# GEOMETRIA PROIETTIVA 2

Titolo nota 04/09/2007

$P$ $\qquad \{X \mid {}^tP X = 0\}$ Retta

$P : [a, b, c] \qquad 0 = (a, b, c) \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = ax + by + cz$

$X : [x, y, z]$

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$A$ matrice $3 \times 3$ simmetrica $(A = {}^tA)$ $\quad \det A \neq 0$

$\mathcal{C} = \{X \mid {}^tX A X = 0\}$ conica

<u>Retta</u> <u>polare</u> : $P \longmapsto \{X \mid {}^tP A X = 0\} = \mathrm{pol}_{\mathcal{C}}(P)$

$${}^tPA = {}^t(AP) = {}^tP\,{}^tA = {}^tPA$$

<u>Caso</u> <u>particolare</u> : $P \in \mathcal{C}$ $\left( \iff {}^tPAP = 0 \right)$

$\iff P \in pol_{\mathcal{C}}(P)$

·) $P \neq Q$ $\quad {}^tP \cdot A \neq {}^tQ A \quad [{}^tPA] \neq [{}^tQA]$

$\Rightarrow pol_{\mathcal{C}}(\cdot)$ iniettiva

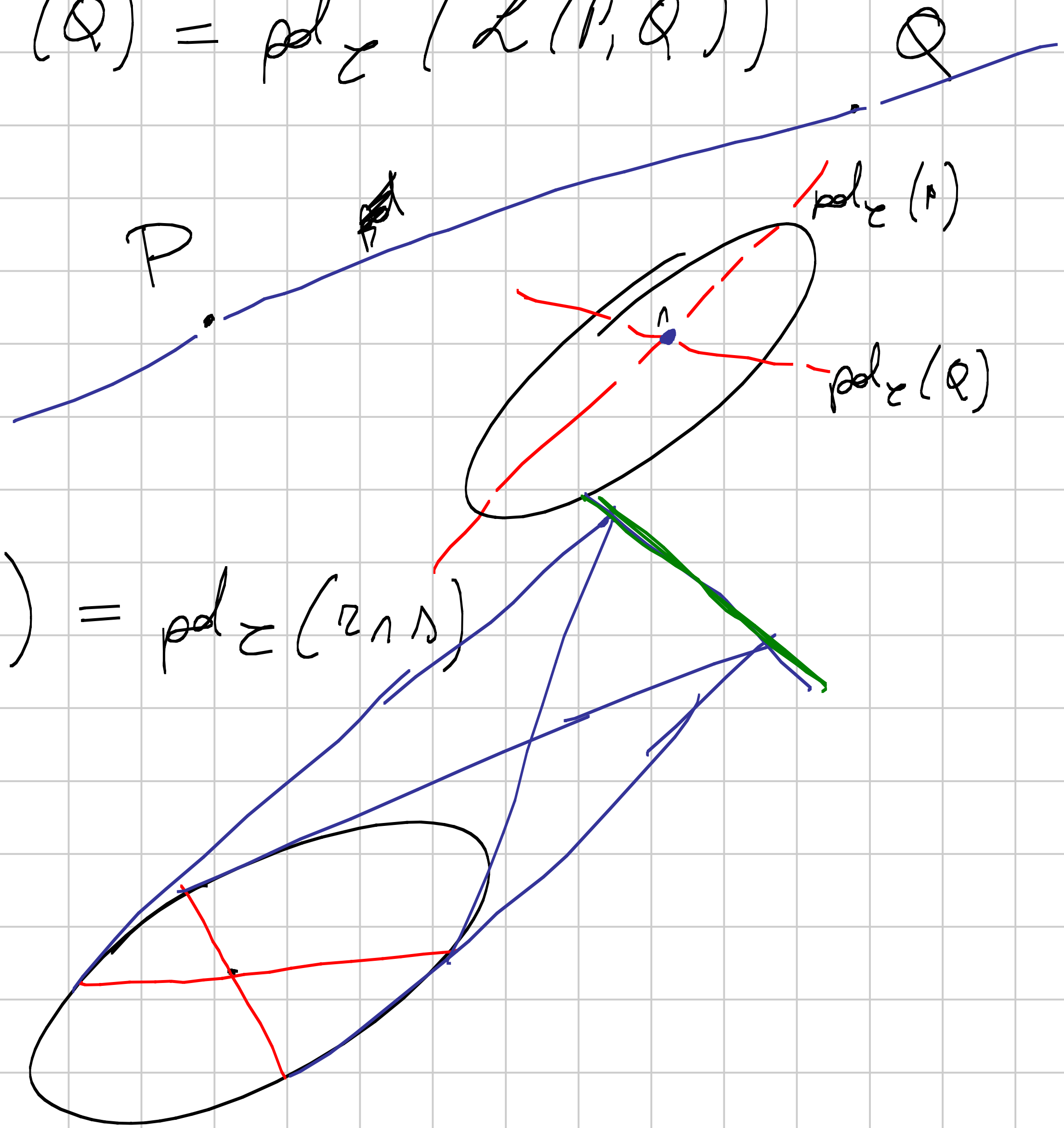
·) $pol_{\mathcal{C}}(P)$ surgettiva.

