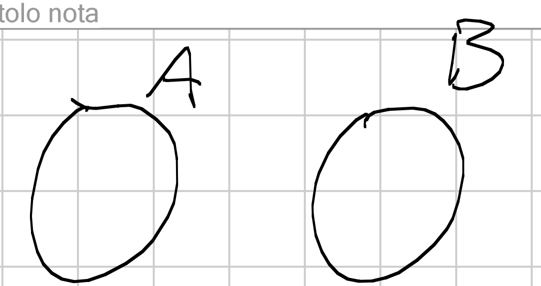
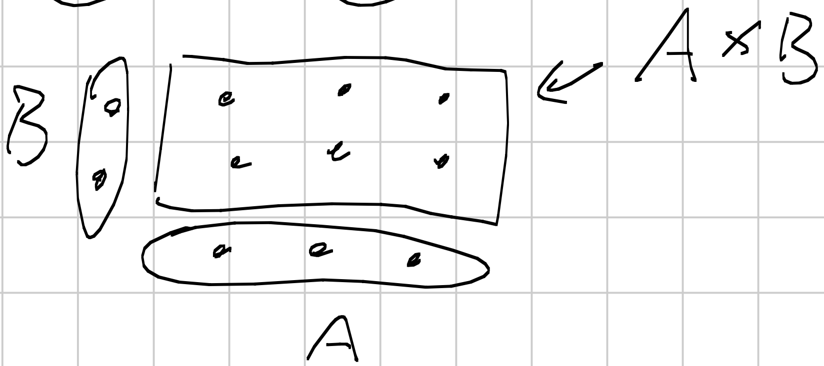
C 1 F. VENEZIANO

Titolo nota 10/09/2008



$$|A \cup B| = |A| + |B|$$



$$|A \times B| = |A| \cdot |B|$$

FATTORIALE

$n!$

$n \in \mathbb{N}$

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1$$

$$0! = 1 \qquad (n+1)! = (n+1) \cdot n!$$

$A = \{1, \dots, n\}$

$$|\{f : A \to A,\ \text{biunivoche}\}| = n!$$

$f(1)$	n
$f(2)$	$n-1$
$f(3)$	$n-2$
$\vdots$	
$f(n)$	1

$n!$

$$\left(\frac{n}{3}\right)^n \leq n! \leq \left(\frac{n}{2}\right)^n \quad \forall n \geq 6$$

$$\left(n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \right)$$

$$\binom{n}{k} \qquad n \geq k \geq 0$$

$\Rightarrow$ Il numero di sottoinsiemi con k elementi di $\{1, 2, \dots, n\}$

$k \quad 0 \qquad\qquad n$

$$
\begin{array}{cccccccccccc}
0 & & 0 & & 1 & & 1 & & & & \leftarrow & 0 \\
0 & & 0 & 1 & & 1 & & 2 & & & \leftarrow & 1 \\
& & & 1 & & 2 & & 1 & & & \leftarrow & 2 \\
& & 1 & & 3 & & 3 & & 1 & & \leftarrow & 3 \\
& 1 & & 4 & & 6 & & 4 & & 1 & & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 &
\end{array}
$$

$$\boxed{\binom{n}{k} = \binom{n-1}{k} + \left[\binom{n-1}{k-1}\right] \qquad \binom{n}{0} = 1 \quad \binom{n}{n} = 1}$$

$$\boxed{\binom{n}{k} = 0 \text{ se } k < 0 \text{ o } k > n}$$

$A = \{1, \ldots, n\} \qquad B \subseteq A,\ n \in B \qquad |B| = k$

$A' = \{1, \ldots, n-1\} \qquad B' = B \setminus \{n\} \subseteq A' \qquad |B'| = k-1$

