GEOMETRIA PROIETTIVA

$(a_1, \ldots, a_n)$ $\qquad a_i \in \mathbb{R}$ $\qquad \mathbb{R}^n$

$$(a_1, \ldots, a_n) \cdot (b_1, \ldots, b_n) = \sum_{i=1}^{n} a_i b_i \in \mathbb{R}$$

Matrice

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \qquad a_{ij} \in \mathbb{R}$$

$$\begin{pmatrix} a_{11} & & a_{1n} \\ & & \\ a_{m1} & & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} a_{11}b_1 + \cdots + a_{1n}b_n \\ a_{21}b_1 + \cdots + a_{2n}b_n \\ \vdots \\ a_{m1}b_1 + \cdots + a_{mn}b_n \end{pmatrix} = \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix}$$

$A, \vec{b}, \vec{c}$ $\qquad A \cdot \vec{b} = \vec{c}$

$\underline{n = 2}$

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \qquad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$\begin{cases} \alpha x_1 + \beta x_2 = b_1 \\ \gamma x_1 + \delta x_2 = b_2 \end{cases} \qquad \underbrace{(\alpha\delta - \beta\gamma)}_{\det A} \neq 0$$

$$\left(\frac{b_1\delta - b_2\beta}{\alpha\delta - \beta\gamma}, \frac{\alpha b_2 - \gamma b_1}{\alpha\delta - \beta\gamma} \right)$$

$\underline{n = 3}$ $\quad A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$$\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} -$$
$$- a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

$\vec{x}$ punto $\longrightarrow A \cdot \vec{x}$

A matrice $\qquad$ Affinità

$\det A \neq 0$

$$A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y} \qquad A(\lambda \vec{x}) = \lambda \cdot (A \cdot \vec{x}) \quad \lambda \in \mathbb{R}$$

$$\begin{pmatrix} a_{11} & a_{1n} \\ a_{n1} & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{1n} \\ b_{n1} & b_{nn} \end{pmatrix} = \begin{pmatrix} c_{ij} \end{pmatrix}$$

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$$B \cdot (A \cdot \vec{x}) = (B \cdot A) \cdot \vec{x} \qquad B \cdot A \neq A \cdot B$$

$$\det(A\cdot B) = \det A \cdot \det B$$

$\vec{x}, \vec{y}, \vec{z}$ $\quad S$

A affinità $\longrightarrow$ $A\vec{x}$ $\quad A\vec{y}$ $\quad A\vec{z}$ $\quad S' = \det A \cdot S$

$$\vec{x} = \begin{pmatrix} \\ \\ \end{pmatrix} \qquad \begin{pmatrix} & & \end{pmatrix} = \vec{x}^T = {}^T\vec{x}$$

$$\vec{A} = (a_{ij}) \longrightarrow \vec{A}^T = (a_{ji})$$

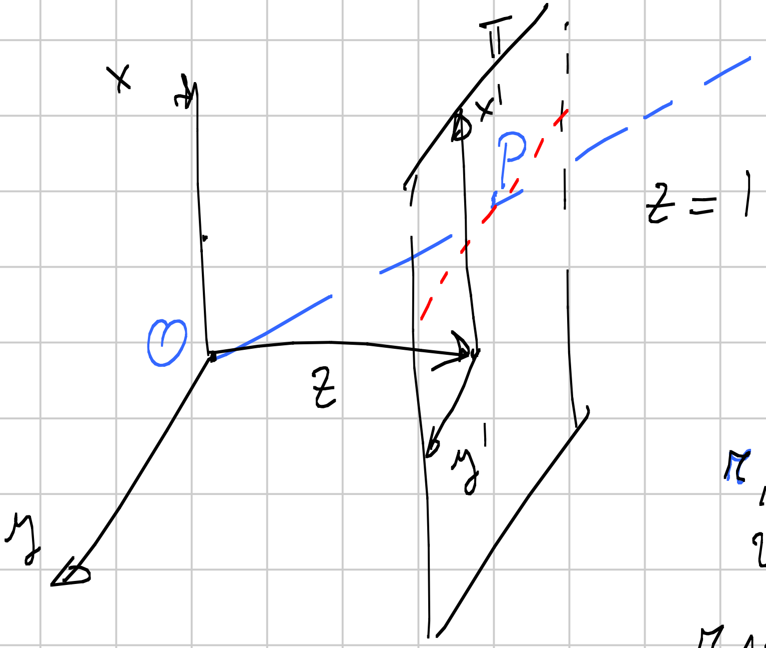
$$(A\vec{x})^T = \vec{x}^T \cdot A^T$$

$A = A^T$ simmetriche.

Assiomi di incidenza

Per 2 punti (DISTINTI) passa una e una sola retta

Due rette distinte, ~~se non sono parallele~~, si incontrano in uno e un solo punto.



