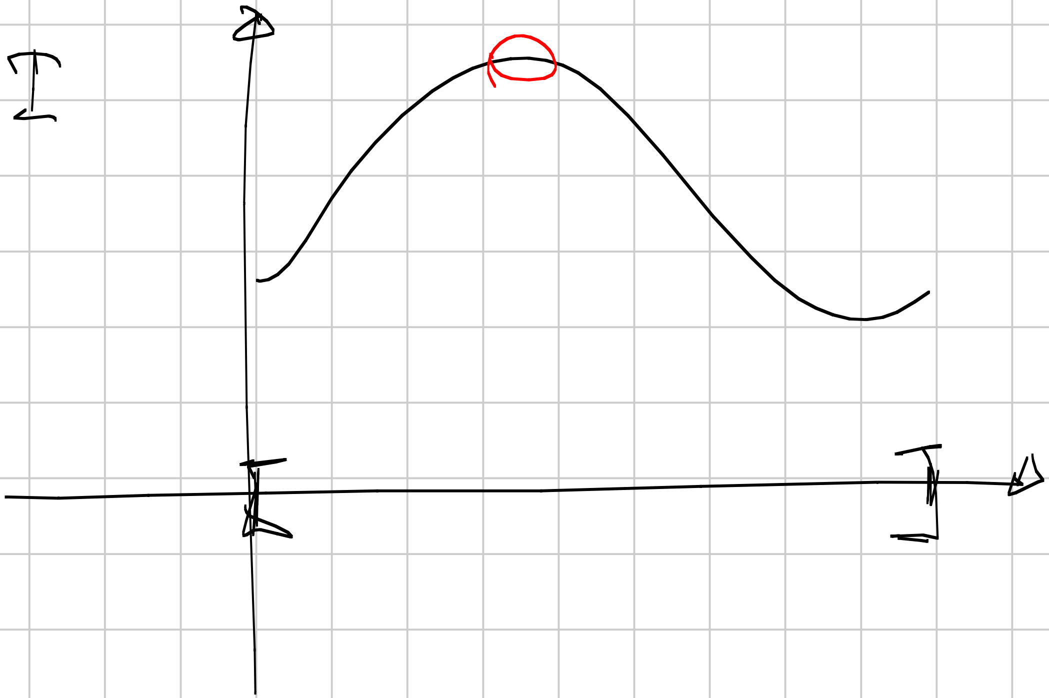
A4

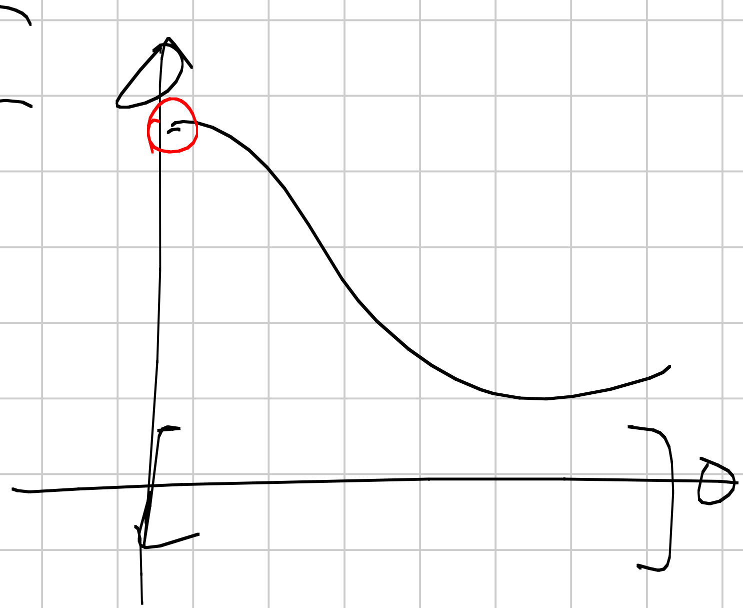
Titolo nota 09/09/2008

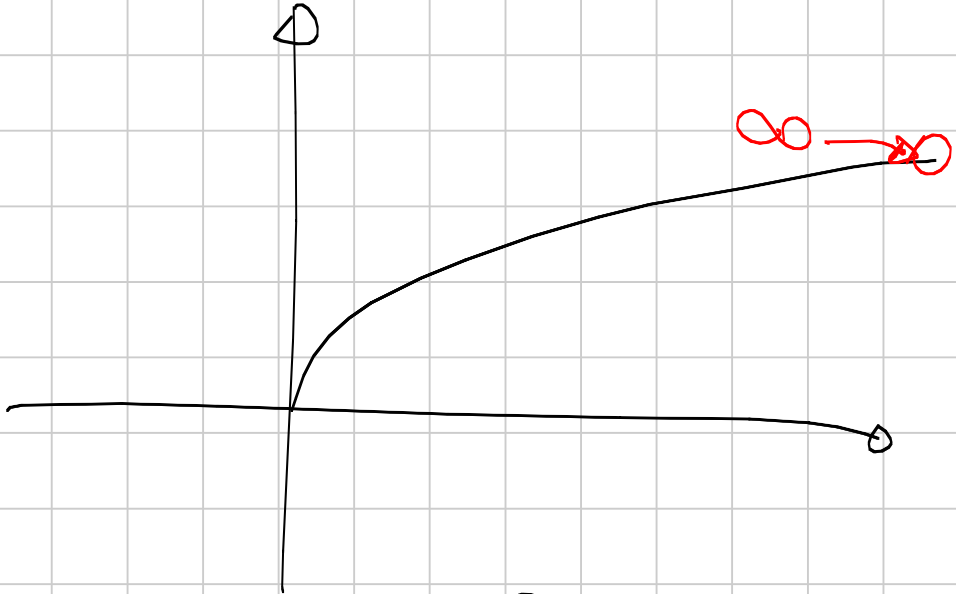
1) tecniche di analisi x disuguaglianze
2) convessità