$r$ retta $\quad pol_{\mathcal{C}}(r) = P \iff pol_{\mathcal{C}}(P) = r$

·) $P \notin \mathcal{C}$ $\quad pol_{\mathcal{C}}(P) \cap \mathcal{C} = \{Q, R\}$

$\Rightarrow$ $PQ$, $PR$ Tangono $\mathcal{C}$

·) $pol_\tau(P) \cap pol_\tau(Q) = pol_\tau(\mathcal{L}(P,Q))$




·) $\mathcal{L}(pol_\tau(r), pol_\tau(s)) = pol_\tau(r \cap s)$

$$\{ x^2 + y^2 - z^2 = 0 \}$$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$P = \left[ \frac{1}{\phantom{1}}, \frac{1}{\phantom{1}}, \sqrt{2} \right]$$

$$\{ {}^tPAX = 0 \} = \{ x + y - z\sqrt{2} = 0 \}$$

$${}^tPA = (1, 1, \sqrt{2}) \begin{pmatrix} A \end{pmatrix} = (1, 1, -\sqrt{2})$$

$$\begin{cases} x' = ax + by \\ y' = dx + ey \end{cases}$$

$$x^2 + y^2 = 1$$

$T : \mathbb{P}^2 \longrightarrow \mathbb{P}^2$ $\quad \Pi$ inv. $\quad P \longrightarrow \Pi P$

$$\mathcal{C} = \{ {}^tXAX = 0 \} \longrightarrow T(\mathcal{C}) = \{ X' \mid {}^t(\Pi^{-1}X') A (\Pi^{-1}X') \} =$$

$$A \longrightarrow {}^t(\Pi^{-1}) A (\Pi^{-1}) \qquad = \{ X' \mid {}^tX' \, {}^t(\Pi^{-1}) A (\Pi^{-1}) X' \}$$

$$x^2+y^2+z^2=0$$

$$x^2+y^2-z^2=0$$

$$\left.\begin{array}{l} x^2+y^2=0 \\ x^2-y^2=0 \\ x^2=0 \end{array}\right] \quad \det A = 0$$

$pol_{\mathcal{C}}(\cdot)$ induce una dualità associata a $\mathcal{C}$.

$\{$terne omogenee$\}$ $\longrightarrow$ $\{$terne omogenee$\}$

$\| \qquad\qquad\qquad\qquad \text{u}$

coord dei punti $\qquad\qquad$ coeff delle rette

$$[a,b,c] \longmapsto [A]\cdot\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\begin{pmatrix} 1 & 2 & -3 \\ -2 & 1 & 0 \\ 3 & 0 & -1 \end{pmatrix}$$

$$x^2+y^2-z^2=0$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

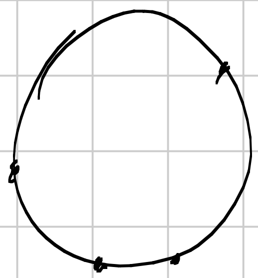
## Birapporto e coniche

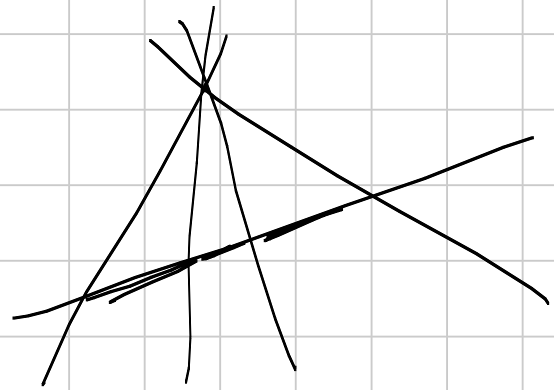
$A, B, C, D, O$

$$(A,B,C,D)_O := (AO, BO, CO, DO)$$

Teo: $A, B, C, D \in \mathcal{C}$ non degenere $\Rightarrow (A,B,C,D)_P$ è cost. al variare di $P \in \mathcal{C}$.

Dim: $T: \mathbb{P}^2 \to \mathbb{P}^2$ $\quad T(\mathcal{C}) = \{x^2+y^2-z^2=0\}$





Teo Pascal

Siano $P_i$, $i=1,\dots,6$ punti su $\mathcal{C}$. Allora, posti

$\mathcal{L}(P_1,P_2)\cap\mathcal{L}(P_4,P_5)=A_1$, $\mathcal{L}(P_2,P_3)\cap\mathcal{L}(P_5,P_6)=A_2$

$\mathcal{L}(P_3,P_4)\cap\mathcal{L}(P_6,P_1)=A_3$, $A_1, A_2, A_3$ sono allineati

Dim: $(P_2,P_4,P_5,P_6)_{P_1} = (P_1P_2, P_1P_4, P_1P_5, P_1P_6) =$

$= (P_1A_1, P_1P_4, P_1P_5, P_1X)$ con $X = P_1P_6 \cap P_4P_5$

$$(P_2, P_4, P_5, P_6)_{P_3} = (P_3P_2,\ P_3P_4,\ P_3P_5,\ P_3P_6) =$$

$$= (P_3A_2,\ P_3Y,\ P_3P_5,\ P_3P_6) \qquad Y = P_3P_4 \cap P_5P_6$$

$$(A_1, P_4, P_5, X) = (A_3A_1,\ A_3P_4,\ A_3P_5,\ A_3X)$$

$$\qquad\quad \parallel \qquad\qquad\qquad\qquad\qquad\quad \parallel$$

$$(A_2, Y, P_5, P_6) = (A_3A_2,\ A_3Y,\ A_3P_5,\ A_3P_6)$$

$A_3, Y, P_4$ allineati

$\Rightarrow A_3 A_1 A_2$ sono all.