$P \in \pi \xhookrightarrow[\text{iniett.}]{} \text{rette per } O$

$P \longrightarrow OP$

$\text{rette} \in \pi \xhookrightarrow{\text{iniett.}} \text{piani per } O$

$r, s \in \pi$ rette $\longrightarrow \alpha_r, \alpha_s$ piani per O

$r \cap s = P \in \pi \longrightarrow OP = \alpha_r \cap \alpha_s$

$P, Q \in \pi \longrightarrow OP, OQ$

$r = \mathcal{L}(P, Q) \longrightarrow \pi(OP, OQ)$

$r \in \pi$

$$\begin{cases} z = 1 \\ ax + by + cz = 0 \end{cases}$$

$$\begin{cases} z = 1 \\ x = 1 \end{cases} \qquad x' = 1$$

$ax' + by' + c = 0$

$$\begin{cases} z = 1 \\ x - z = 0 \end{cases}$$

$r \longrightarrow \alpha_r$

$ax' + by' + c = 0 \longrightarrow ax + by + cz = 0$

$(p', q') \longrightarrow (p', q', 1) \longrightarrow \{(tp', tq', t) \mid t \in \mathbb{R}\}$

rette — piani

$x' = 0 \longrightarrow x = 0$

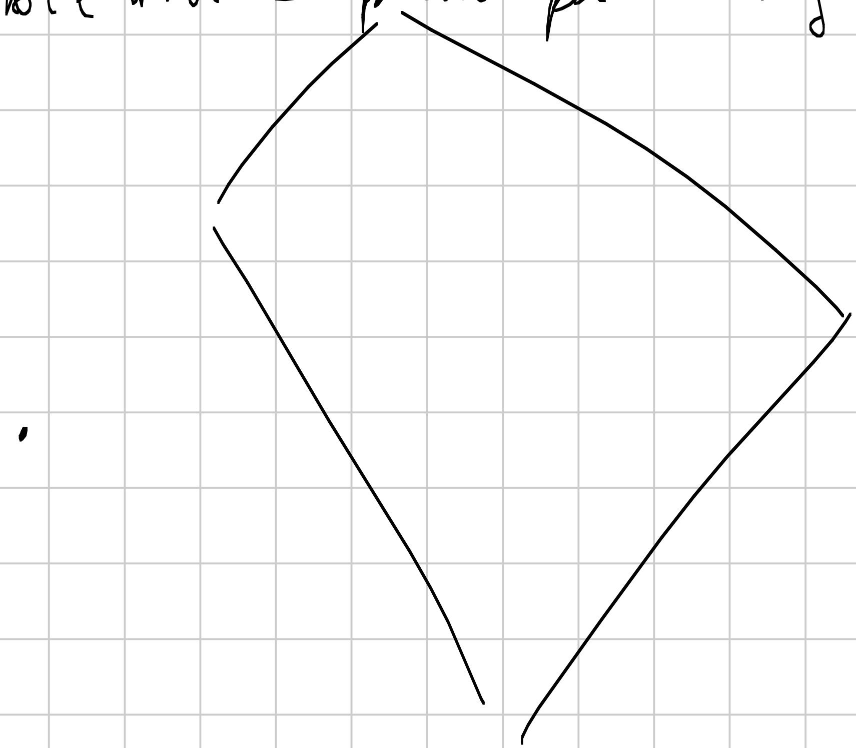
$x' = 1 \longrightarrow x - z = 0$

$\cap \downarrow \qquad \downarrow \cap$

$\cancel{\emptyset} \qquad (0, y, 0) \quad y \in \mathbb{R}$

punti proiettivi = rette per l'origine di $\mathbb{R}^3$

rette proiettive = piani per l'origine di $\mathbb{R}^3$



Teme omogenee

$$[a, b, c] = \{ (\lambda a, \lambda b, \lambda c) \mid \lambda \in \mathbb{R}^* \} \qquad a^2+b^2+c^2 \neq 0$$

$$\begin{bmatrix} a_{11} & \cdots & a_{13} \\ a_{31} & & a_{33} \end{bmatrix} = \left\{ \begin{pmatrix} \lambda a_{11} & \lambda a_{13} \\ \lambda a_{31} & \lambda a_{33} \end{pmatrix} \middle| \lambda \in \mathbb{R}^* \right\}$$

$$[A] \cdot [v] = [A(\lambda) \cdot v(\mu)] = [\lambda A(1) \cdot \mu v(1)] =$$

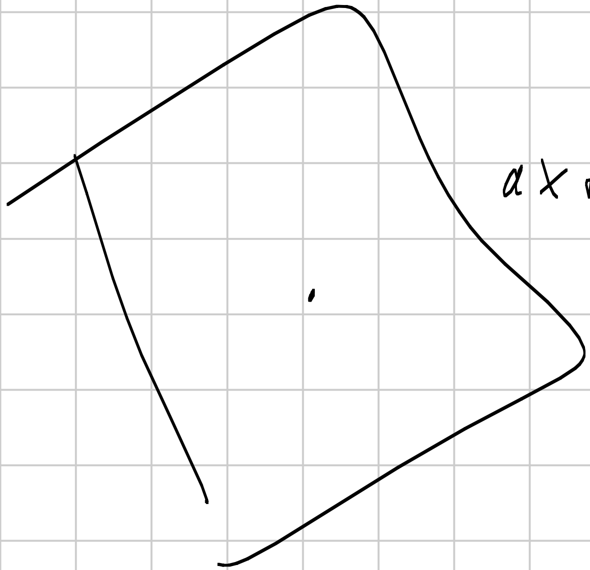
$$= [\lambda \cdot \mu \; A(1) v(1)] =$$

$$= [A(1) \; v(1)]$$

$$\begin{array}{cc} \downarrow & \downarrow \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \begin{bmatrix} 2 \\ 3 \end{bmatrix} \end{array}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix}0 & 2\\ 2 & 0\end{pmatrix} \cdot \begin{pmatrix}6\\ 9\end{pmatrix} = \begin{pmatrix}18\\ 12\end{pmatrix} \qquad [18, 12] = [3, 2]$$