I

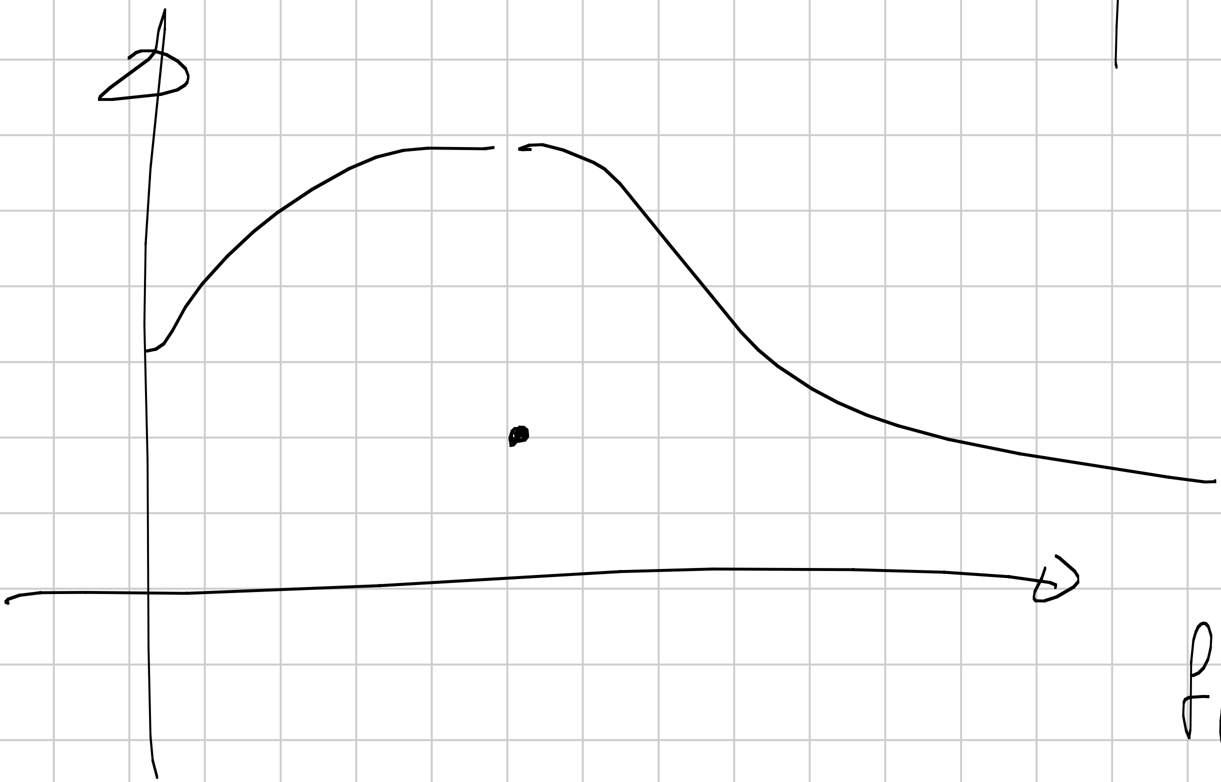


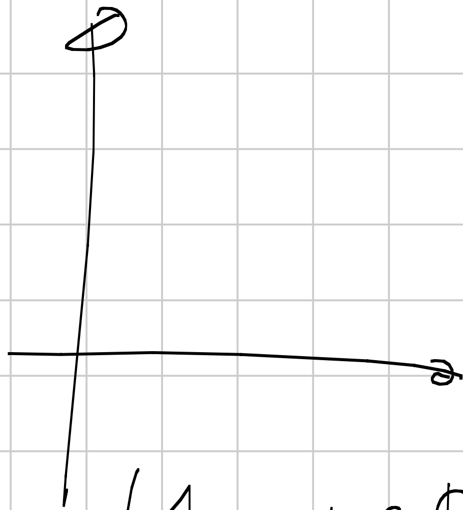
II





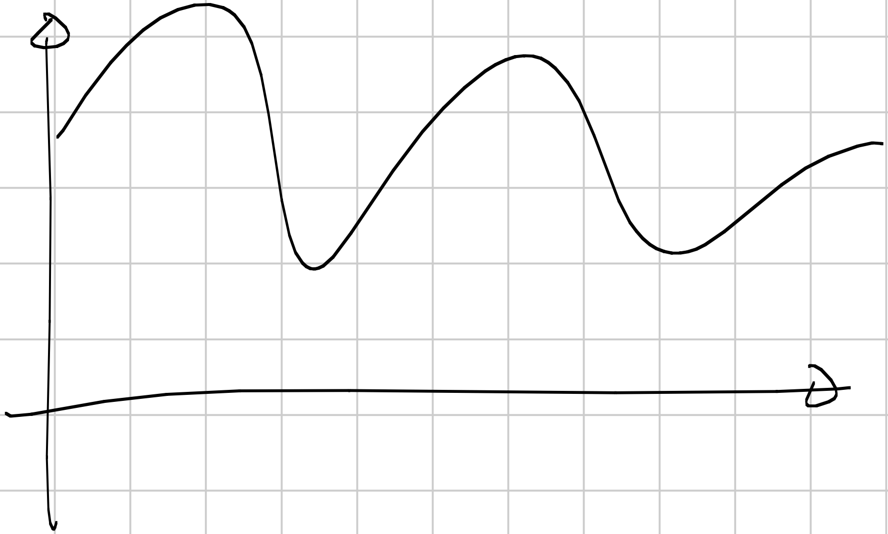
III





$$f(x)\begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

1) FUNZIONE CONTINUA



"TRATTO DI PENNA"

2) TUTTE LE "FORMULE"

+ , − , · , % ,

elevamenti a potenze

radici

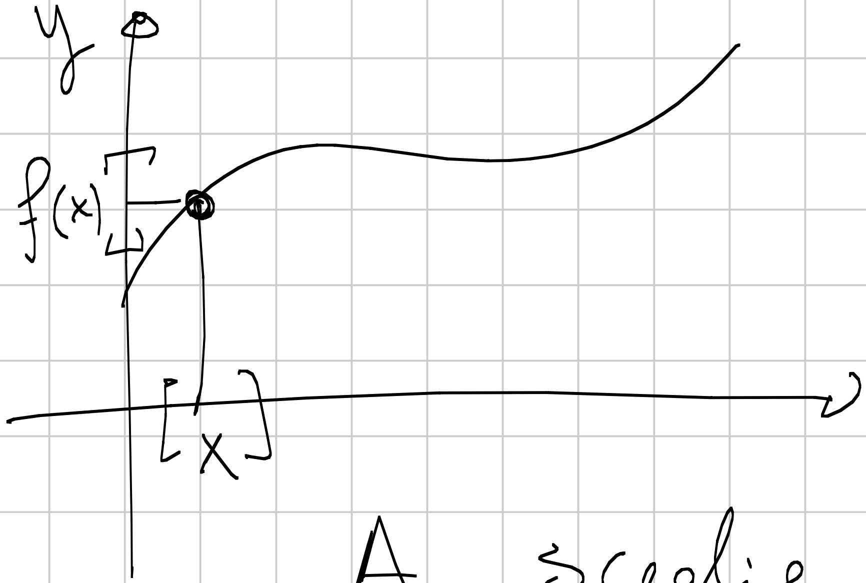
valori assoluti

exp log trigonometriche

$$f(x) = \frac{\sin e^{\cos|x|} + \operatorname{tg} x}{x^2 - 32}$$

FORMULE CONTINUE DOVE DEFINITE

3)



f

A B

A sceglie $[a,b] \ni f(x)$

B sceglie $[c,d] \ni x$

tale che $f([c,d]) \subseteq [a,b]$

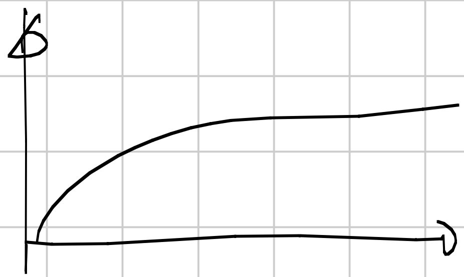
Se vince B funzione è continua

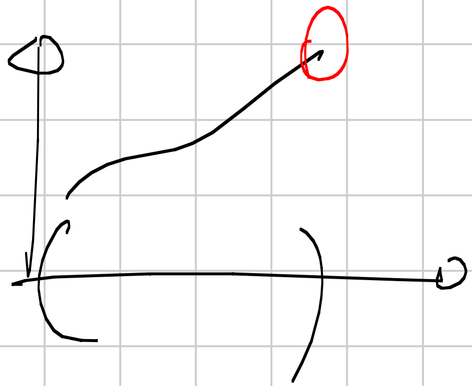
Teo (Weierstrass)

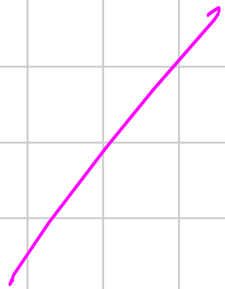
f continua su $[a,b]$ **allora**

esiste $\max f$

Quando W **fallisce**?