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

$C \subseteq A \qquad |C| = k \quad n \notin C \qquad C \subseteq A'$

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} \qquad 0 \le k \le n \qquad \frac{n!}{0!\,n!} = \frac{n!}{n!\,0!} = 1$$

$$\frac{n!}{k!\,(n-k)!} \overset{?}{=} \frac{(n-1)!}{k!\,(n-k-1)!} + \frac{(n-1)!}{(k-1)!\,(n-k)!}$$

$$\frac{n}{k\,(n-k)} \overset{?}{=} \frac{1}{k} + \frac{1}{n-k}$$

A BC   B AC   C AB

A CB   BCA   CBA

$\cancel{A}\ A\ B$ $A\ B\ A$ $B\ A\ A$

$A\ A\ B$ $A\ B\ A$ $B\ A\ A$

$$\frac{6!}{2!}$$

$n \quad a_1 \quad a_2 \quad a_k$

$$\frac{n!}{a_1!\,a_2!\cdots a_k!}$$

MATEMATICA

$$\frac{10!}{3!\,2!\,2!}$$

$\underbrace{A \cdots A}_{k}\ \underbrace{B \cdots B}_{n-k}$

$$\frac{n!}{k!\,(n-k)!}$$

$1^\circ \cdot$ n°

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

$$\binom{n}{i} x^i y^{n-i}$$

$$\underbrace{(x+y)(x+y)\cdots(x+y)}_{n}$$

$$\binom{n}{k} = \binom{n}{n-k}$$

$$\binom{n}{k} \cdot \frac{n+1}{k+1} = \binom{n+1}{k+1}$$

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \frac{n^k}{k!}$$

$\binom{n}{k} \sim \frac{n^k}{k!}$ k fissato n molto grande

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n \xleftarrow{x=1} (1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

||

$$|\mathcal{P}(\{1, \ldots, n\})| = 2^n \leftarrow \{1, \ldots, n\}$$

$$\{A \mid A \subseteq \{1, \cdots, n\}\}$$

$$f : \{1, \ldots, n\} \to \{0, 1\}$$

$$\Updownarrow$$

$$A = \{x \in \{1, \cdots n\} \mid f(x) = 1\}$$

$$A \xrightarrow{f} B$$

$$\underbrace{|B| \cdot |B| \cdot |B| \cdots}_{|A|}$$

$$|B^A| = |B|^{|A|}$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$$

$n \geq k$

$n > m$

$(k > m)$

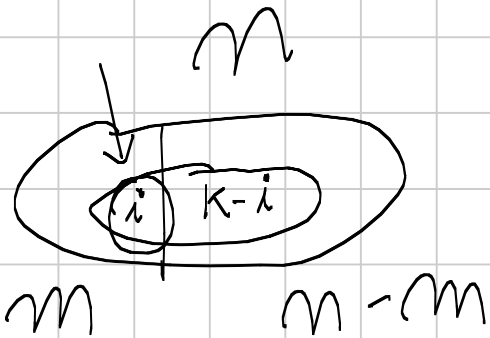
$$\sum_{j=0}^{m} \binom{m}{j} \binom{n-m}{k-j} = \binom{n}{k}$$

$$\boxed{\binom{m}{k} = 0 \text{ se } k > m}$$

$$(1+x)^n = (1+x)^m (1+x)^{n-m}$$

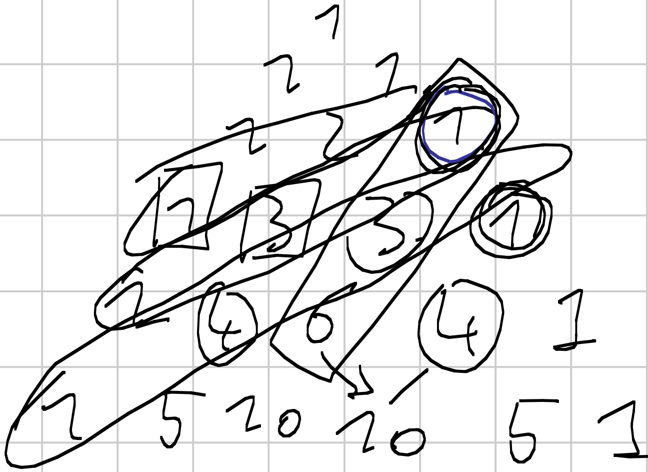
$$\sum_{i=0}^{n} \binom{n}{i} x^i = \left(\sum_{j=0}^{m} \binom{m}{j} x^j\right) \cdot \left(\sum_{l=0}^{n-l} \binom{n-m}{l} x^l\right)$$

