$$k = \frac{\mu}{1-\mu} \Rightarrow \mu(1+k) = k \quad \boxed{\mu = \frac{k}{1+k}}$$

$$\frac{PR}{RQ} = \frac{\mu}{1-\mu}$$

$$\overrightarrow{OP}(1-\mu) + \overrightarrow{OQ}\,\mu = \overrightarrow{OR}$$

$$\overrightarrow{OM} = \frac{1}{2}\overrightarrow{OP} + \frac{1}{2}\overrightarrow{OQ}$$

A ———————— M

G t.q. $\dfrac{AG}{GM} = 2 = k$

$$\overrightarrow{A}\left(\frac{1}{3}\right) + \left(\frac{2}{3}\right)\overrightarrow{M} = \overrightarrow{G}$$

$$\mu = \frac{2}{3}$$

(Un po' di 3 - Numeri Complessi)

(a, b) $\quad a+ib$

$\mathbb{C}$ con lo zero = piano con l'origine

numero complesso $\longrightarrow$ vettore

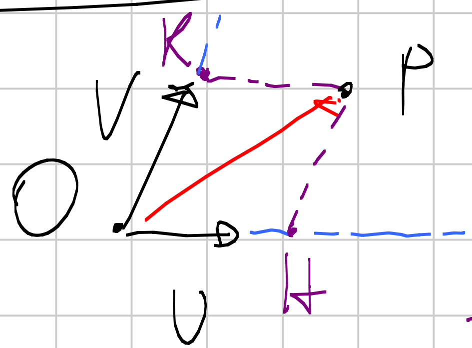
$z, w \longrightarrow \overrightarrow{OP}, \overrightarrow{OQ}$

$z+w \longrightarrow \overrightarrow{OP} + \overrightarrow{OQ}$

$\lambda \in \mathbb{R} \quad \lambda z \longrightarrow \lambda \overrightarrow{OP}$

$\lambda z + (1-\lambda) w$

Torniamo ai vettori



Vogliamo $h, k \in \mathbb{R}$ t.c. $h\vec{U} + k\vec{V} = \vec{P}$

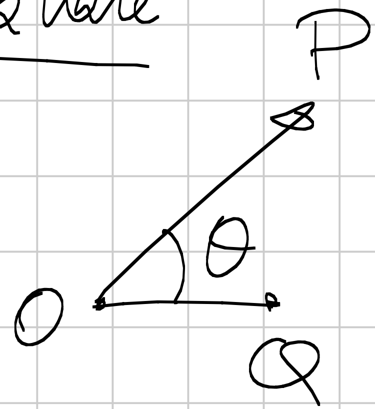
$\exists\, h \in \mathbb{R}$ t.c. $\vec{H} = h\vec{U}$ e $\exists\, k$ t.c. $\vec{K} = k\vec{V}$

$$\Rightarrow \quad h\vec{U} + k\vec{V} = \vec{P}$$

(h, k) coord di $\vec{P}$ rispetto a $\vec{U}, \vec{V}$

Prodotto scalare

$\vec{P}, \vec{Q}$



$\vec{P} \cdot \vec{Q} =$

$= (\vec{P}, \vec{Q}) =$

$= \langle \vec{P}, \vec{Q} \rangle =$

$= \|\vec{OP}\| \cdot \|\vec{OQ}\| \cdot \cos\theta$

$\vec{P} \cdot \vec{P} = \|\vec{OP}\| \cdot \|\vec{OP}\| =$

$= \|\vec{OP}\|^2$

$P : (x, y) \qquad Q : (z, w)$

$\vec{P} \cdot \vec{Q} = xz + yw$

$\vec{P} \cdot \vec{Q} = \vec{Q} \cdot \vec{P}$

$$\vec{P} \cdot (\lambda \vec{Q}) = (\lambda \vec{P}) \cdot \vec{Q} = \lambda (\vec{P} \cdot \vec{Q})$$

$$\vec{P} \cdot (\vec{Q} + \vec{R}) = \vec{P} \cdot \vec{Q} + \vec{P} \cdot \vec{R}$$

$$\vec{P} \cdot (\lambda \vec{Q} + \mu \vec{R}) = \lambda \vec{P} \cdot \vec{Q} + \mu \vec{P} \cdot \vec{R}$$