<u>Duale di una conica</u>

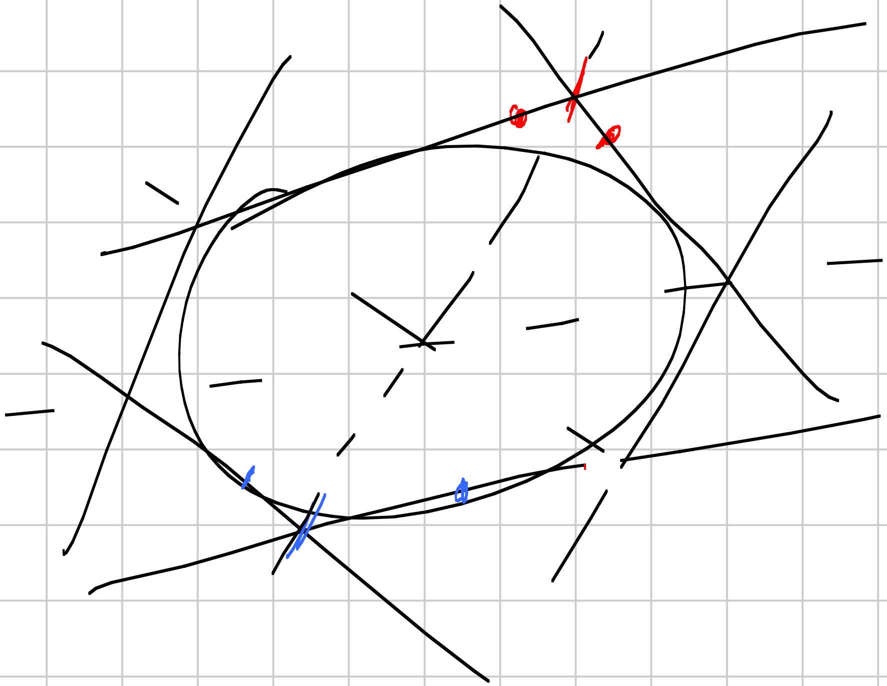
$$\mathcal{C} \text{ conica} \xrightarrow{M} \{\text{rette tangenti a } \mathcal{C}'\}$$

$$\{{}^tXAX=0\} \qquad\qquad \{{}^tX\,{}^t(M^{-1})AM^{-1}X=0\}$$

Duale di Pascal: Siano $r_i$ $i=1,\dots,6$ rette Tg a $\mathcal{C}$

$\mathcal{L}(r_j \cap r_{j+1},\ r_{j+3} \cap r_{j+4}) = l_j$  (ind. mod 6)  $j=1\dots3$

Allora $l_i$ concorrono



Teor. di Brianchon vero xchè duale di Pascal

Fasci di rette $\quad \lambda, \mu \in \mathbb{R}$

in $\mathbb{R}^2$ $\begin{cases} x + y - \sqrt{2} = 0 \\ 3x + \pi\sqrt{3}y - e = 0 \end{cases}$ $\quad \{ 0 = x(\lambda + 3\mu) + y(\lambda + \pi\sqrt{3}\mu) - \sqrt{2}\lambda - e\mu \}$

in $\mathbb{P}^2$ $\quad r : \{ {}^tPX = 0 \}$ $\quad \mathcal{F}(s, r) = \{ {}^t(\lambda P + \mu Q) X = 0 \mid [\lambda, \mu] \}$

$s : \{ {}^tQX = 0 \}$

$$\begin{array}{ccc} T : \mathcal{F}_1 & \longrightarrow & \mathcal{F}_2 \\ \downarrow & & \\ [\lambda, \mu] & \longrightarrow & [N] \begin{bmatrix} \lambda \\ \mu \end{bmatrix} = \begin{bmatrix} \lambda' \\ \mu' \end{bmatrix} \end{array}$$

$N$ matr. $2 \times 2$

$\det N \neq 0$

$[1, 0, 0]$

$\{ t\cancel{x}y + sz = 0 \}$

$[t, s]$

$[0, 1, 0]$

$\{ px + qz = 0 \}$

$[p, q]$

$1+ \quad \{ y = 0 \}$

$2 = \quad \{ z = 0 \}$

$3 \quad \{ y + z = 0 \}$

$$N = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$$

$\{ x + z = 0 \} \quad 1 +$

$\{ x - z = 0 \} \quad 2 =$

$\{ 2x = 0 \} \quad 3$

$\{ -2x - 2z + 10x - 10z = 8x - 12z = 0 \}$

$\{ 2x - 3z = 0 \}$

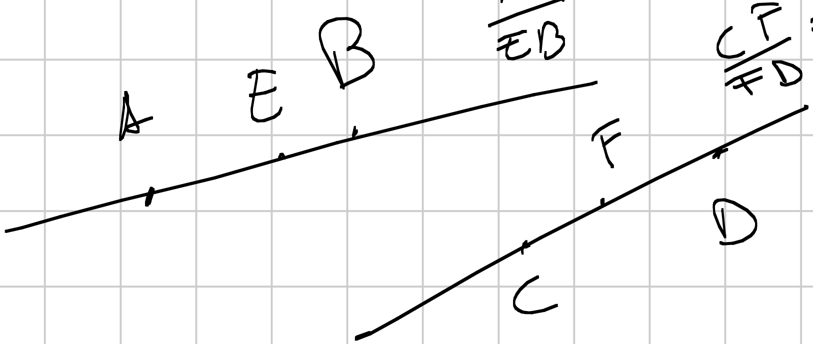
$\{ 3y + 5z = 0 \}$

$[3, 5]$

$$[3, 5] \xrightarrow{N \cdot \binom{3}{5}} [-2, 10]$$

$\frac{AE}{EB} = k$

$\frac{CF}{FD} = \frac{1}{k}$



# Teo di Steiner.

$T : \mathcal{F}_1 \to \mathcal{F}_2$ proiett. Allora:

$$\mathcal{C} = \{ r \cap T(r) \mid r \in \mathcal{F}_1 \}$$ è una conica.