$$= \underbrace{\left[\sum_{\alpha=0}^{i} \binom{m}{\alpha} \cdot \binom{n-m}{i-\alpha}\right]}_{= \binom{n}{i}} x^i$$



$$\sum_{i=0}^{m} \binom{m}{i} \binom{n-m}{k-i}$$

$$\sum_{n=k}^{k+h} \binom{n}{k}$$



$$= \binom{n+1}{k+1} \qquad \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} = F(n+1)$$

$$\sum_{k=0} \binom{n-2-k}{k} + \underbrace{\sum_{j=0} \binom{n-1-j}{j}} \overset{?}{=} \sum_{h} \binom{n-h}{h}$$

$$\sum_{k=0} \binom{n-2-k}{k} \qquad \sum_{j'=1} \binom{n-j'-2}{j'+1}$$

$$j' = j - 1$$

$$\sum_{j'} \binom{n-j'-1}{j'+1} = \sum_{h=0} \binom{n-h}{h}$$

$$j'+1 = h$$

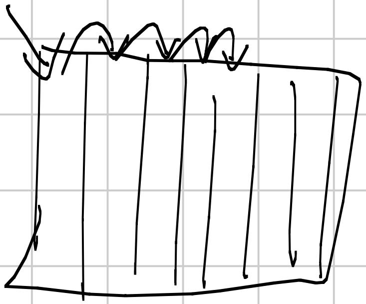
$$\sum_{k=0} \binom{n-2-k}{k} + \sum_{j=0} \binom{n-1-j}{j}$$

$j = k+1$

$$\sum_{j=1} \binom{n-1-j}{j-1} + \sum_{j=1} \binom{n-1-j}{j} + 1$$

$$\sum_{j=1} \left(\binom{n-1-j}{j-1} + \binom{n-1-j}{j} \right) + 1$$

$$\sum_{j=1} \binom{n-j}{j} + 1 = \sum_{j=0} \binom{n-j}{j}$$



$3 \cdot 2^{12}$

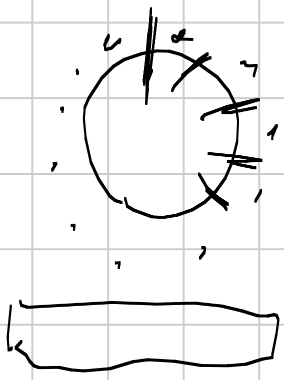
30 scienziati da 6 città (5 da ciascuna)

6 tavoli da 5 posti non devono esserci 2 scienziati della stessa città allo stesso tavolo

(tavoli distinguibili, non conta l'ordine ad ogni tavolo)

$6! \cdot (5!)^6$

n



$(n-1)!$

$\|$

$n \quad \frac{n!}{n}$

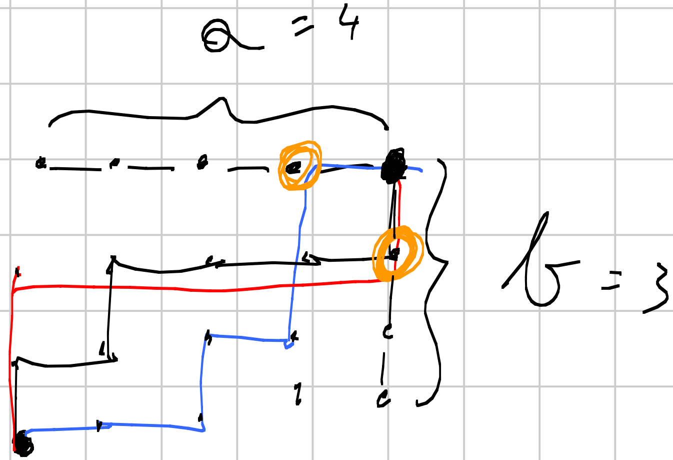
7 interi distinti $\{1, \ldots, 20\} = A$