$ax+by+cz$

$$\{[x,y,z] \mid ax+by+cz=0\}$$

$$\left[-\frac{2}{3}, 1\right] \qquad [-2, 3]$$

$$x = -\frac{2}{3}$$

$$\{[x,y] \mid 3x+2=0\}$$

$$\{[x,y,z] \mid p(x,y,z)=0\}$$ Va bene quando p è omogeneo

$$(p', q') \longrightarrow \{(tp', tq', t) \mid t \in \mathbb{R}\}$$

$$= \{(0,0,0)\} \cup [p', q', 1] \qquad [p', q', 0]$$

$$ax' + by' + c = 0 \longrightarrow ax + by + cz = 0$$

$$= \{(0,0,0)\} \cup \{[x, y, z] \mid ax + by + cz = 0\}$$

$$\{p(x', y') = 0\} \longrightarrow p^H(x, y, z) = z^d \, p\left(\frac{x}{z}, \frac{y}{z}\right)$$

$d = \deg p$

$$p(x', y') = P(x', y', 1) \qquad P(x, y, z) \text{ omog}$$

$\{ax+by+cz=0\}$ (risolto in terne om.)
$\{dx+ey+fz=0\}$ (") distinte

ovvero

$[a,b,c] \neq [d,e,f]$

$$\begin{cases} ax+by+cz=0 \\ dx+ey+fz=0 \end{cases}$$

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ d & e \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = -\begin{pmatrix} c \\ f \end{pmatrix}$$

$$\left(\frac{-ce+fb}{ae-db}, \frac{-af+dc}{ae-db} \right)$$

$$[-ce+fb, -af+dc, -db+ae]$$

$$\left[\begin{vmatrix} b & c \\ e & f \end{vmatrix},\ \underbrace{-\begin{vmatrix} a & c \\ d & f \end{vmatrix}}_{\begin{vmatrix} c & a \\ f & d \end{vmatrix}},\ \begin{vmatrix} a & b \\ d & e \end{vmatrix} \right]$$

$$[a, b, c] \neq [d, e, f]$$

$$\alpha x + \beta y + \gamma z = 0$$

$$\begin{array}{c} [\alpha, \beta, \gamma] \\ \| \\ \left[\begin{vmatrix} b & c \\ e & f \end{vmatrix},\ \begin{vmatrix} c & a \\ f & d \end{vmatrix},\ \begin{vmatrix} a & b \\ d & e \end{vmatrix} \right] \end{array}$$

$$\begin{cases} \alpha a + \beta b + \gamma c = 0 \\ \alpha d + \beta e + \gamma f = 0 \end{cases}$$

$$\mathbb{P}^2(\mathbb{R}) = \{ [a,b,c] \mid a^2+b^2+c^2 \neq 0,\ a,b,c \in \mathbb{R} \}$$