Dim: $P$, $Q$ centro di $\mathcal{F}_1$, $\mathcal{F}_2$

Trovo una proiett. $F : \mathbb{P}^2 \to \mathbb{P}^2$ t.c.

$$F(P) = [1,0,0] \qquad F(Q) = [0,1,0]$$

$$\mathcal{F}_2 \to \{ y = k \mid k \in \mathbb{R} \} \qquad \mathcal{F}_2 \to \{ x = h \mid h \in \mathbb{R} \}$$

$\{y=0\}$ $\{\lambda y + \mu z = 0\} \to \{ y = -\frac{\mu}{\lambda} \}$

$\{z=0\}$

$\{x=0\}$ $\lambda'$ $\to \{ x = -\frac{\mu'}{\lambda'} \}$

$\{z=0\}$ $\mu'$

$$[\lambda, \mu] \to [a_{11}\lambda + a_{12}\mu,\ a_{21}\lambda + a_{22}\mu]$$

$$y = -\frac{\mu}{\lambda} \longrightarrow x = -\frac{a_{21}\lambda + a_{22}\mu}{a_{11}\lambda + a_{12}\mu} = -\frac{a_{21} + a_{22}\frac{\mu}{\lambda}}{a_{11} + a_{12}\frac{\mu}{\lambda}}$$

$$y = -k \ \wedge\ x = -\frac{a_{21} + a_{22}k}{a_{11} + a_{12}k} \longrightarrow \left(-\frac{a_{21} + a_{22}k}{a_{11} + a_{12}k},\ -k\right)$$

che è un p.

$\square$

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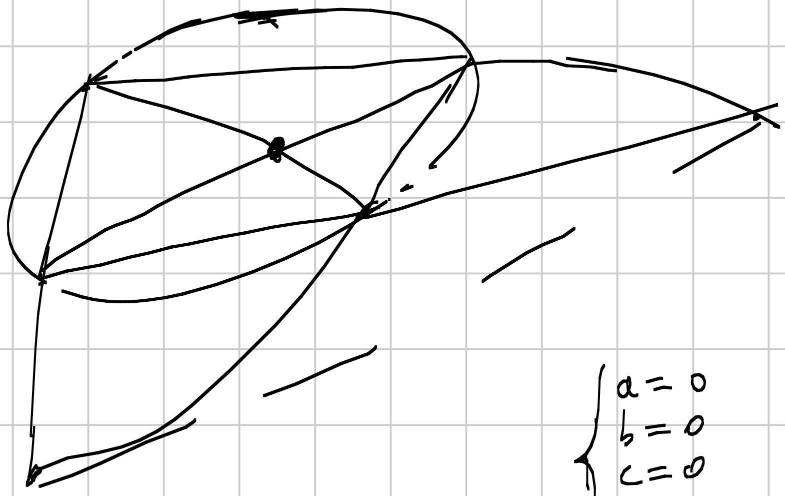
**Teo**: $A, B, C, D$ su $\mathcal{C}$, $E = AB \cap DC$, $F = BC \cap AD$.

Allora $AC \cap BD = \mathrm{pol}_{\mathcal{C}}(EF)$



Dim: $\exists T$ t.c.

$T(A) = [1,0,0]$

$T(B) = [0,1,0]$

$T(C) = [0,0,1]$

$T(D) = [1,1,1]$

$E = [1,1,0]$

$F = [0,1,1]$

$P = AC \cap BD = [1,0,1]$

$$\begin{cases} a = 0 \\ b = 0 \\ c = 0 \\ 2d + 2e + 2f = 0 \end{cases}$$

$f = 1$

$pol_{\mathcal{C}}(P) = \{x + z - y = 0\}$

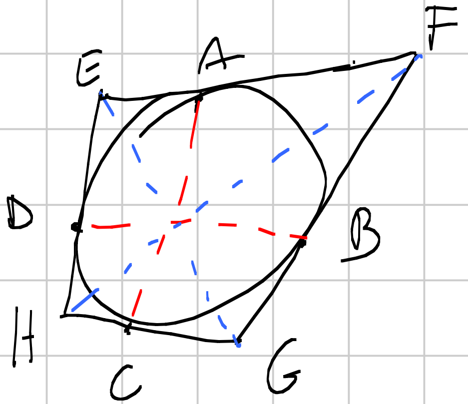
$\Downarrow$

ok

$\mathcal{C} = \{2(e+1)xy - 2exz - 2yz = 0\}$

Teo duale: fatevelo!!