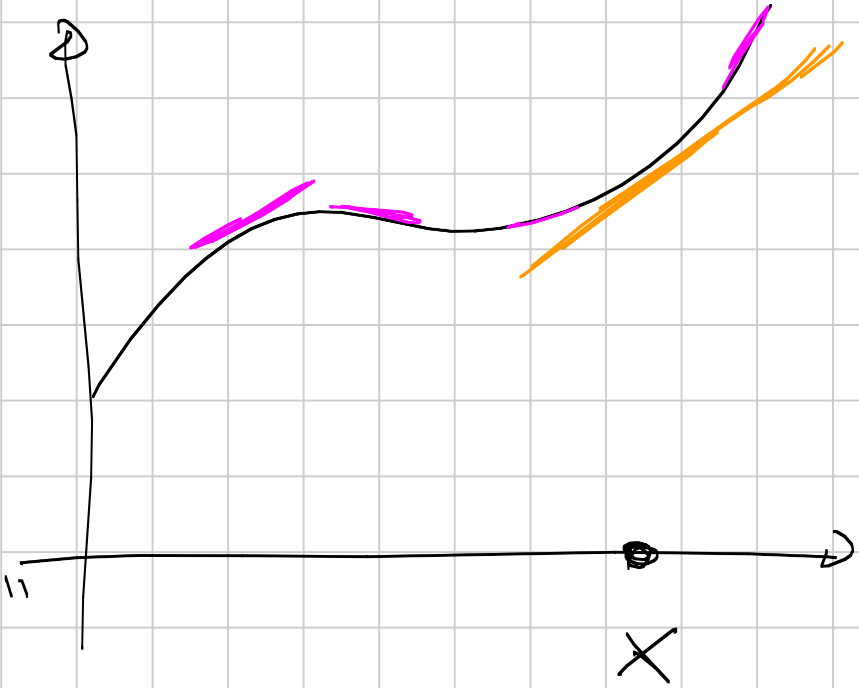
$(0,1)$

OK, c'è massimo

I) BORDO

II) INTERNO

DERIVATA



$x \mapsto$ pendenza

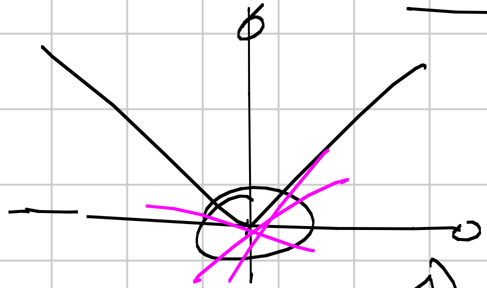
è detta "derivata di x"

" $\frac{dy}{dx}$ "

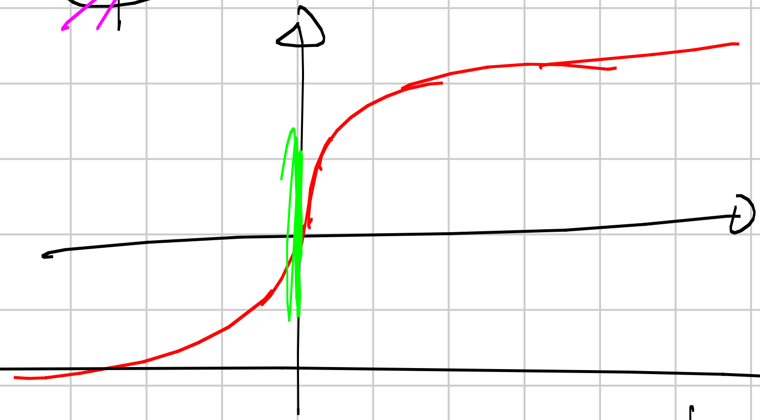
COSA È DERIVABILE E COSA NO:

TUTTE LE "FORMULE" ~~TRANNE~~

1) $|x|$

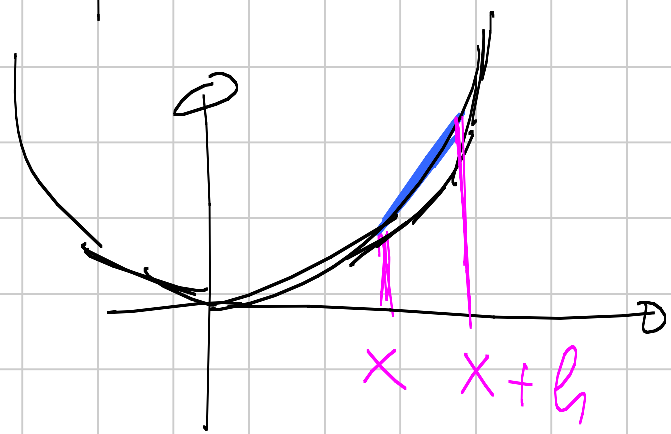


2) $x^r \quad r<1$



ESEMPIO STUPIDO

$$f(x) = x^2$$



$$\frac{(x+h)^2 - x^2}{x+h-x} = 2x+h \xrightarrow{h \text{ piccolo}} 2x$$

BESTIARIO

1) $D\,K f(x) = K\,D f(x)$

2) $D(f+g) = Df + Dg$

3) $D(fg) = Df \cdot g + f \cdot Dg$

4) $$\frac{d f(g(x))}{d\cancel{x}}$$

$$z = f(y) \qquad y = g(x)$$

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$$

DERIVATE NOTEVOLI

$$x^r \Rightarrow r x^{r-1} \qquad x^2 \to 2x \qquad \frac{1}{x} \to -\frac{1}{x^2}$$

$$\sqrt{x} \to \frac{1}{2} \frac{1}{\sqrt{x}}$$

$$\sin x \to \cos x$$

$$\cos x \to -\sin x$$

$$\operatorname{tg} x = \frac{\sin x}{\cos x} = \sin x \cdot \frac{1}{\cos x} =$$