$\mathbb{P}^2$

piano proiettivo

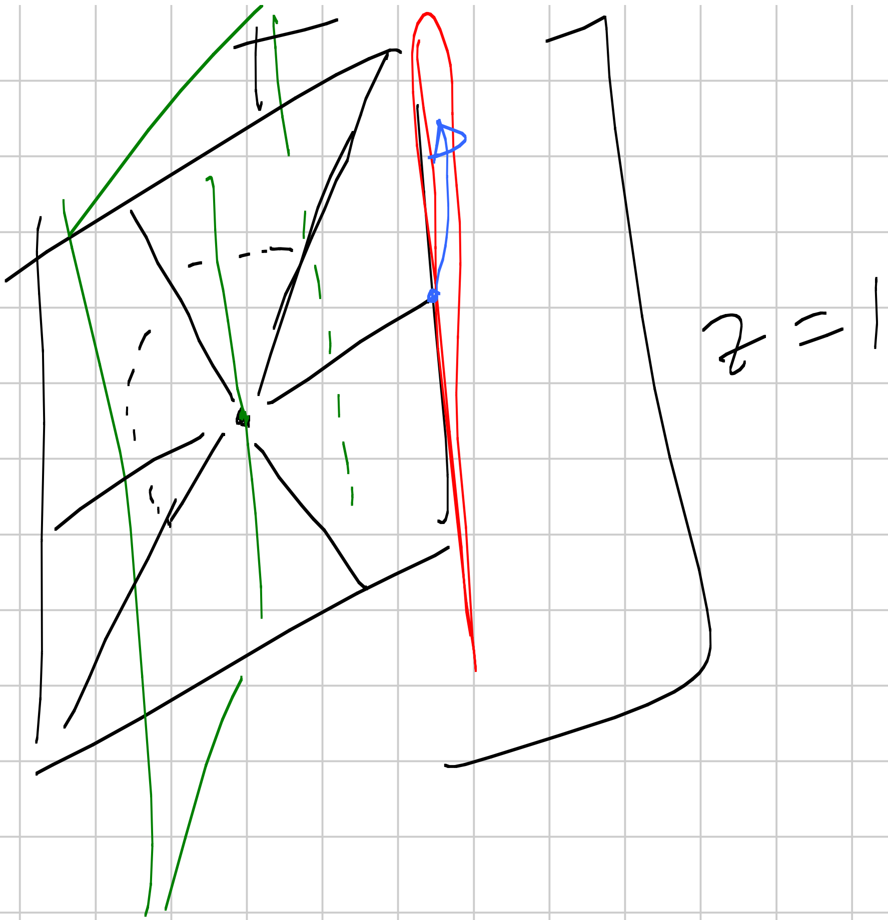
punti $[a,b,c]$

rette $\{ [a,b,c] \mid \alpha a + \beta b + \gamma c = 0 \}$

α, β, γ coefficienti a meno di fattore comune non nullo.

$$U_z = \{ [x,y,z] \mid z \neq 0 \} = \mathbb{P}^2 - \{ z=0 \}$$

$\cong \mathbb{E}^2 = \mathbb{R}^2 =$ il piano "normale"

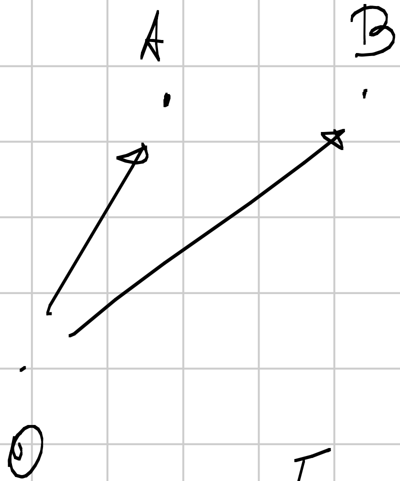


$$\alpha x + \beta y + \gamma z + \delta = 0 \qquad \delta \neq 0$$

$[x, y, z]$

$z = 1 \qquad (x, y, 1)$

$z = 2 \qquad (x, y, 2)$



$$\left\{ \lambda\vec{A} + (1-\lambda)\vec{B} \;\middle|\; \lambda \in \mathbb{R} \right\}$$

retta per $\vec{A}, \vec{B}$

$$[0, 1] + [1, 0] = [1, 1]$$

??

$$[0, 1] + [2, 0] = [2, 1]$$

$$\underbrace{[a, b, c]}_{S} + \underbrace{[d, e, f]}_{T} = \left\{ \lambda(a, b, c) + \mu(d, e, f) \;\middle|\; \lambda, \mu \in \mathbb{R},\ \lambda^2 + \mu^2 \neq 0 \right\}$$

Scelgo $[p, q, r]$ su $\mathcal{L}(S, T) \subset \mathbb{P}^2$ $\qquad \mathcal{L}$

$\quad\; \parallel$

$\quad\; U$

Chiedo: $\boxed{U = S + T}$ $\qquad S = [S']$

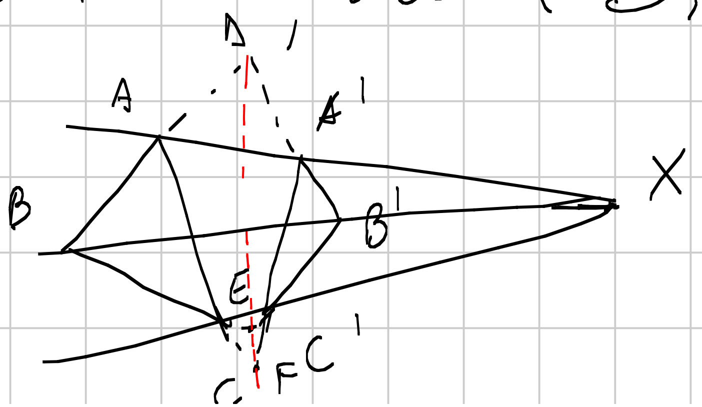
$S' + T' = U'$

$[\lambda S' + \mu T']$

$[\lambda, \mu] \longmapsto [\lambda S + \mu T]$

Teo Desargues

Se AA', BB', CC' concorrono, posto $D = AB \cap A'B'$, $E = BC \cap B'C'$, $F = CA \cap C'A'$, allora D, E, F sono allineati.