Probabilità che il secondo più piccolo sia 5

$$\boxed{4\binom{15}{5} \Big/ \binom{20}{7}}$$

$$\frac{4 \cdot \cancel{15} \cdot \cancel{14} \cdot 13 \cdot 12 \cdot 11}{\cancel{5!}} \cdot \frac{\cancel{7!} \, 6 \cdot 7}{\cancel{16} \cdot 17 \cdot 20 \cdot 19 \cdot 18 \cdot \cancel{15 \cdot 14}}$$

griglia $a \times b$



$f(a, b) = ?$

$$\binom{a+b}{a} = \binom{a+b}{b}$$

$$f(a,b) = f(a-1,b) + f(a,b-1)$$

$$f(a,b) = \binom{a+b}{a} \longrightarrow \binom{a+b-1}{a-1} + \binom{a+b-1}{a}$$

$$\binom{a+b}{a} =$$

$$\uparrow \quad \rightarrow \qquad \underbrace{\uparrow\ \uparrow \ldots \uparrow}_{b} \ \underbrace{\rightarrow \qquad \rightarrow}_{a} \qquad \binom{a+b}{a}$$

$$n = a_1 + a_2 + \cdots + a_k \qquad a_i \geq 0$$

$n = 3$

$k = 2$

$3 = 3 + 0$ —

$ 2 + 1$ —

$ 1 + 2$ —

$ 0 + 3$ —

$n=3$

$k=3$

$3 = 3+0+0$

$2+1+0$

$2+0+1$

$1+0+2$

$1+1+1$

$1+2+0$

$0+3+0$

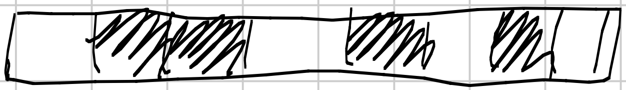
$0+2+1$

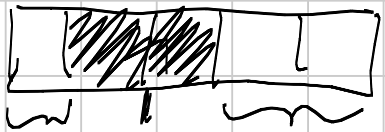
$0+1+2$

$0+0+3$

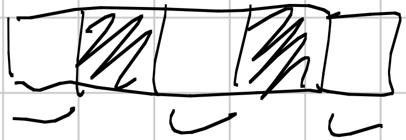
$$\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$$

$n+k-1$





1 0 2



$3 = 1+1+1$

Funzioni iniettive

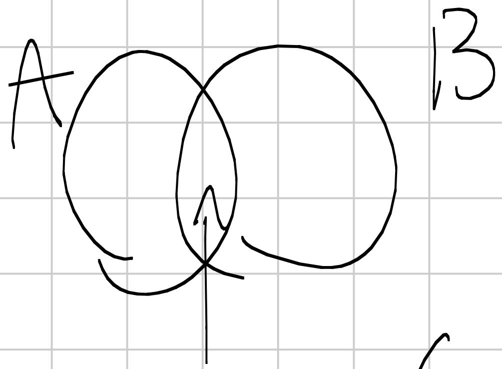
$A \xrightarrow{f} B$ $\qquad |B| \geq |A|$

$\binom{|B|}{|A|} |A|!$

Funzioni suriettive

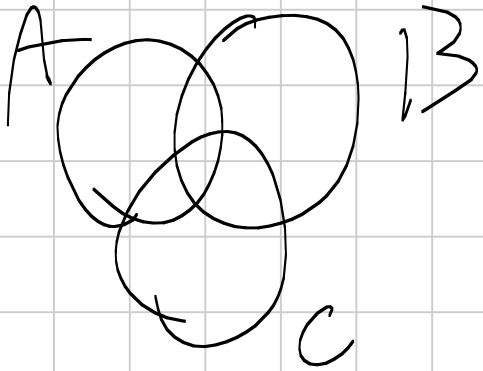
PRINCIPIO DI INCLUSIONE – ESCLUSIONE

$A_1, \quad , A_n \qquad |A_1 \cup \cdots \cup A_n|$



$|A \cup B| = |A| + |B| - |A \cap B|$

$A \cap B \neq \emptyset$



$$|A \cup B \cup C| = |A| + |B| + |C|$$
$$- |A \cap B| - |A \cap C| - |B \cap C|$$
$$+ |A \cap B \cap C|$$

$$|A \cup B| + |C| - |\underbrace{(A \cup B) \cap C|}_{|(A \cap C) \cup (B \cap C)|}$$

$$|A| + |B| - |A \cap B| + |C| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$|A_1 \cup \cdots \cup A_m| = \sum_{k=1}^{m} (-1)^{k+1} S(k)$$

$$S(k) = \sum_{1 \le i_1 < i_2 < \dots < i_k \le n} |A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}|$$