$$= \cos x \cdot \frac{1}{\cos x} + \sin x \cdot D \frac{1}{\cos x} =$$

$$= 1 + \frac{\sin^2 x}{\cos^2 x} = 1 + \operatorname{tg}^2 x$$

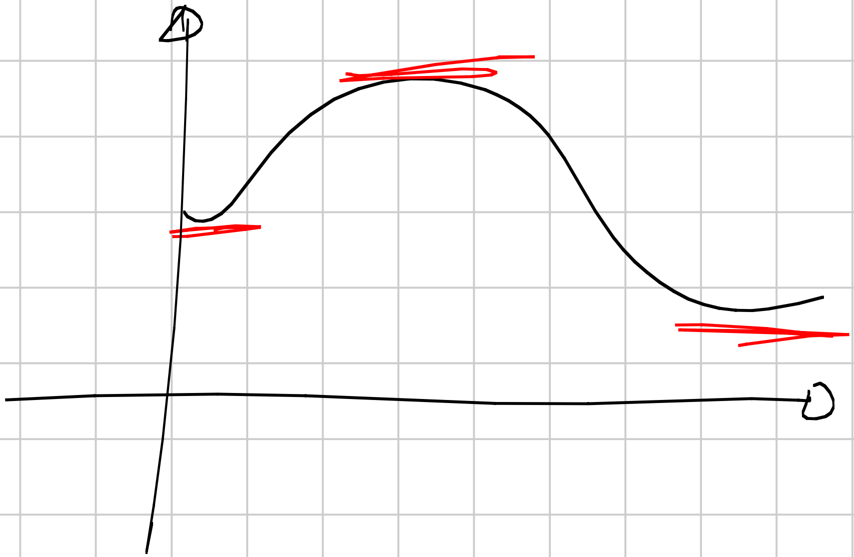
$$z = \operatorname{inv}(\cos x)$$

$$z = y^{-1} \qquad y = \cos x$$

$$\frac{dz}{dy} = -\frac{1}{y^2} \qquad \frac{dy}{dx} = -\sin x$$

$$\frac{dz}{dx} = + \frac{\sin x}{\cos^2 x}$$

IDEA



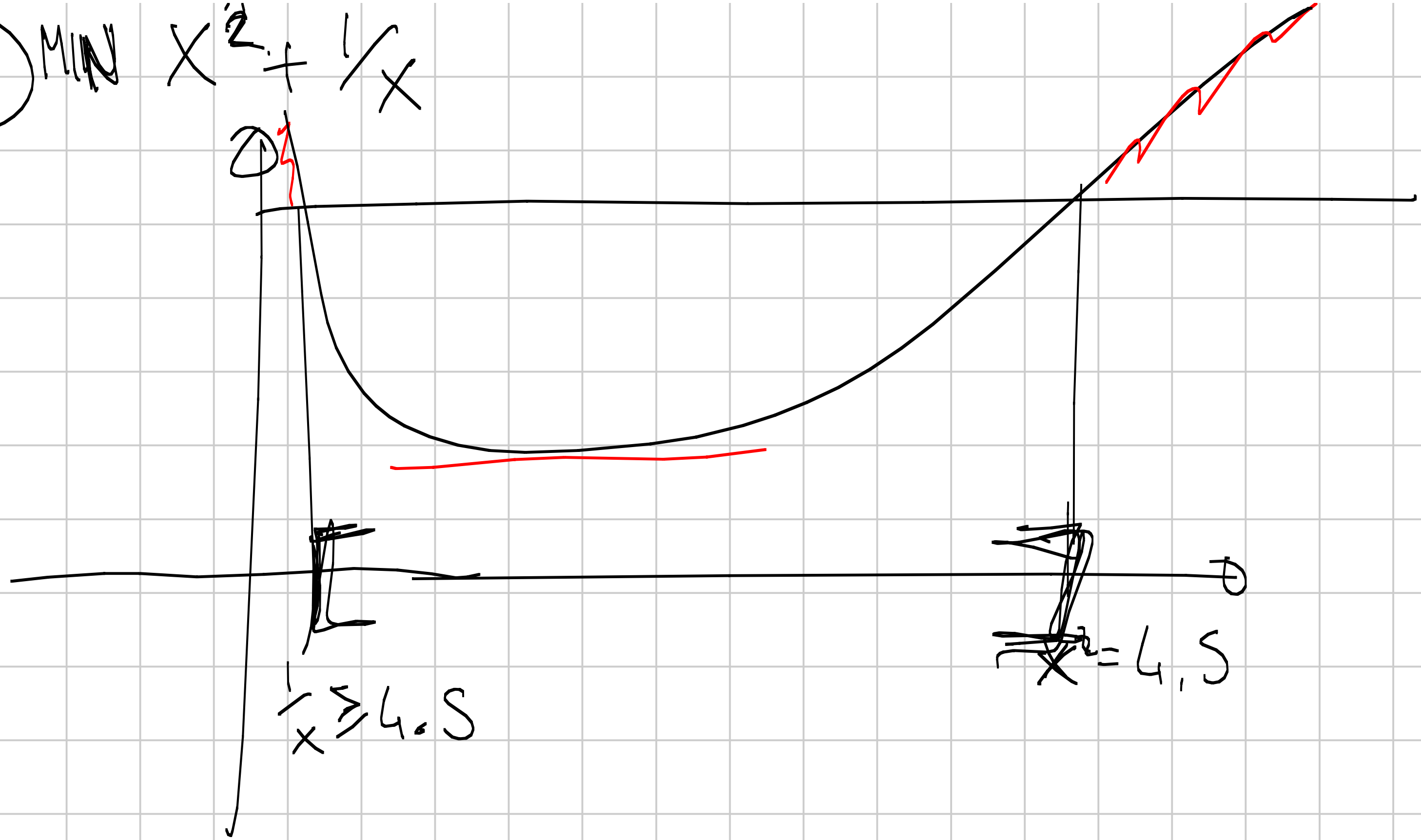
PROGETTO: A) MAX ESISTE PER WEIERSTRASS

1) DOVE $\frac{dy}{dx} = 0$

2) dove $\frac{dy}{dx}$ non esiste

3) ai bordi dell'intervallo

(E) MIN $x^2 + 1/x$



ESEMPIO DI DIM. DI DOV'È IL MINIMO

1) MIN ≤ 4.5

2) ~~PER~~ MIN STA IN $\left[\frac{1}{4.5}, \sqrt{4.5}\right]$

3) WEIER: MIN ESISTE

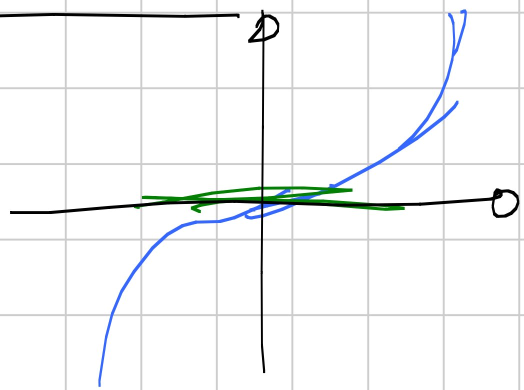
4) DERIVO BRUTALMENTE

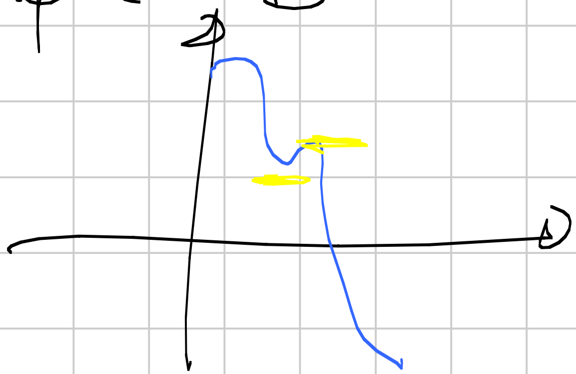
$$D\left(x^2+\frac{1}{x}\right) = 2x - \frac{1}{x^2} = 0$$

$$2x = \frac{1}{x^2} \qquad 2x^3 = 1$$

$$x = \sqrt[3]{1/2}$$

OCCHIO non sempre $D=0$ vuol dire MIN/MAX





5) CONTROLLO DOVE D NON ESISTE

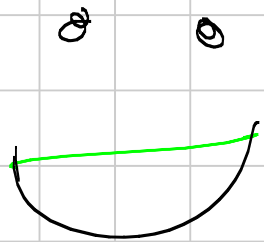
6) CONFRONTO I "CANDIDOTI"

$$x = \sqrt[3]{1/2}$$

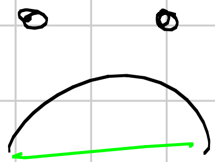
$$1/4.5$$

$$\sqrt{4.5}$$

7) IDEA: $f'' > 0$



$f'' < 0$



$$xy = 1 \quad , \quad \min_{x,y} \frac{x+y}{2}$$

$$\downarrow$$

$$y = 1/x \qquad \min_{x} \frac{x + 1/x}{2}$$

PIÙ VARIABILI

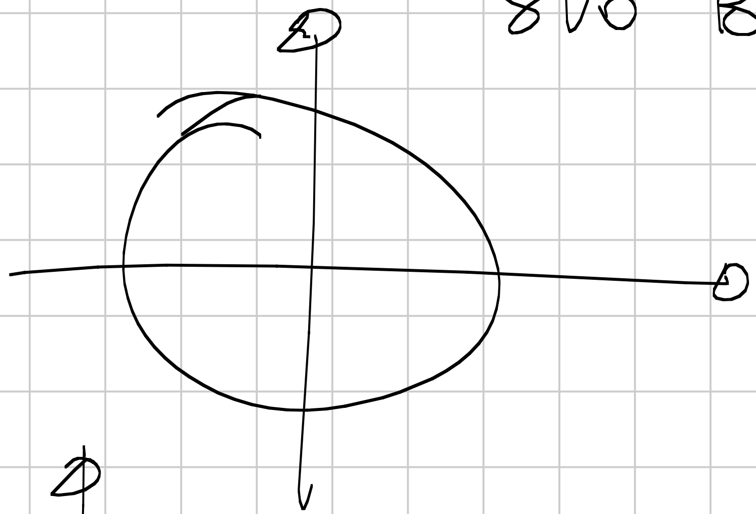
<u>FACILE</u> come si estende Weierstrass