Dim: $$X = \underbrace{\lambda_A A + \mu_A A' = \lambda_B B + \mu_B B'}_{} = \lambda_C C + \mu_C C'$$

Fissando punti di misura su AA', BB', CC'.

$$\lambda_A A - \lambda_B B = \mu_B B' - \mu_A A' = D$$

$$\lambda_B B - \lambda_C C = \mu_C C' - \mu_B B' = E \qquad D + E = F$$

$$\lambda_A A - \lambda_C C = \mu_C C' - \mu_A A' = F$$

$\square$

$$[0,1,2] \qquad [0,1,3] \qquad [0,1,1]$$

$$\lambda(0,1,2) + \mu(0,1,3) = (0,1,1)$$

$$\lambda + \mu = 1 \qquad 2\lambda + 3\mu = 1 \qquad \lambda = 1 - \mu$$

$2 - 2\mu + 3\mu = 1 \quad 2 + \mu = 1 \quad \mu = -1$

$\lambda = 2$

$(0, 2, 4) \quad (0, -1, -3)$

$$\left\{ \left[\alpha(0, 2, 4) + \beta(0, -1, -3) \right] \mid [\alpha, \beta] \right\}$$

Riferimenti Proiettivi

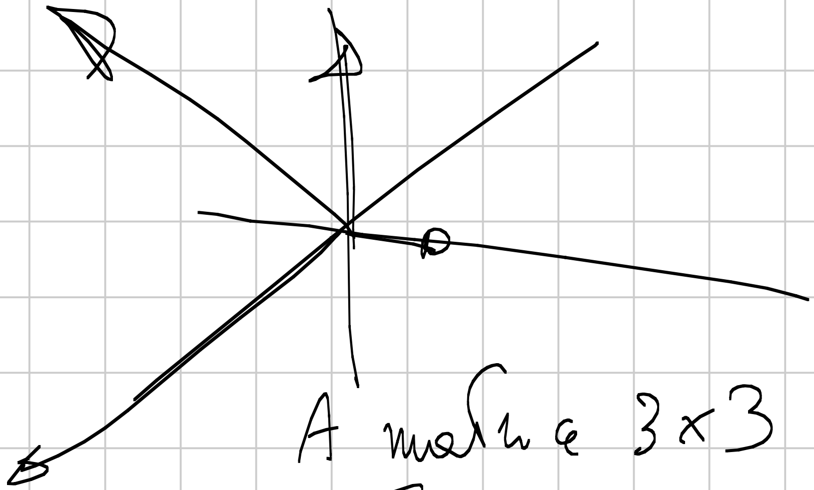
Scelgo 4 punti a 3 a 3 non allineati

A_1, A_2, A_3, U

$A_1 = [1, 0, 0] \quad A_2 = [0, 1, 0] \quad A_3 = [0, 0, 1]$

$U = [1, 1, 1]$

$k, h, j \in \mathbb{R}^*$



$$\begin{matrix} x \\ y \\ z \end{matrix} \longrightarrow \begin{matrix} kx \\ hy \\ jz \end{matrix}$$

<u>Proiettività</u>

A matrice 3x3

$[A]$

$[a, b, c] \longmapsto \left[A \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right]$

Concorrenze, Allineamento, <u>Birapporto</u>

Siano A, B, C, D allineati.

Fissiamo una param. della loro retta t.c. $A+B=C$

allora $D = \lambda A + \mu B$

$\frac{\lambda}{\mu} = (A, B, C, D)$ birapporto della quaterna ordinata

Se ho preso una qualsiasi param. di $\mathcal{L}(A, B)$ dove

$C = \alpha A + \beta B \qquad D = \gamma A + \delta B$

$$(A, B, C, D) = \frac{\beta\gamma}{\alpha\delta}$$

Se $T : \mathbb{P}^2 \longrightarrow \mathbb{P}^2$ proiettività

$$(T(A), T(B), T(C), T(D)) = (A, B, C, D)$$

Siano A_i $i = 1 \dots 4$ 4 pt all.

B_j $j = 1 \dots 4$ 4 pt all.

$\exists$ $T : \mathbb{P}^2 \to \mathbb{P}^2$ p. t.c. $T(A_i) = B_i$

$\Leftarrow$

$(A_1, A_2, A_3, A_4) = (B_1, B_2, B_3, B_4)$

$r \xrightarrow{[M]} \{x=0\}$ $\qquad$ $s \xrightarrow{[N]} \{x=0\}$

$A_i \longrightarrow \overline{A}_i$ $\qquad$ $B_i \longrightarrow \overline{B}_i$

$[0, y_i, z_i]$

$$\begin{pmatrix} h & 0 & 0 \\ 0 & \boxed{} & \\ 0 & & \end{pmatrix}$$

A, B, C, D pt. euclidei

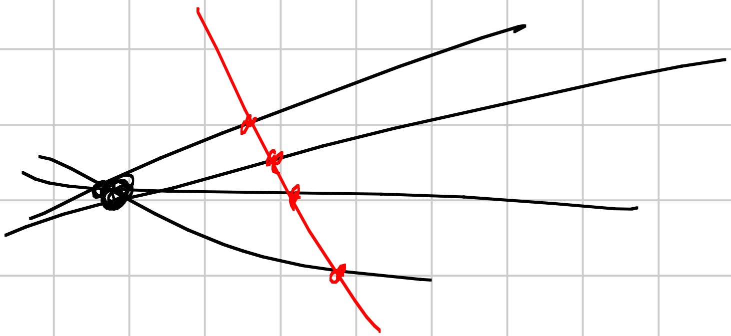
$$(A,B,C,D) = \frac{\frac{AC}{CB}}{\frac{AD}{DB}}$$

$D = \infty$

$$(A,B,C,D) = -1 \Rightarrow \frac{AC}{CB} = 1$$

$$(A,B,C,D) = \lambda \Rightarrow \frac{AC}{CB} = -\lambda$$

$(A,B,C,D) = -1$ quaterna armonica



Dualità punto - retta

punti $\longleftrightarrow$ rette

intersez. $\longleftrightarrow$ unione

allin. $\longleftrightarrow$ concorr.