$x \in A_1 \cup \dots \cup A_n$

$x \in$ ad esattamente m degli A_i

$$m - \frac{m(m-1)}{2} + \frac{m(m-1)(m-2)}{6}$$

$$\sum_{i=1}^{m} (-1)^{i+1} \binom{m}{i} = 1$$

$k \ge m$

$$\{1, \dots, k\} \longrightarrow \{1, \dots, n\}$$

$$A_i = \{ f : \{1, \dots, k\} \to \{1, \dots, n\} \text{ t.c. } i \notin \operatorname{Im} f \}$$

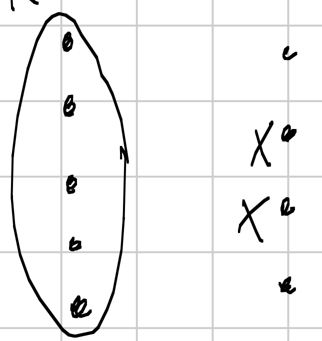
$1 \le i \le n$

$$A_1 \cup \dots \cup A_n = \{ f \text{ NON suriettive} \}$$

$i_1, \dots, i_h$ distinti

$$|A_{i_1} \cap \dots \cap A_{i_h}| = (n-h)^k$$

$k = 5$ $\quad$ $n = 4$



$i_1 = 2$

$i_2 = 3$

$$(n-h)^k$$

$$S(h) = (n-h)^k \cdot \binom{n}{h}$$

$$|A_1 \cup \dots \cup A_n| = \sum_{h=1}^{n} (-1)^{h+1} (n-h)^k \binom{n}{h}$$

$\#$ funz. suriettive $= n^k + \sum_{h=1}^{n} (-1)^h (n-h)^k \binom{n}{h}$

$= \sum_{h=0}^{n} (-1)^h (n-h)^k \binom{n}{h}$

<u>$n = k$</u> $\quad \sum_{h=0}^{n} (-1)^h (n-h)^n \binom{n}{h} = n!$

$\{1, \quad , n\} \qquad A_i = \{ \sigma \in S_n \mid \sigma(i) = i \}$

$|A_1 \cup \quad \cup A_n|$

$i_1, \ . \ , i_h \qquad |A_{i_1} \cap \dots \cap A_{i_h}| = (n-h)!$

distinti

$$S(h) = (n-h)! \cdot \binom{n}{h} = \frac{n!}{h!}$$

$$|A_1 \cup \cdots \cup A_n| = \sum_{h=1}^{n} (-1)^{h+1} \frac{n!}{h!}$$

permutazioni di $\{1, \dots, n\}$ senza punti fissi è

$$n! - \sum_{h=1}^{n} (-1)^{h+1} \frac{n!}{h!} = \boxed{\sum_{h=0}^{n} (-1)^{h} \frac{n!}{h!}}$$

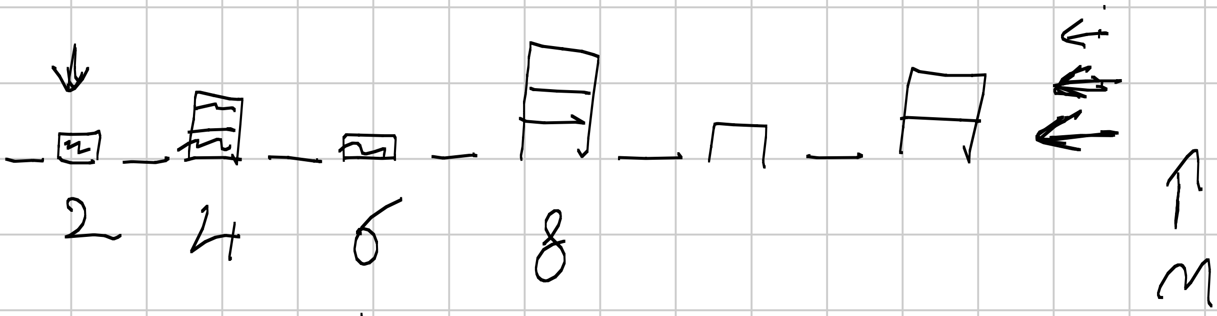
$$n!$$

$$\text{Prob.} = \underbrace{\sum_{h=0}^{n} \frac{(-1)^h}{h!}} \xrightarrow[n \to \infty]{} 1/e \simeq 37\%$$