W: f CONTINUA SU UN <u>COMPATTO</u>

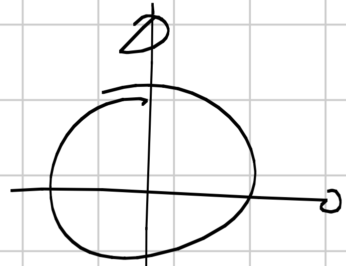
$\Rightarrow$ esiste MAX (MIN)

COMPATTO = CONTIENE I PUNTI SUL SUO BORDO

E: $x^2+y^2 \leq 1$ ✓



$x^2+y^2<1$ X



$xy \geq 1$ X

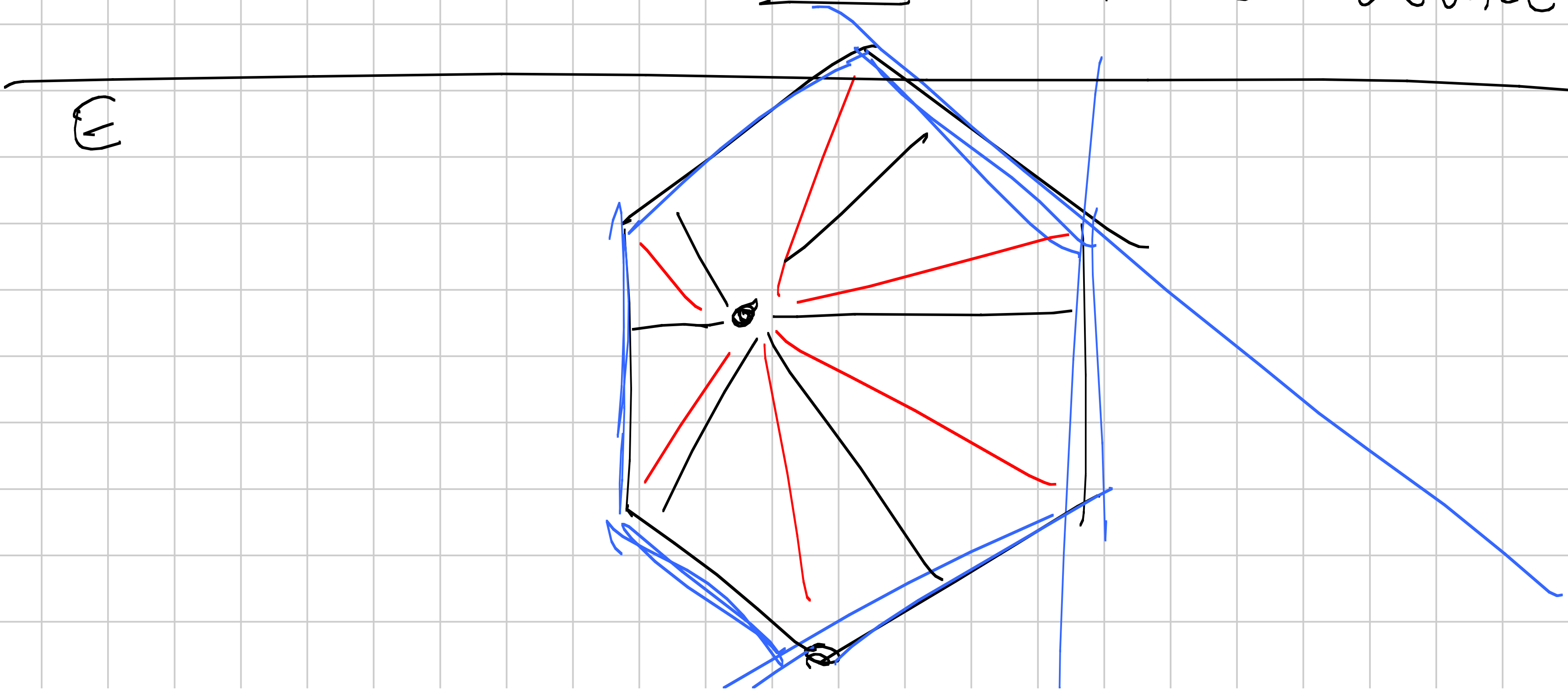
DEFINIZ. "OPERATIVA"

1) LIMITATO $(x \in S, |x| \leq 100'000)$

2) CHIUSO — DEFINITO DA FORMULA ≥ 0

≤ 0

<u>CON</u> SEGNO DI UGUALE



(x, y)

$$\sqrt{(x - x_p)^2 + (\quad - - \quad)} + \sqrt{}$$

CALCOLO DERIVATE:

SE $f(x, y) = \text{MIN}$

ROSSO FISSARE $\cancel{x} = a$ e $f(a, y)$ HA MAX

$$D_y f(a, y) = 0$$

se (a, b) MIN

$$D_x f(y, b) = 0$$

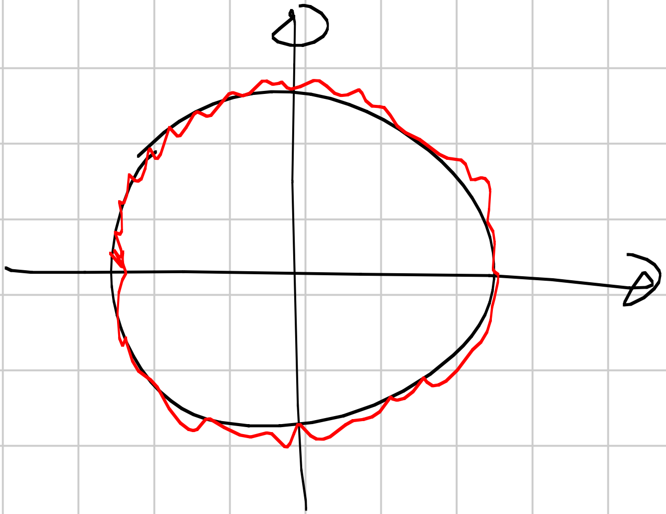
$$\text{MIN} \quad \frac{x}{y} + \frac{y}{x} + f(x)$$

$$\frac{\partial}{\partial y} \quad \frac{x}{y} + \frac{y}{x} = -\frac{x}{y^2} + \frac{1}{x} + 0$$

$$\frac{\partial}{\partial x} = -\frac{y}{x^2} + \frac{1}{y}$$

DEVONO ESSERE ENTRAMBE 0

COSA DIFFICILE: PROVARE I PUNTI SUL BORDO



$$\frac{\min}{\max} = \frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a}$$

a "grosso"

b, c "piccoli"

$1 + \square + 0$

c piccolo

b grosso

$1 + 1 + 0$ $\quad$ $1 + 0 + 0$

a smoothamente grosso

$$\frac{\partial}{\partial a}\left(\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}\right)=$$