Siano X, Y, Z tre punti allineati. Sia $\ell = \mathcal{L}(r_1 \cap s_1, r_2 \cap s_2)$

$$\underset{r_1 \cap r_2}{X} , \quad \underset{s_1 \cap s_2}{Y} , \quad \underset{t_1 \cap t_2}{Z}$$

$m = \mathcal{L}(s_1 \cap t_1, s_2 \cap t_2)$ $\quad n = \mathcal{L}(t_1 \cap r_1, t_2 \cap r_2)$.

Allora ℓ, m, n concorrono.

$[1, 0, 1]$ $\quad$ $[2, 1, 2]$

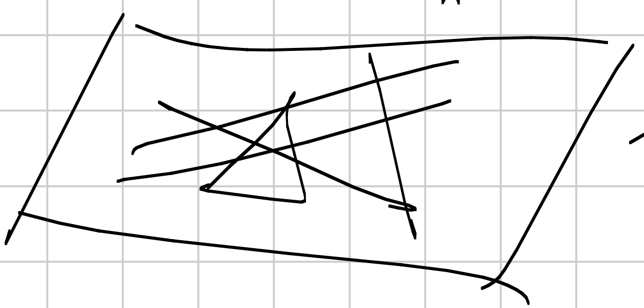
$$x - z = 0$$

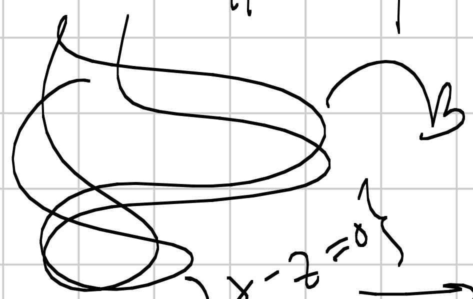
$[1, 0, 0]$ $\quad$ $[0, 1, 0]$

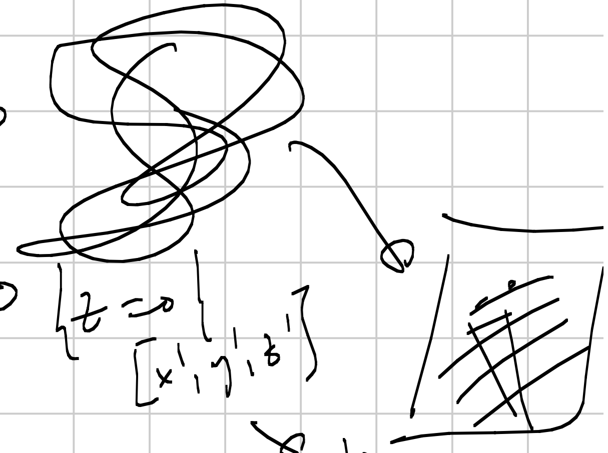
$$\begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} = ($$