<u>Oss</u>: $\vec{P} \cdot \vec{Q} = 0 \iff \begin{cases} P = O \\ Q = O \\ OP \perp OQ \end{cases}$

$$\|\vec{A} + \vec{B}\|^2 = (\vec{A} + \vec{B}) \cdot (\vec{A} + \vec{B}) = \vec{A} \cdot (\vec{A} + \vec{B}) + \vec{B} \cdot (\vec{A} + \vec{B}) =$$

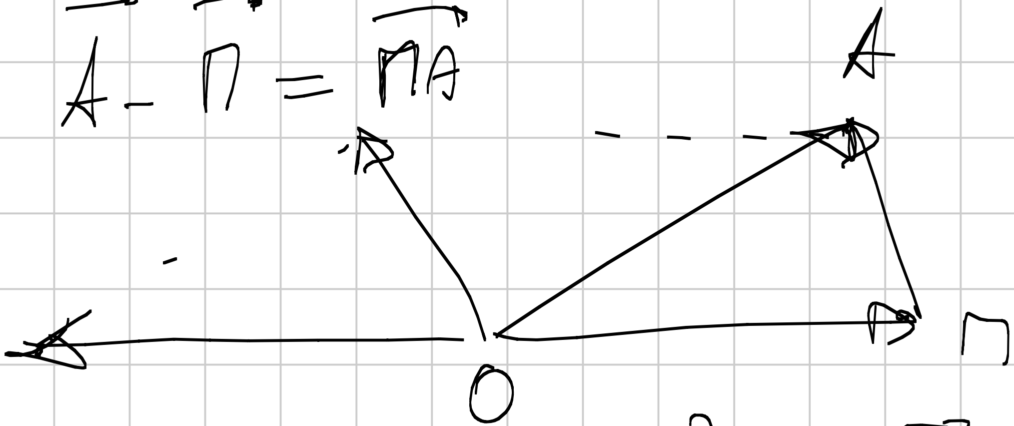
$$= \vec{A} \cdot \vec{A} + \vec{A} \cdot \vec{B} + \vec{B} \cdot \vec{A} + \vec{B} \cdot \vec{B} =$$

$$= \|A\|^2 + \|B\|^2 + 2 \vec{A} \cdot \vec{B}$$

Es: A, B, C $\qquad O$

M pt. mid di BC $\qquad \vec{M} = \frac{1}{2}\vec{B} + \frac{1}{2}\vec{C}$

$\vec{A} - \vec{M} = \vec{MA}$



$$d(M,A)^2 = \|\vec{MA}\|^2 = \|\vec{A} - \vec{M}\|^2 = \|A\|^2 + \|M\|^2 - 2\vec{A}\cdot\vec{M} =$$

$$= \|\vec{A}\|^2 + \left\|\frac{\vec{B}+\vec{C}}{2}\right\|^2 - 2\vec{A}\cdot\left(\frac{\vec{B}+\vec{C}}{2}\right) =$$

$$= \|\vec{A}\|^2 + \frac{\|\vec{B}\|^2}{4} + \frac{\|\vec{C}\|^2}{4} + \frac{\cancel{2}\vec{B}\cdot\vec{C}}{\cancel{4}\,2} - \frac{\cancel{2}\vec{A}\cdot\vec{B}}{\cancel{2}} - \frac{\cancel{2}\vec{A}\cdot\vec{C}}{\cancel{2}} =$$

$$\|A-B\|^2 = \|A\|^2 + \|B\|^2 - 2(A\cdot B)$$

$$\|A-M\|^2 = \left\|A - \frac{B+C}{2}\right\|^2 = \left\|\frac{A-B}{2} + \frac{A-C}{2}\right\|^2$$