Teo: $A, B, C, D$ on $\mathcal{C}$, $EF$ tg $\mathcal{C}$ in $A$, $FG$ tg $\mathcal{C}$ in $B$, $GH$ tg $\mathcal{C}$ in $C$, $HE$ tg $\mathcal{C}$ in $D$.




Allora $AC, BD, FH, EG$ concorrono.

Dim: $P = FG \cap EH \quad Q = EF \cap HG$

$$pol_{\mathcal{C}}(PQ) = EG \cap FH$$

$pol_{\mathcal{C}}(F) = AB \quad pol_{\mathcal{C}}(FG) = B \quad pol_{\mathcal{C}}(EH) = D$

$pol_{\mathcal{C}}(P) = BD \quad pol_{\mathcal{C}}(Q) = AC$

$$pol_{\mathcal{C}}(PQ) = pol_{\mathcal{C}}(P) \cap pol_{\mathcal{C}}(Q) = AC \cap BD = EG \cap FH. \quad \square$$

Oss: $(U, V, W, Z) = -1$, $r$ passa per $Z$ e non per gli altri

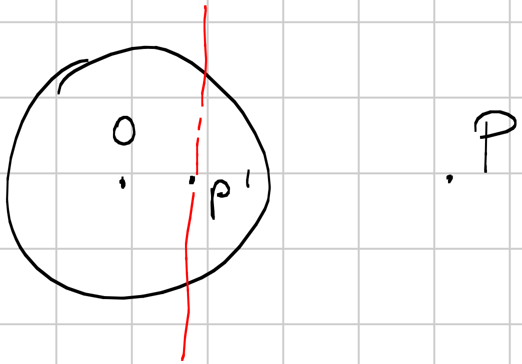
se "mando $r$ all'$\infty$", nell'affine $\overline{UW} = \overline{VW}$ e $W$ sta tra $U$ e $V$.

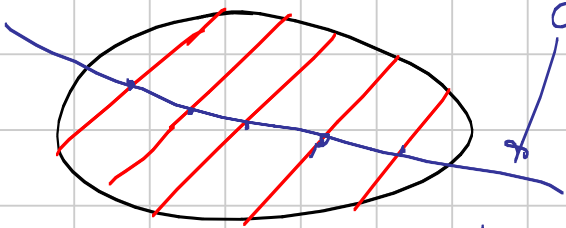
$$\frac{UW}{VW} \Big/ \frac{UZ}{VZ}$$

## Teo delle corde e delle polare

Siano $A, B \in \mathcal{C}$, $P \in \mathcal{L}(A, B)$.
Sia $Q = \text{polare}(P) \cap AB$. Allora $(A, B, P, Q) = -1$.

<u>Cor</u>: La pol. di $P$ rispetto a una cfr. di centro $O$ è la retta $\perp OP$ passante per l'inv. di $P$.



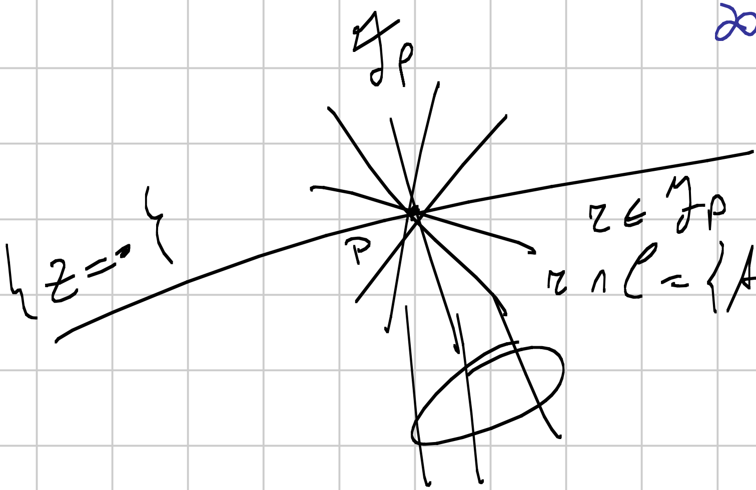


diametro: biseca corde parallele

i diam. concorrono nel centro

oppure

sono paralleli

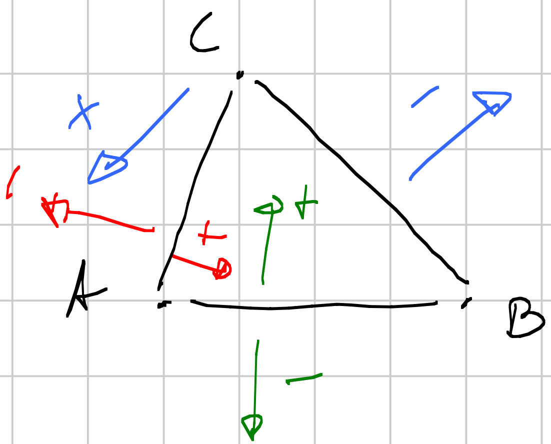



$r \cap \mathcal{C} = \{A, B\}$ trovo $C$ t.c. $(A, B, C, P) = -1$

# COORDINATE TRILINEARI

Def: Sia $ABC$ un triangolo, definiamo le coord. trilineari esatte del punto $P$ come

$$\{x, y, z\} \quad \text{con} \quad x = d(P, BC) \quad y = d(P, CA) \quad z = d(P, AB)$$