$x, y \to$ $ax + by + c = x'$
$dx + ey + f = y'$

$\mathbb{R}^2$ $\quad$ $\mathbb{P}^2 \xrightarrow{T} \mathbb{P}^2$







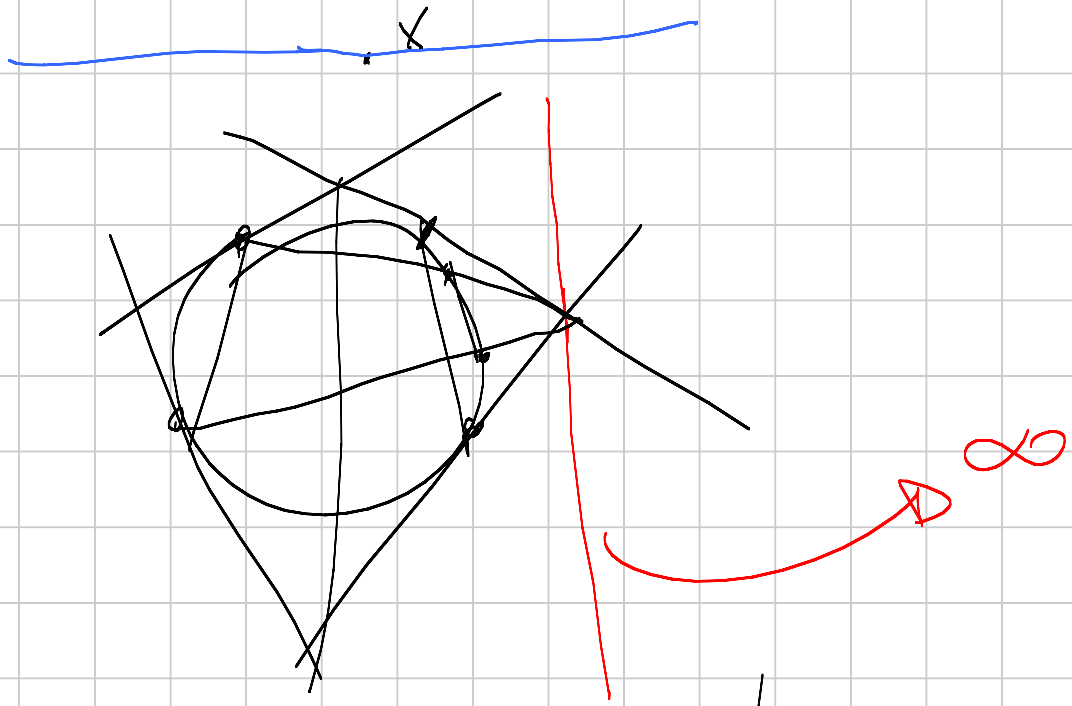
$\{x - z = 0\} \to \{z = 0\}$

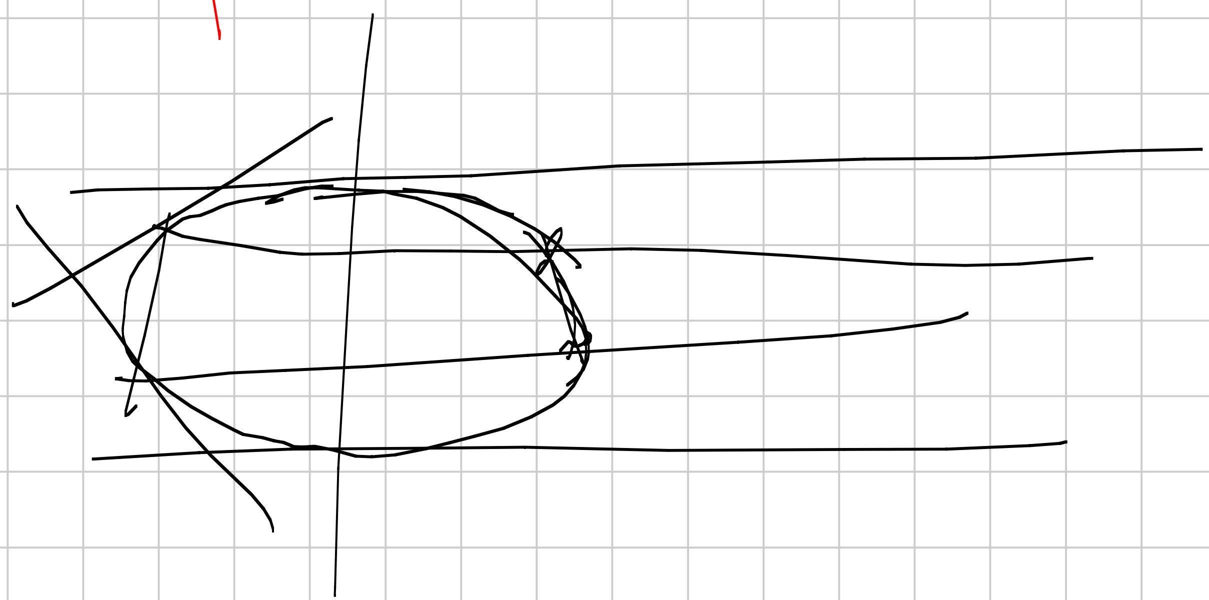
$[x', y', z']$

$\left(\frac{x'}{z'}, \frac{y'}{z'}\right)$

$(x, y) \to [x, y, 1]$

$ax + by + c = 0 \to \{ax + by + cz = 0\}$





A matrice simmetrica $\det A \neq 0$

$$\mathcal{C} = \left\{ [x,y,z] \;\middle|\; (x,y,z) \cdot A \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \right\}$$

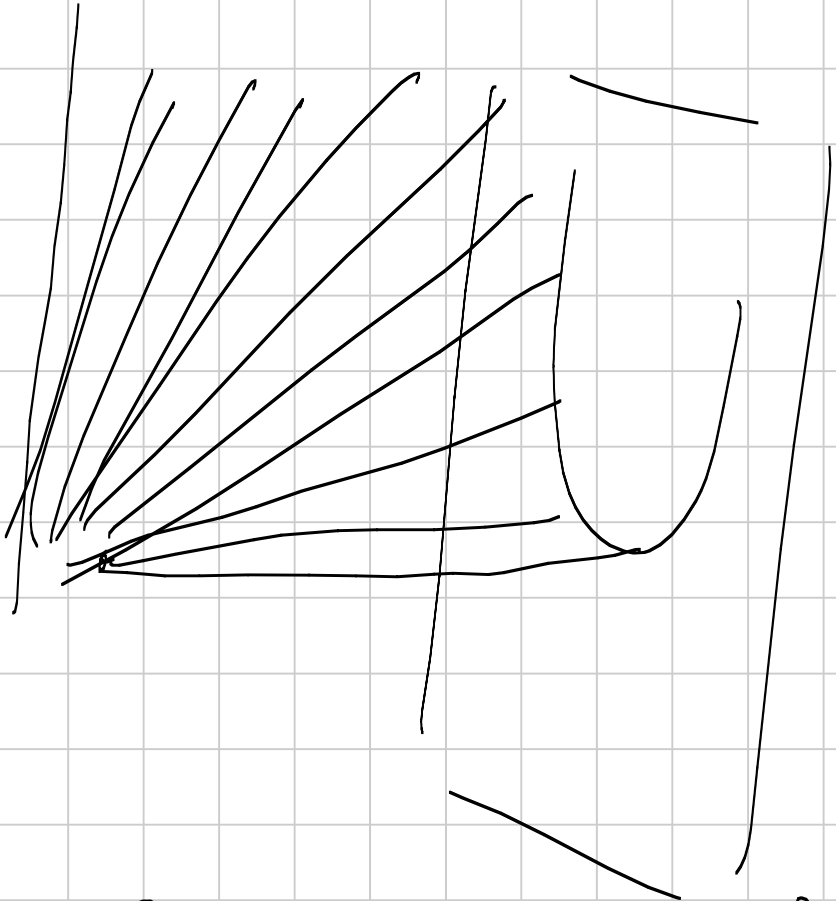
$$(x,y,z) \begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix} = \begin{pmatrix} xa + yd + ze \\ xd + yb + zf \\ xe + yf + zc \end{pmatrix}$$