$1-\frac{b}{a+b}$

$$=\begin{cases}\frac{b}{(b+a)^2} \quad 0 \quad -\frac{c}{(c+a)^2} = 0\\ \text{e cicliche}\end{cases}$$

$$b(c+a)^2 = (b+a)^2 c$$

$$bc^2 + ba^2 = b^2c + a^2c$$

$$a^2\cancel{(b-c)} = bc\cancel{(b-c)}$$

$$a \cdot a^2 = bc \cdot a$$

$$a^3 = abc$$

$$b^3 = abc$$

$$c^3 = abc$$

$$\boxed{a = b = c}$$

3 CASI:

$a = b = c$ ✓

$a = b \neq c$ ✗

$a \neq b \neq c$ ✗

$$f(a,a,a) = \frac{3}{2}$$

$$\cancel{\frac{3}{2}} \leq \qquad < 2$$

PUNTI INTERNI = FACILE
BORDO = PIÙ FATICOSO

$\downarrow$

(HOME, DISOMOGENEIZZANDO)

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3) MOLTIPLICATORI DI LAGRANGE

(*) $f(x, y, z)$

$g_1(x, y, z) = 0$

$g_2(x, y, z) = 0$

<u>ES</u> : MAX $x^3 + y^3 + z^3$

$xyz = \frac{1}{2}$

$xy + yz + zx = 1$

Th (Lagrange)

Il problema $(*)$ ha un massimo (interno)

$\iff$ $f(x) + \lambda_1 g_1(x) + \lambda_2 g_2(x)$

(VISTA COME FUNZIONE DELLE VARIABILI + DEI λ)

HA TUTTE LE DERIVATE NULLE

$$\to x^3 + y^3 + z^3 + \lambda_1\left(xyz - \frac{1}{2}\right) + \lambda_2\left(xy + yz + zx - 1\right)$$

DEVE AVERE LE $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$ NULLE

min/max $ab+bc+ca$ su $\boxed{a^2+b^2+c^2=1}$

$$ab+bc+ca+\lambda(a^2+b^2+c^2-1)$$

$\frac{\partial}{\partial a}$ $\boxed{b+c+\lambda\cdot 2a=0}$ e cicliche

e cicliche

$$2(a+b+c)+\lambda(2(a+b+c))=0$$

$$\lambda=-1$$

(a meno che $a+b+c=0$)

$$\boxed{b+c=2a}$$

$$a+b=2c$$

$$a+c=2b$$

$$b+c=2\lambda a=2\lambda(-b-c)$$

$$(b+c)(1+2\lambda)=0$$

~~$\alpha = 0$~~

$b + c = 0 \Rightarrow a = 0$

QUANDO TUTTI I VINCOLI SONO $= 0$
(+ COMPATTEZZA)
NO BORDO, LAGRANGE TROVA
TUTTI I MAX/MIN

QUANDO CI SONO DEI ≥ 0
C'È BORDO
(QUANDO ALMENO UN $\geq$ DIVENTA $=$)

E VA CONTROLLATO
(DI SOLITO CON UN ALTRO LAGRANGE)

ESEMPIO PARTICOLARMENTE CATTIVO

$$\sum \frac{a}{\sqrt{a^2+8bc}} \geq 1$$

$$\sum \frac{1}{\sqrt{1+8\frac{a^2}{bc}}} \geq 1$$

$\frac{a^2}{bc} = x \quad \frac{b^2}{ca} = y \quad = z$

$$\sum \frac{1}{\sqrt{1+8x}} \geq 1 \qquad \text{con vincolo } xyz = 1$$

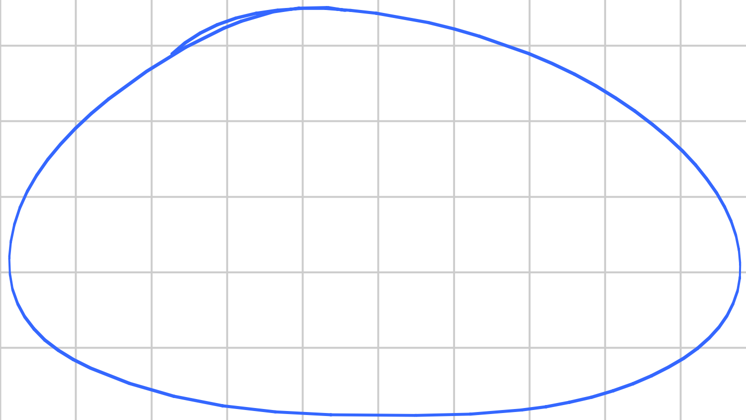
1) UGUALE IN MEZZO $\quad x = y = z = 1$

2) UGUALE ALL'INFINITO $\quad x = y = \text{grosso}$
$z = \text{piccolo}$

$$\sum \frac{1}{\sqrt{1+8x}} \geq 1 - \varepsilon \qquad \forall \varepsilon > 0$$

0,99

FISSATO ε



$$\frac{1}{\sqrt{1+8x}} \geq \sqrt{1-8x} \geq 1-\frac{8}{2}x \geq 1-4x$$

+ SISTEMONE (TANTI AUGURI)

- Convessa $\iff f'' \geq 0$
- convessità in + variabili
- convessità come: "max sul bordi"
- Karamata.

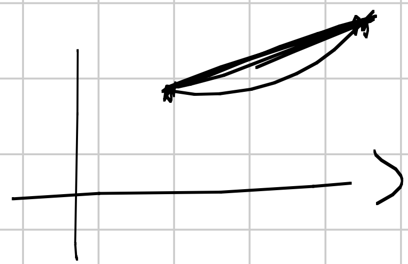
$$f \text{ convessa} \iff f''(x) \geq 0 \ \forall x$$

$\Updownarrow$ R cresce in ogni var.

$\Updownarrow$ f' crescente

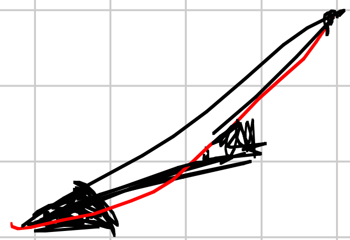
$$f(\underbrace{\lambda x + (1-\lambda) y}_{z}) \leq \lambda f(x) + (1-\lambda) f(y)$$

$$R(x,y) = \frac{f(x) - f(y)}{x-y}$$



$x < z < y$

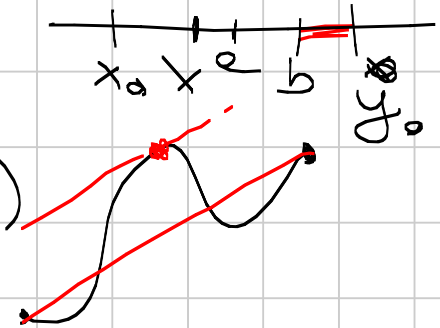
$$f(z) - f(x) \leq \frac{z-y}{x-y} f(x) + \frac{x-z}{x-y} f(y) - f(x)$$