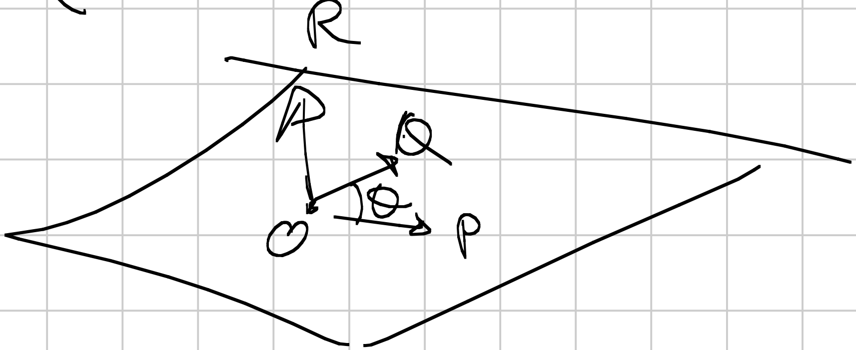
$$= \frac{\|A-B\|^2}{2} + \frac{\|A-C\|^2}{2} - \left\|\frac{B-C}{2}\right\|^2 = \frac{c^2}{2} + \frac{b^2}{2} - \frac{a^2}{4} = AM^2$$

<u>Prodotto vettore</u>: $\overrightarrow{OP}, \overrightarrow{OQ}$

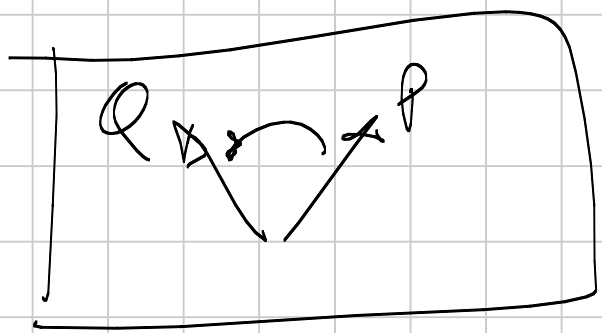
$$\overrightarrow{OP} \wedge \overrightarrow{OQ} = \overrightarrow{OR}$$

$$\|\overrightarrow{OR}\| = \|\overrightarrow{OP}\|\,\|\overrightarrow{OQ}\|\,|\sin\theta|$$

$$OR \perp OP, \quad OR \perp OQ$$



$$\vec{OP} \wedge \vec{OQ} = -\vec{OQ} \wedge \vec{OP}$$



$$\vec{P} \wedge (\lambda \vec{Q} + \mu \vec{R}) =$$
$$= \lambda\, \vec{P} \wedge \vec{Q} + \mu\, \vec{P} \wedge \vec{R}$$

$$\vec{P} \cdot (\vec{Q} \wedge \vec{R}) = \vec{R} \cdot (\vec{P} \wedge \vec{Q}) = \vec{Q} \cdot (\vec{R} \wedge \vec{P})$$
$$= -\vec{P} \cdot (\vec{R} \wedge \vec{Q}) = -\vec{Q} \cdot (\vec{P} \wedge \vec{R}) = -\vec{R} \cdot (\vec{Q} \wedge \vec{P}) =$$
$$= \pm 6 \cdot \mathrm{Vol}(OPQR)$$

$$\| \vec{OP} \wedge \vec{OQ} \| = 2\, \mathrm{Area}\,(\overset{\triangle}{OPQ})$$

3 – Numeri Complessi

$z \longrightarrow i \cdot z$ = rotaz. di $\frac{\pi}{2}$

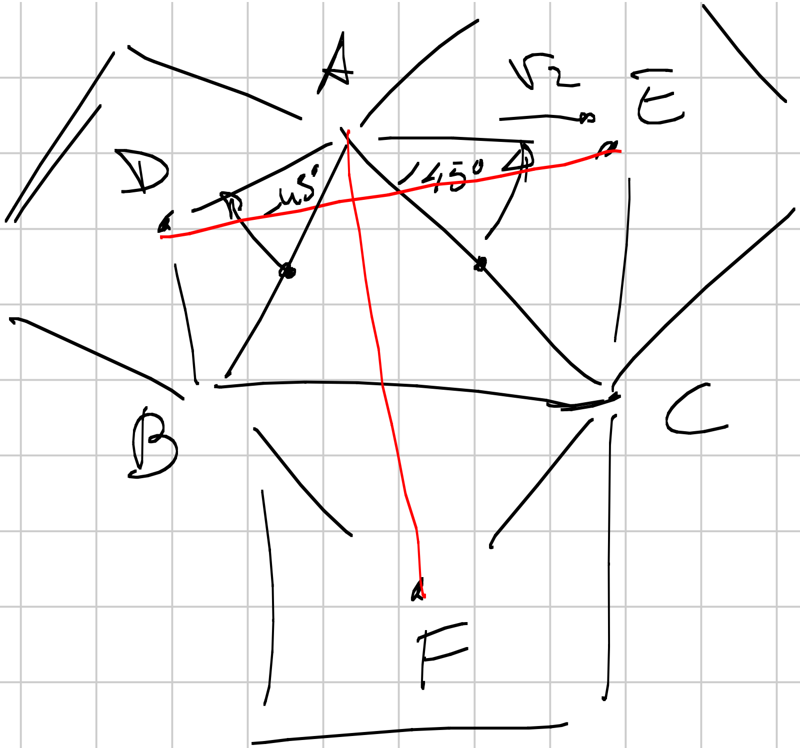
$\omega = \cos\theta + i \operatorname{sen}\theta = e^{i\theta}$