Oss: $xa + yb + zc = 2\Delta$

Oss 2: Dati $A, B, C, x, y$ posso calcolare $z$

$$z = \frac{2\Delta - yb - xa}{c}$$

$[x, y, z]$

Oss 3: Per ogni $(x, y, z) \in \mathbb{R}^3$ t.c. $ax + by + cz \neq 0$ esiste uno e un solo $P$ t.c. $x = \lambda\, d(P, BC)$ $y = \lambda\, d(P, CA)$

$z = \lambda\, d(P, AB)$ per un qlc. $\lambda \in \mathbb{R}$

$ax + by + cz = T \neq 0$ $\qquad x' = \frac{x}{T} 2\Delta \qquad y' = \frac{y}{T} 2\Delta$

$\{x', y', z'\}$ $\qquad z' = \frac{z}{T} 2\Delta$

$ax' + by' + cz' = 2\Delta$

<u>Def 2</u>: $[x, y, z]$ si dicono coord Trilineari di P.

<u>Oss</u>: $T : \mathbb{P}^2 \longrightarrow \mathbb{P}^2$

$T(\{\cancel{a}x + \underbrace{by + cz}_{=z} = 0\}) = \{z = 0\}$

| A, | B, | C, | I |
|---|---|---|---|
| 1 | 0 | 0 | 1 |
| 0 | 1 | 0 | 1 |
| 0 | 0 | 1 | 1 |

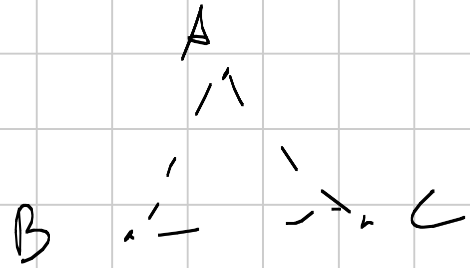
$AI \cap BC = [0, 1, 1] = D$ $\qquad (B, C, D, U) = -\frac{c}{b}$

$BC \cap r = U$

$E = AC \cap BI$ $\qquad (A, C, E, V) = -\frac{c}{a}$

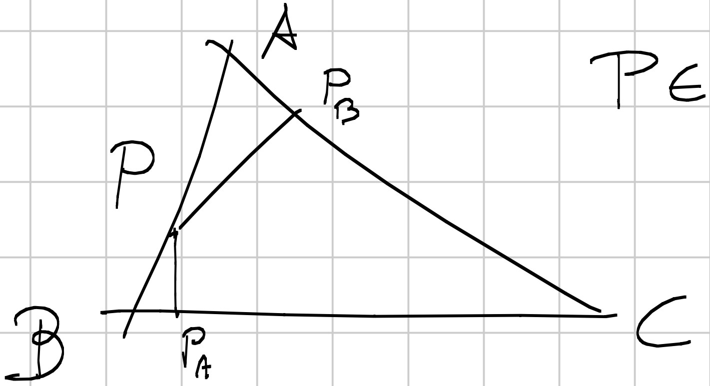
$V = AC \cap r$

$F = AB \cap CI \quad W = AB \cap r \quad (A, B, F, W) = -\frac{b}{a}$



$$[x,y,z] \xrightarrow{T} \xrightarrow{z=1} \xrightarrow{f:\to\Delta} \begin{pmatrix} p \\ q \end{pmatrix}$$

## Calcolo delle coord Trilineari



$P \in AB$ t.c. $\frac{AP}{PB} = k \quad k \in \mathbb{R} \setminus \{0,-1\}$

$$PP_A = PB \sin B$$
$$PP_B = PA \sin A$$

$$PA = AB\frac{k}{1+k} \qquad PB = AB\frac{1}{1+k} \qquad \sin A = \frac{BC}{2R}$$

$$\sin B = \frac{AC}{2R}$$

$$[PP_A, PP_B, 0] = [b, ka, 0]$$

$$[c, 0, ak] \qquad [0, c, bk]$$

$R, S \quad U = RS \cap r = [\ \text{——}\ ] \qquad (R, S, P, U) = -k$

**Punti ovvi**

Incentro $\{r, r, r\}$ — $[1, 1, 1]$

Bisettrice: $\{y - z = 0\}$ $AI$ $\quad \{x - z = 0\}$ $BI$ $\quad \{x - y = 0\}$ $CI$

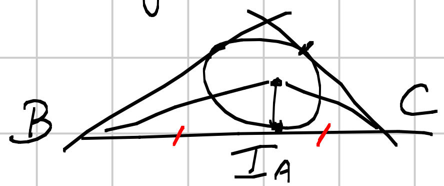
$[0, 1, 1] \qquad [1, 0, 1] \qquad [1, 1, 0]$

Excentri: $\{-r_a, r_a, r_a\}$ $\quad [-1, 1, 1]$

$\{r_b, -r_b, r_b\}$ $\quad [1, -1, 1]$

$\{r_c, r_c, -r_c\}$ $\quad [1, 1, -1]$

Punti di tg del cerchio inscritto:




$$\frac{BI_A}{I_AC} = \frac{tg\left(\frac{C}{2}\right)}{tg\left(\frac{B}{2}\right)}$$