$n!$

p primo

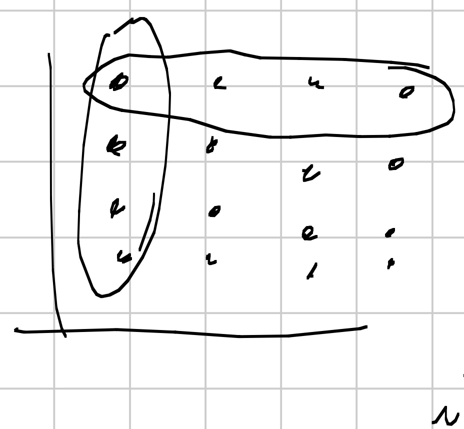
$p = 2$



$$\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \dots$$

$$\sum_{i=1}^{\text{"}\infty\text{"}} \left\lfloor \frac{n}{p^i} \right\rfloor = \sum_{i=1}^{\lfloor \log_p n \rfloor} \left\lfloor \frac{n}{p^i} \right\rfloor$$

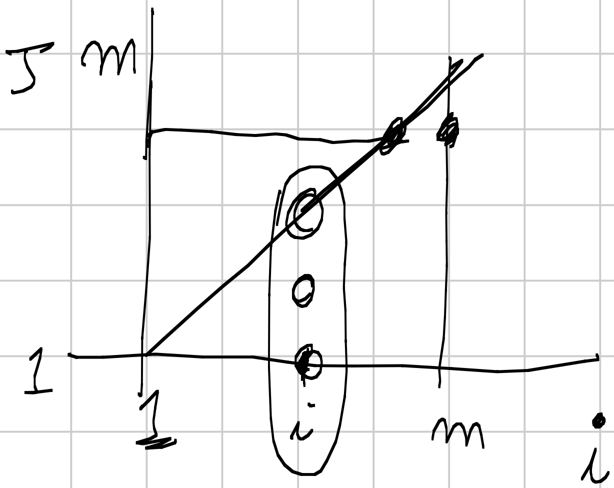
$$\sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} = \sum_{j=1}^{m} \sum_{i=1}^{n} a_{ij}$$