$z \longrightarrow \omega z$ = rotaz di θ

$w = \rho e^{i\theta}$

$z \longrightarrow zw = \rho z e^{i\theta}$ = rotaz di θ + dilat. di ρ

$z \longrightarrow \bar{z}$ = simm. risp. a $\{ \operatorname{Im} z = 0 \}$



$$A = 0$$

$$\frac{c}{2}\sqrt{2}\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right) = e = \frac{c}{2}(1+i)$$

$$\frac{b}{2}\sqrt{2}\left(\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right) = d = \frac{b}{2}(1-i)$$

$$\left(\frac{b+c}{2}-b\right)\left(\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)\sqrt{2}+b =$$

$$= \left(\frac{c-b}{2}\right)(1-i)+b = f$$

$$e-d = \frac{c}{2}+\frac{ic}{2}-\frac{b}{2}+\frac{ib}{2} =$$

$$= \left(\frac{c-b}{2}\right)(1+i)+ib \qquad \Rightarrow \quad if = e-d$$

$$if = \left(\frac{c-b}{2}\right)(i+1)+ib \qquad \Rightarrow \text{FINE}$$

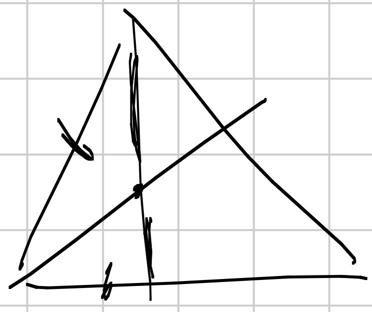
$$\|a+b\|^2 = \|a\|^2 + \|b\|^2 + a\bar{b} + \bar{a}b$$

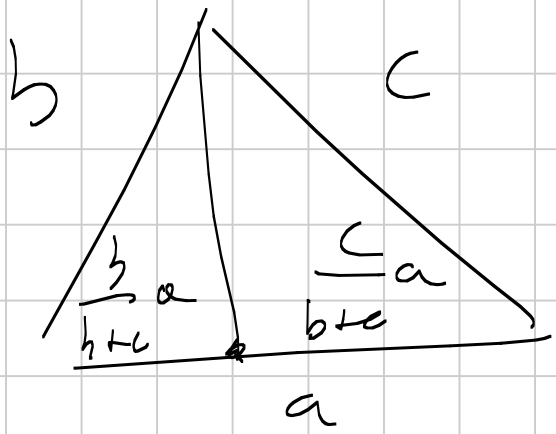
<u>In un Triangolo</u>

$$\vec{G} = \frac{\vec{A} + \vec{B} + \vec{C}}{3} \qquad \vec{I} = \frac{a\vec{A} + b\vec{B} + c\vec{C}}{a+b+c}$$

Se l'origine è nel circocentro (O)

$$\vec{H} = \vec{A} + \vec{B} + \vec{C}$$





Se l'origine è nel circocentro

$$\|\vec{A}\|^2 = R^2 = \|\vec{B}\|^2 = \|\vec{C}\|^2$$

$$-\vec{A}\cdot\vec{B} = \frac{\|\vec{A}-\vec{B}\|^2 - \|A\|^2 - \|B\|^2}{2} =$$

$$-\vec{A}\cdot\vec{B} = \frac{c^2 - 2R^2}{2} \qquad \vec{A}\cdot\vec{B} = \frac{2R^2 - c^2}{2}$$

Esercizi: 1, 5, 6, 12