$\Rightarrow$ (formule di prima) $I_A = \left[0, c, b \frac{tg\left(\frac{C}{2}\right)}{tg\left(\frac{B}{2}\right)}\right]$

$$I_A = \left[0, \cos^2\left(\frac{C}{2}\right), \cos^2\left(\frac{B}{2}\right)\right]$$

$$\begin{cases} a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{23}yz + 2a_{13}xz = 0 \\ x = 0 \end{cases} \qquad a_{22}y^2 + a_{33}z^2 + 2a_{23}yz = 0 \qquad a_{23}^2 = a_{22}a_{33}$$

$$a_1x^2 + a_2y^2 + a_3z^2 \pm 2\sqrt{a_1a_2}\,xy \pm 2\sqrt{a_2a_3}\,yz \pm 2\sqrt{a_1a_3}\,xz = 0$$

$a_1 = \cos^4\left(\frac{A}{2}\right) \quad a_2 = \cos^4\left(\frac{B}{2}\right) \quad a_3 = \cos^4\left(\frac{C}{2}\right)$ (tutti $-$)

<u>Exscritto opposto ad A</u>

$a_1 = \cos^4\left(\frac{A}{2}\right) \quad a_2 = \sin^4\left(\frac{B}{2}\right) \quad a_3 = \sin^4\left(\frac{C}{2}\right)$

$+2 \quad -2 \quad -2$

Circocentro: $\Pi_a, \Pi_b, \Pi_c$

$$O\Pi_a = OB \sin(\widehat{OBC}) = R \cos A$$

$$O : [\cos A, \cos B, \cos C]$$

Cfr cerchio: $a_{11} = 0$, $a_{22} = 0$, $a_{33} = 0$

1. $O = \mathrm{pol}_\gamma(r)$
2. passaggio per un pt. (es: asse n lorentz)

$$\{2 \sin A\, zy + 2 \sin B\, xz + 2 \sin C\, yx = 0\}$$

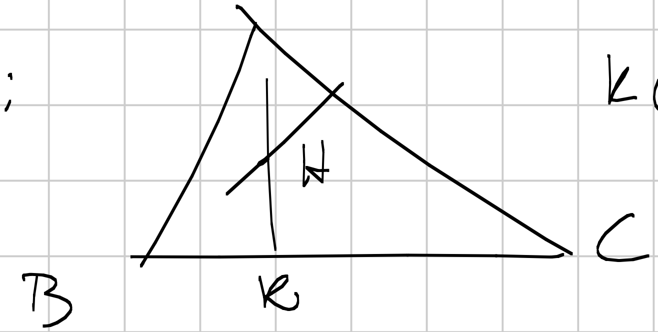
$$\{a\, zy + b\, xz + c\, yx = 0\}$$

Baricentro: $\left[\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right] : G$

Eq. retta di Eulero: $\frac{\cos A}{\frac{1}{a}}, \frac{\cos B}{\frac{1}{b}}, \frac{\cos C}{\frac{1}{c}}$ $\left[\frac{\cos B}{c} - \frac{\cos C}{b}, \frac{\cos C}{a} - \frac{\cos A}{c}, \frac{\cos A}{b} - \frac{\cos B}{a}\right]$

$$\{ \sin(2A)\sin(B-C)\,x + \sin(2B)\sin(C-A)\,y + \sin(2C)\sin(A-B)\,z = 0 \}$$

<u>Ortocentro</u> :




$KC = b\cos C \qquad HK = KC\cot(B)$

$$HK = \frac{b\cos C\cos B}{\sin B} = 2R\cos C\cos B$$

$$\left[\frac{1}{\cos A}, \frac{1}{\cos B}, \frac{1}{\cos C}\right]$$

$G : [\,\mathrm{sen}\,B\,\mathrm{sen}\,C,\ \mathrm{sen}\,A\,\mathrm{sen}\,C,\ \mathrm{sen}\,A\,\mathrm{sen}\,B\,]$

$H : [\,\cos B\cos C,\ \cos A\cos C,\ \cos A\cos B\,]$

$O = G - H$

$$E_\infty = \begin{bmatrix} \cos A - 2\cos B\cos C \\ \cos B - 2\cos C\cos A \\ \cos C - 2\cos A\cos B \end{bmatrix}$$

retta Eul

$$E_\infty = G - 3H \qquad -\frac{OG}{OH} = (G, H, O, E_\infty) = \frac{1 \cdot 1}{(-1)(-3)} = \frac{1}{3}$$

## Concorrenze di 3 ceviane

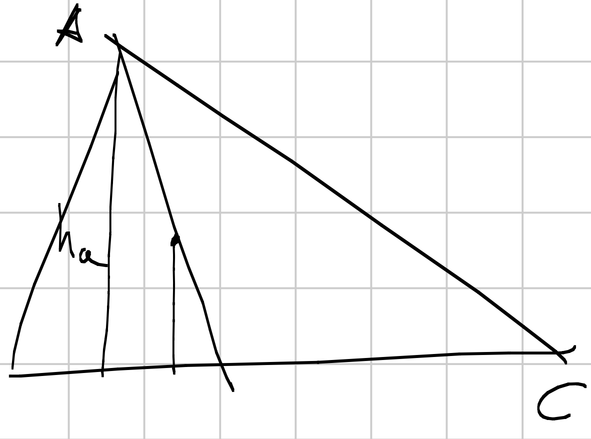
$AD, BE, CF$

$$h k j = 1$$

$$\frac{BD}{DC} = h \qquad \frac{CE}{EA} = k \qquad \frac{AF}{FB} = j$$

$P = AD \cap BE \cap CF$

Teo Van Obel




$$\frac{AP}{PD} = \frac{AF}{FB} + \frac{AE}{EC}$$

$$= j + \frac{1}{k}$$

$$\frac{AD}{PD} = j + \frac{1}{k} + 1$$

$$d(P, BC) = \frac{k h_a}{jk+k+1} = \frac{2\Delta}{a} \frac{k}{jk+k+1}$$

$$P : \left[ \frac{k}{a(jk+k+1)}, \frac{j}{b(hj+j+1)}, \frac{h}{c(kh+h+1)} \right]$$

## Teorema di Routh

$ABC$, $D \in BC$, $E \in CA$, $F \in AB$.