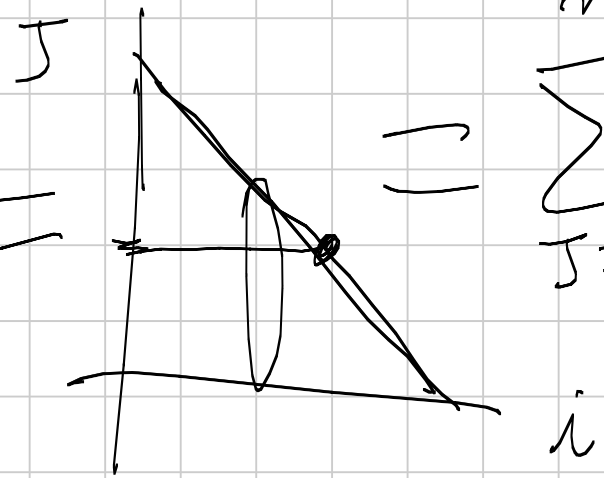
$$\sum_{i=1}^{n}\sum_{j=1}^{i} a_{ij}$$

$$\Vert$$

$$\sum_{j=1}^{n}\sum_{i=j}^{n} a_{ij}$$



$$\sum_{i=1}^{n}\sum_{j=1}^{n-i} a_{ij} =$$



$$= \sum_{j=1}^{n}\sum_{i=1}^{n-j} a_{ij}$$

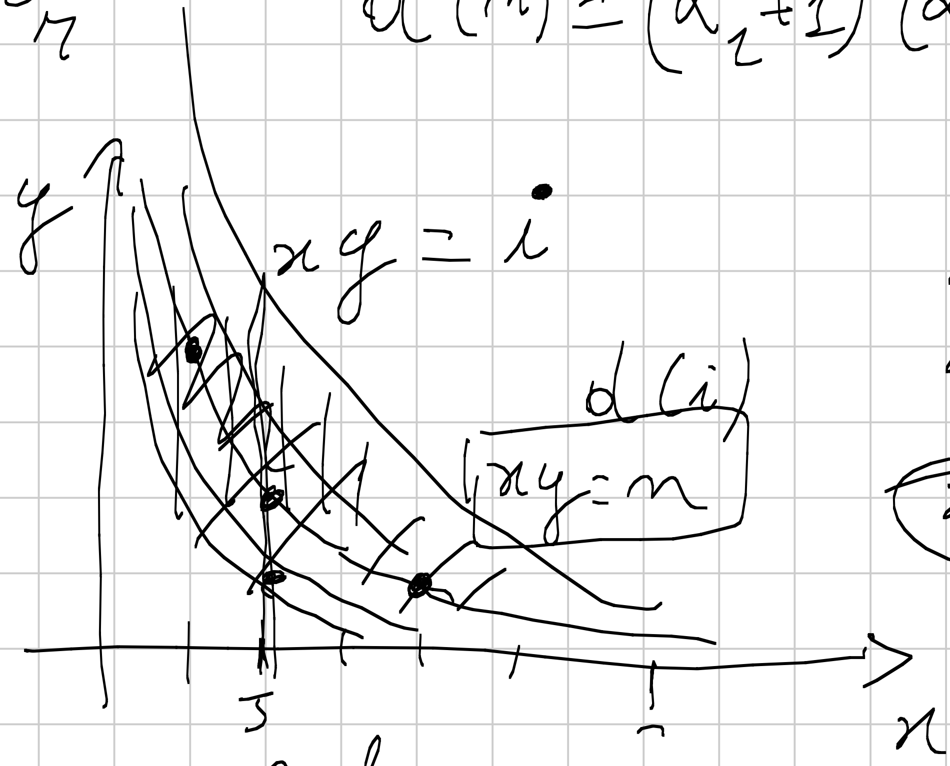
$d(n) =$ # di divisori di n

$n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ $\qquad d(n) = (\alpha_1+1)(\alpha_2+1)\cdots(\alpha_r+1)$

$$\sum_{i=1}^{n} d(i)$$

$$\|$$

$$\sum_{j=1}^{n} \left\lfloor \frac{n}{j} \right\rfloor$$

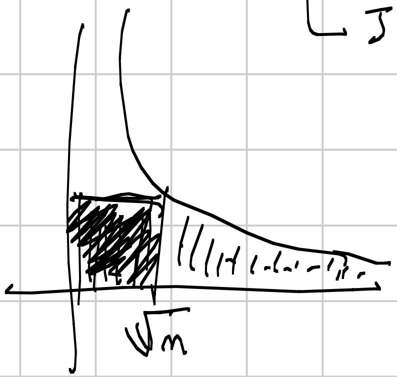


$$\left.\begin{matrix} xy = 1 \\ xy = 2 \\ xy = 3 \\ \vdots \\ xy = n \end{matrix}\right\}$$

a, b

$\left\lfloor \frac{n}{j} \right\rfloor$

$xy = ab \le n$



$$\sum_{i=1}^{n} d(i) \le f(n)$$

$$\sum_{i=1}^{n} d(i) \le \left(\sum_{i=1}^{n} \frac{n}{i}\right) = n \underbrace{\sum_{i=1}^{n} \frac{1}{i}}_{} \sim n \log n$$