$$\cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x^2 a + xyd + xze + yxd + y^2 b + yzf + xze + yzf + z^2 c =$$

$$= ax^2 + by^2 + cz^2 + 2dxy + 2exz + 2fyz$$

$$\begin{cases} \mathcal{C} \\ z = 1 \end{cases}$$

$$ax^2 + by^2 + 2dxy + 2ex + 2fy + c^2$$



$y = x^2$ $\qquad$ $x^2 - y = 0$ $\qquad$ $z^2\left(\frac{x^2}{z^2} - \frac{y}{z}\right) = x^2 - zy$

$\begin{cases} x^2 - zy = 0 \\ z = 0 \end{cases}$ $\qquad$ $\begin{matrix} x^2 = 0 \\ z = 0 \end{matrix}$ $\qquad$ $[0, 1, 0]$

$$\begin{cases} (\qquad\qquad) = 0 \\ t = 0 \end{cases}$$

$$a x^2 + b y^2 + 2 d x y = 0 \qquad a \frac{x^2}{y^2} + 2 d \frac{x}{y} + b = 0$$

$$d^2 - ab \gtreqless 0$$

Polarità

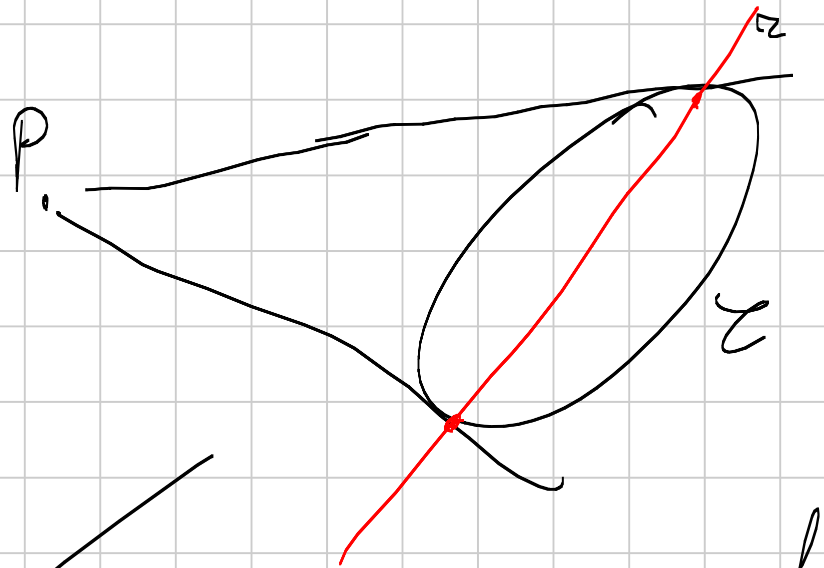
$\det A \neq 0$

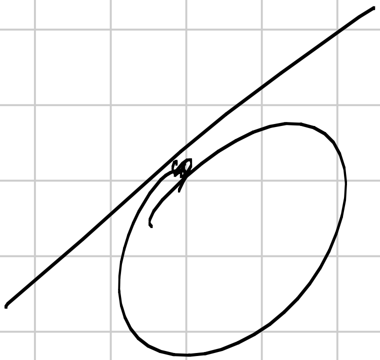
$[a, b, c]$ $\quad \mathcal{C} = \{ {}^t X A X = 0 \}$

$\Downarrow$

$P \qquad P \longrightarrow \{ {}^t P A X = 0 \}$

$$\left\{ [x, y, t] \;\middle|\; (a, b, c) \, A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \right\}$$





$pol_{\mathcal{C}}(r) = P$

$r = pol_{\mathcal{C}}(P)$

$pol_{\mathcal{C}}(P) \cap pol_{\mathcal{C}}(Q) =$

$= pol_{\mathcal{C}}(\mathcal{L}(P, Q))$

$pol_{\mathcal{C}}(P)$ tg a $\mathcal{C}$

se e solo se

$P \in \mathcal{C}$ e allora $pol_{\mathcal{C}}(P) \cap \mathcal{C} = P$