$L = BE \cap CF \quad M = AD \cap CF \quad N = AD \cap BE$

$$\left( \frac{BD}{DC}, \frac{CE}{EA}, \frac{AF}{FB} \right) = (\lambda, \mu, \nu)$$

$$\frac{[LMN]}{[ABC]} = \frac{(\lambda\mu\nu - 1)^2}{(\mu + \mu\nu + 1)(\nu + \nu\lambda + 1)(\lambda + \lambda\mu + 1)}$$

Coord baricentriche o areolari

$$P \longrightarrow \left\{ \overset{\alpha}{[PBC]}, \overset{\beta}{[PAC]}, \overset{\gamma}{[PAB]} \right\}$$

$$\{x, y, z\} \longrightarrow \left\{ \frac{xa}{2}, \frac{yb}{2}, \frac{zc}{2} \right\}$$

$$\alpha + \beta + \gamma = [ABC]$$

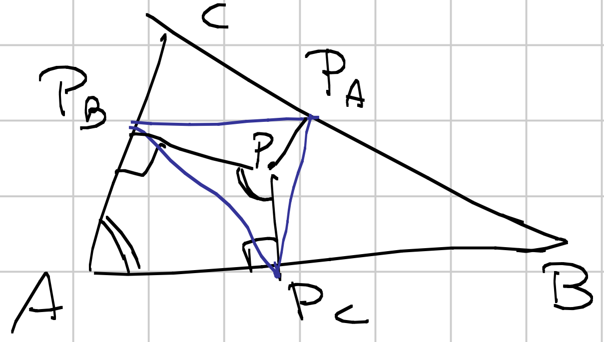
Se $P, Q, R$ hanno coord bar.

$$(p_1 : p_2 : p_3), \quad (q_1 : q_2 : q_3), \quad (r_1 : r_2 : r_3)$$

$$\frac{[PQR]}{[ABC]} = \det \begin{vmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{vmatrix}$$

# Teo di Eulero sui triangoli pedali

$P$ pt interno ad $ABC$. $P_A, P_B, P_C$ proiez di $P$ sui lati




$$\frac{[P_AP_BP_C]}{[ABC]} = \frac{R^2 - OP^2}{4R^2} = \frac{\text{pow}_\Gamma(P)}{4R^2}$$

$\Gamma$ circ. circoscritta.

$$[PP_CP_B] = PP_B \cdot PP_C \cdot \operatorname{sen} P_B\widehat{P}P_C \cdot \frac{1}{2} = \frac{1}{2} yz \operatorname{sen} A = \frac{1}{2}\frac{ayz}{2R}$$

$P : \{x, y, z\}$ tri es.

$$\frac{[P_AP_BP_C]}{[ABC]} = \frac{1}{2R} \frac{ayz + bxz + cxy}{ax + by + cz} =$$

$$= \frac{1}{abc}(ayz + bxz + cxy)$$

$$\vec{P} = \frac{ax\vec{A} + by\vec{B} + cz\vec{C}}{2\Delta}$$

$$|\overrightarrow{OP}|^2 = \frac{R^2}{4\Delta^2}\sum_{cyc}\left(c^2z^2 + 2abxy\cos(2C)\right) =$$

$$= R^2 - \frac{R^2}{\Delta^2}\sum_{cyc} abxy\sin^2 C$$

$$OP^2 - R^2 = \frac{4R^2}{abc}(ayz + bxz + cxy) = 4R^2\,\frac{[P_AP_BP_C]}{[ABC]}$$

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Coniugato isogonale.

$$[1,0,0] \quad \{ty + sz = 0\} \quad [t,s]$$

bisett. $\{y - z = 0\}$ è fissa $\quad \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}k\\-k\end{pmatrix} \quad j = h$

$y = 0 \to z = 0$

$z = 0 \to y = 0$

$$0 = \begin{pmatrix}\alpha\\ \gamma\end{pmatrix} = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix} \qquad \begin{pmatrix}0 & j\\ h & 0\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix} =$$

$$0 = \begin{pmatrix}\beta\\ \delta\end{pmatrix} = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} \qquad = \begin{pmatrix}-j\\h\end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} t \\ s \end{pmatrix} = \begin{pmatrix} s \\ t \end{pmatrix}$$

simm di $\{ty + sz = 0\}$ è $\{sy + tz = 0\}$

$P = [p, q, r]$ $\qquad PA = \{ry - qz = 0\}$ $\qquad$ Coniugato

$\downarrow \qquad \downarrow$ Isogonale

$PB = \{\cancel{q}\, rx - pz = 0\}$ $\qquad \{rz - qy = 0\}$

$\{rz - px = 0\}$ $\qquad \left[\frac{1}{p}, \frac{1}{q}, \frac{1}{r}\right]$

$PC = \{qx - py = 0\} \longrightarrow \{px - qy = 0\}$