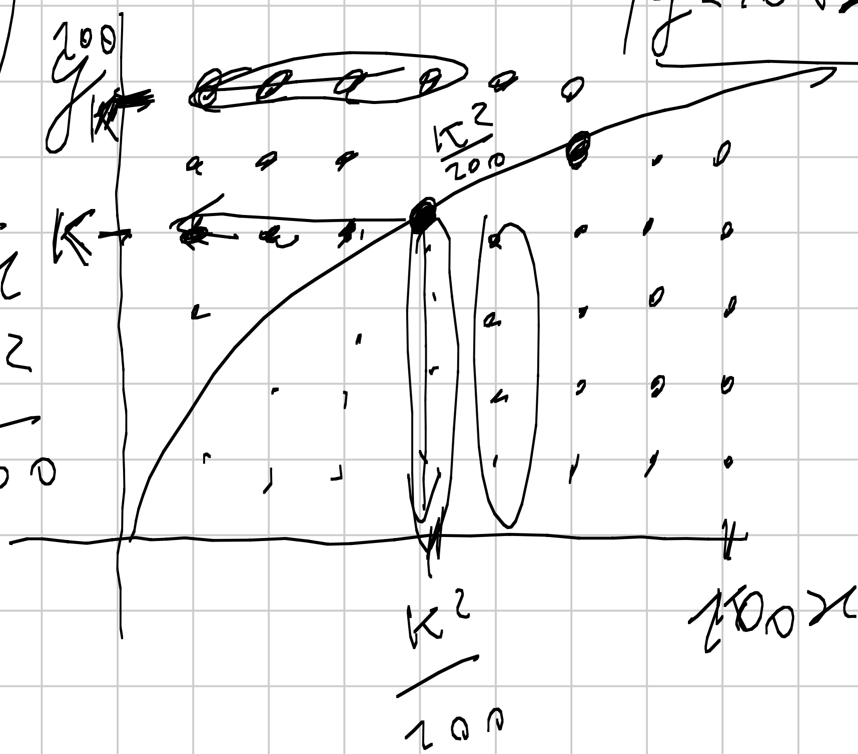
$$\sum_{K=1}^{100}\left(\left\lfloor 10\sqrt{K} \right\rfloor + \left\lfloor \frac{K^2}{100} \right\rfloor\right)$$

$$= 100^2 + 10$$

$$\frac{x^2}{100} \qquad 10\sqrt{x} \qquad \frac{K^2}{100}$$

$$x = \frac{y^2}{200}$$

$$y = 10\sqrt{x}$$

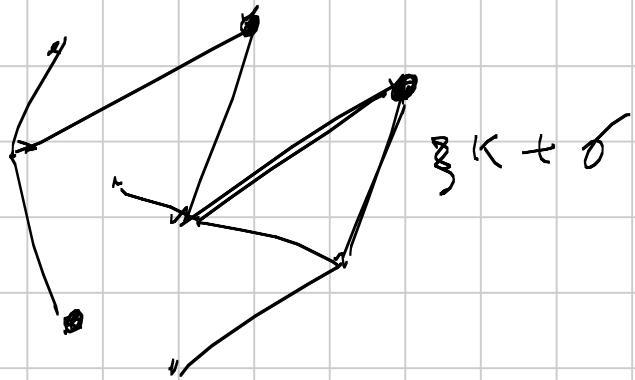


12 K persone

ogni persona stringe la mano a $\underline{3K+6}$ persone

N   A, B   ci sono $\underline{N}$ persone che hanno stretto la mano ad entrambi

12 K



$$\cancel{12K}\,\frac{(3K+6)(3K+5)}{\cancel{2}} = \frac{\cancel{12K}\,(12K-1)}{\cancel{2}}\,N$$

$$N = \frac{(3K+6)(3K+5)}{12K-1}$$

$$12K-1 \mid (3K+6)(3K+5)$$

$$4(9K^2+33K+30) - (12K-1)3K$$

$$4\cdot 33K + 120 + 3K \qquad 3(45K+40)$$

$$12K-1 \mid 45K+40$$

$$4(45K+40) - (12K-1)15$$

$$160 + 15 = 175 = 5\cdot 5\cdot 7$$

$$12K-1 \mid 5\cdot 5\cdot 7$$

$\cancel{1}$ $\quad$ $\cancel{7}$

$\cancel{5}$ $\quad$ $35 \leftarrow$

$\cancel{25}$ $\quad$ $\cancel{175}$

$12K - 1 = 35$

$\boxed{K = 3}$

$N = \frac{15 \cdot 14}{35} = 6$