$$\frac{f(z)-f(x)}{z-x} \leq \frac{f(y)-f(z)}{y-z}$$

$$f(z) - f(x) \leq \frac{z-x}{x-y} f(x) - \frac{z-x}{x-y} f(y)$$

$$\frac{f(x)-f(y)}{x-y} \leq$$

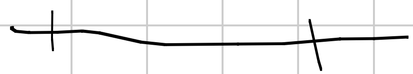


$$f^*(y_0) \geq R(y,y_0) \geq R(a,b) \geq R(x,x_0) \geq f'(x_0)$$

f

$$\frac{f(b)-f(a)}{b-a} = f'(\xi)$$

$$\xi \in [a,b]$$



f'' è cresc $\Longrightarrow$ il rapp incr. è cresc

$$R(a,b) \leq R(c,d)$$

$$\frac{f(b)-f(a)}{b-a} \leq \frac{f(d)-f(c)}{d-c} = f'(\xi_2)$$

$$\| \qquad\qquad \xi_2 \in [c,d]$$

$$f'(\xi_1) \qquad \xi_1 \in [a,b]$$

$$f(a,b,c)=\sum \frac{a}{1+bc} \leq 2 \qquad a,b,c \in [0,1]$$

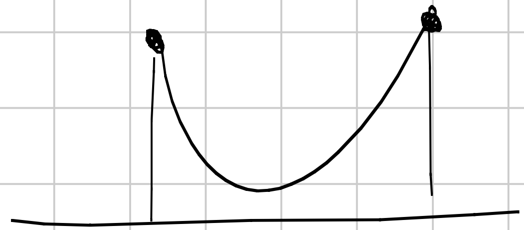
$$[0,1]^3$$

$$\frac{1}{x} \qquad \frac{1}{\cancel{cx}+1}$$

$$\frac{a}{1+bc}+\frac{b}{1+ac}+\frac{c}{1+ab}$$

$$(a_0, b_0, c_0)$$

$$a_0 \in \{0,1\}$$



$a=0 \quad b=1 \quad c=1$

$a=\varepsilon \quad b=1 \quad c=1$

$x, y, z \geq 0 \qquad x+y+z=3$

$$x^2+y^2+z^2+xyz \geq 4$$

$+2yz \quad -2yz$

$$x^2+(y+z)^2+(x-2)yz \geq 4$$

$$x^2+(3-x)^2+(x-2)yz \geq 4$$

$$0 \leq yz \leq \left(\frac{y+z}{2}\right)^2 = \frac{(3-x)^2}{4}$$

$yz=0 \qquad yz=\frac{(3-x)^2}{4}$

$$x^2+(3-x)^2 \geq 4$$

$$x^2+(3-x)^2+(x-2)\frac{(3-x)^2}{4} \geq 4$$

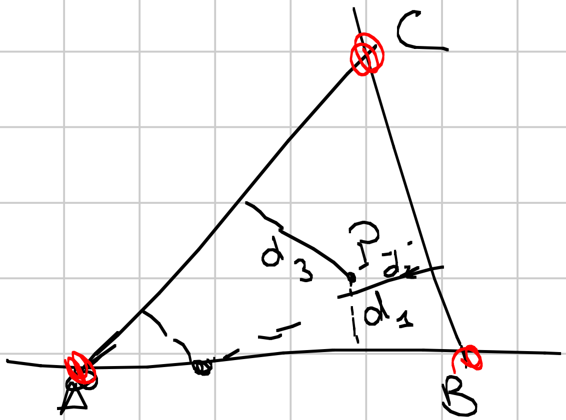
$$f'(x) = \underline{3x^2 - 3} = 0 \qquad \begin{matrix} 1 \\ -1 \end{matrix}$$

$\sum$ convesse è convesse

$$f(x,y) = xy$$

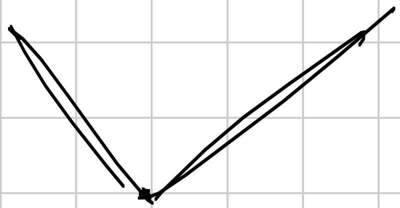
$$d_1 + d_2 + d_3$$

$\sum$ distanze dai lati



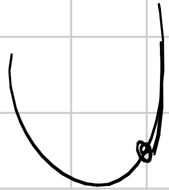
$P \longrightarrow$ distanze da ΔC

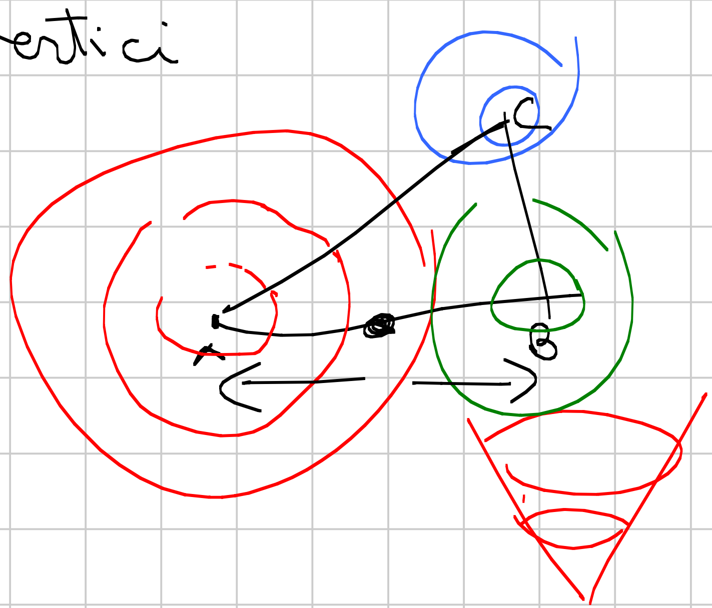
$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$$



$\sum$ distanze dai vertici

MAX





$(a_i) \succ (b)$

$a_1 \geq a_2 \geq \dots \geq a_n$

$b_1 \geq b_2 \geq \dots \geq b_n$

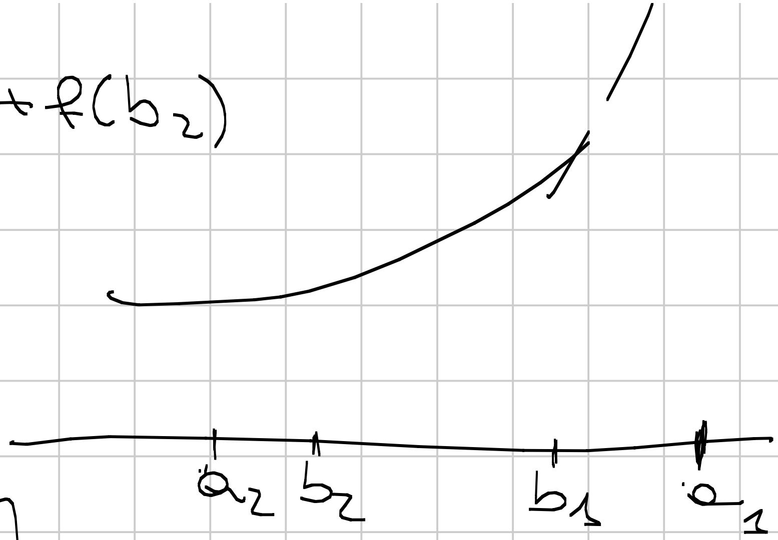
$a_1 + \dots + a_k \geq b_1 + \dots + b_k$

$\sum a_i = \sum b_i$

$f \in C^2$ convessa $\Longrightarrow$

$f(a_1) + \dots + f(a_n) \geq f(b_1) + \dots + f(b_n)$

$$f(a_1)+f(a_2) \geq f(b_1)+f(b_2)$$



$$b_2 \in [a_1, a_2]$$

$$b_2 = \lambda a_1 + (1-\lambda) a_2 \quad \lambda \in [0,1]$$

$$b_1 = \mu a_1 + (1-\mu) a_2 \quad \mu \in [0,1]$$

$$b_1 + b_2 = a_1 + a_2 = (\mu+\lambda) a_1 + (2-\lambda-\mu) a_2$$

$$\mu + \lambda = 1$$

$$b_2 = \lambda a_1 + (1-\lambda) a_2 \qquad b_1 = (1-\lambda) a_1 + \lambda a_2$$

$$f(b_2) \leq \lambda f(a_1) + (1-\lambda) f(a_2)$$

$$f(b_1) \leq (1-\lambda) f(a_1) + \lambda f(a_2)$$

$$\sum f(a_i) - \sum f(b_i) = \sum f(a_i) - f(b_i)$$

$$\frac{f(a_i) - f(b_i)}{a_i - b_i} = f'(\xi_i) \qquad \xi \in [a, b]$$

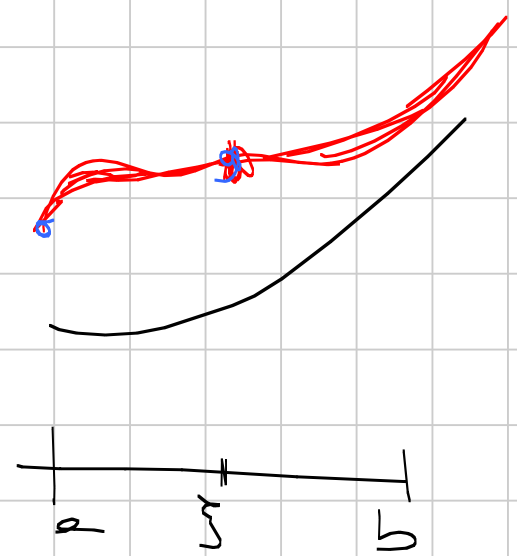
f' è cresc

$a > b$

$$\frac{f(a) - f(b)}{a - b} = f'(\xi) \geq f'(b)$$

$$f(a) - f(b) \geq f'(b)(a - b)$$

$b > a$



$$\frac{f(a)-f(b)}{a-b} = f'(\xi) \geq f'(a) \quad \text{INUTILE}$$

$$\leq f'(b)$$

$$f(a)-f(b) \geq f'(b)(a-b)$$

$$f(x)-f(y) \geq f'(y)(x-y) \quad \text{con } f \text{ conv.}$$

$$\sum^k a_i \geq \sum^k b_i$$

$$\sum f(a_i)-f(b_i) \geq \sum f'(b_i)(a_i-b_i)$$

$$= f'(b_1)(a_1-b_1) + f'(b_2)(a_2-b_2) + \dots$$

$$= (a_1-b_1)\big(f'(b_1)-f'(b_2)\big)+(a_1+a_2-b_1-b_2)$$

$$\big(f'(b_2)-f'(b_3)\big) + \dots \quad + \Big(\sum^n a_i - \sum^n b_i\Big) f'(b_n)$$

≥ 0

---

f convesso $\Rightarrow$

$$3\left[f(a)+f(b)+f(c)+f\left(\frac{a+b+c}{3}\right)\right] \geq 4\left[f\left(\frac{a+b}{2}\right)+f\left(\frac{b+c}{2}\right)+f\left(\frac{c+a}{2}\right)\right]$$

$a > b > c$

$$\left(a, a, a, b, \frac{a+b+c}{3}, \frac{a+b+c}{3}, \frac{a+b+c}{3}, b, b, c, c, c\right)$$

$$\left(\frac{a+b}{2}, \frac{a+b}{2}, \frac{a+b}{2}, \frac{a+c}{2} \times 3, \frac{b+c}{2} \times 3\right)$$

$$\left(\alpha, \alpha, \alpha, \alpha, \beta, \beta, \beta, \beta, \gamma, \gamma, \gamma, \gamma\right)$$

INSULTALO

(a)

(b)

$$\sum a_i \geq \sum b_i$$

(c)

lui che ha inventato il mangiare e il bevere, lui era **grande persona!**

$a_1, \dots, a_n \geq 0$

$$(1+a_1)(1+a_2)\cdots(1+a_n) \leq \left(1+\frac{a_1^2}{a_2}\right)\cdot \ldots \cdot\left(1+\frac{a_n^2}{a_1}\right)$$

Sam e' brutto!

SAM & MARIA SVIZZERA!

TORNA A CASA TUA

(PADRONI A CASA NOSTRA)

$$(1+a_1)\cdots(1+a_n) \le \left(1+\frac{a_1^2}{a_2}\right)\cdots\left(1+\frac{a_n^2}{a_1}\right)$$

$$\sum \ln(1+a_i) \le \sum \ln\left(1+\frac{a_i^2}{a_{i+1}}\right)$$

$c_i = a_{i_k}$ $(c) \le (d)$ $d_i = \frac{a_i^2}{a_{i+1}}$

c_i decresc $d_1 \ge \max(a_i)$

$$d_1 = \frac{\max(a_i)^2}{a_{i+1}}$$

$$\frac{a_i^2}{a_{i+1}}$$

$$\boxed{2a_i - a_{i+1}}$$

$c_i =$ a_i ordinati

$d_1 = 2a_{i_1} - a_{i_1+1}$

$c_1 = \max a_i \geq c_2 \ldots \geq c_m$

$$C = (a_{i_1}, a_{i_2}, \ldots a_{i_m})$$
$$a_{i_1} = c_1, \quad a_{i_2} = c_2, \quad a_{i_m} = c_m$$

$c_k = \ln a_{i_k}$

$d_1 = 2\ln a_{i_1} - \ln a_{i_1+1}$

$d_k = 2\ln a_{i_k} - \ln a_{i_k+1}$

$\ln a_1 \longrightarrow \ln 1 + a_1$

$x \longrightarrow \ln(1+e^x)$

$$\prod (1+a_i) \le \prod \left(1+\frac{a_i^2}{a_{i+1}}\right)$$

$$(1+a_1)^2 \le \left(1+\frac{a_1^2}{a_2}\right) \cancel{(1+a_2)}$$

$$(1+a_2)^2 \le \left(1+\frac{a_2^2}{a_3}\right) (1+a_3)$$

$$\vdots$$

$$\frac{1}{1+e^x}\cdot e^x = 1 - \frac{1}{1+e^x}$$

Altro uso dei moltiplicatori

$$a_1^2 + a_2^2 + \cdots + a_{100}^2 = 1 \qquad a_1, \dots, a_{100} \in \mathbb{R}_{\geq 0}$$

$$a_1^2 a_2 + a_2^2 a_3 + \cdots + a_{100}^2 a_1 < \frac{12}{25}$$

$$\begin{cases} a_{100}^2 + 2a_1 a_2 = 2\lambda a_1 \\ a_1^2 + 2a_2 a_3 = 2\lambda a_2 \\ \vdots \\ a_{99}^2 + 2a_{100} a_1 = 2\lambda a_{100} \end{cases}$$

$$a_{k-1}^2 + 2a_k a_{k+1} = 2\lambda a_k$$

$$\left(\sum \left(e_{k-1}^2 + 2e_k e_{k+1}\right) e_k\right)^2 \le \Big(\underbrace{\sum e_k^2}_{\text{}}\Big)^{\overset{1}{\shortparallel}} \left(\sum \left(e_{k-1}^2 + 2e_k e_{k+1}\right)^2\right)$$

$$\shortparallel$$

$$\left(\sum e_{k-1}^2 e_k + \sum 2e_k^2 e_{k+1}\right)^2$$

$$(3T)^2 \le \sum \left(e_{k-1}^2 + 2e_k e_{k+1}\right)^2$$

$$= \sum e_{k-1}^4 + \sum 4e_{k-1}^2 e_k e_{k+1} + \sum 4e_k^2 e_{k+1}^2$$

AM-GM su $e_{k-1}e_k$ e $e_{k-1}e_{k+1}$

$$\le \sum \left(e_{k-1}^4 + 2e_{k-1}^2 e_{k+1}^2 + 6e_k^2 e_{k+1}^2\right)$$

$$2e_k^2 e_{k+1}^2$$

$$\le \sum \left(\sum e_k^2\right)^2 + 4e_k^2 e_{k+1}^2$$

$$\leq 4\left(\sum a_{2i}^2\right)\left(\sum a_{2i+1}^2\right) + 1 \leq 2$$

$$(3T)^2 \leq 2 \Rightarrow T \leq \frac{\sqrt{2}}{3} \overset{?}{<} \frac{12}{25}$$

$$\frac{2}{9} \overset{?}{<} \frac{144}{625} \to 1250 \overset{?}{<} 1296$$

$\underline{ok!}$