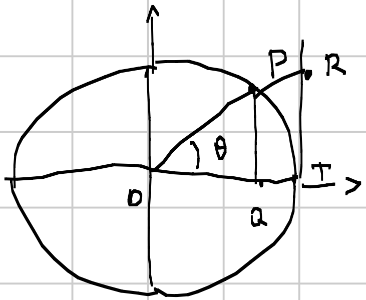
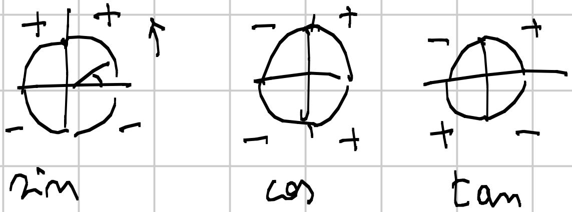
G1 - Trig

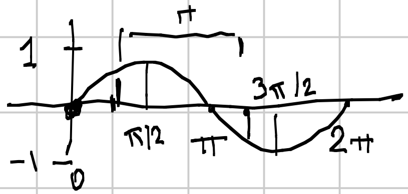
Jack (elianto84)

Titolo nota 08/09/2008



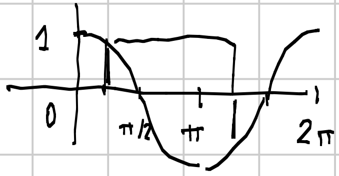
$\sin\theta \doteq \dfrac{PQ}{OP}$ $\quad \cos\theta = \dfrac{OQ}{OP}$ $\quad \tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{RT}{OP}$





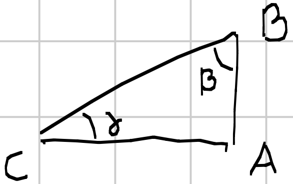
$f(-x) = -f(x)$

$\sin(\pi + x) = -\sin(x)$



$f(x) = f(-x)$

$\cos(\pi + x) = -\cos x$



$\beta + \gamma = \pi/2$

$\sin\gamma = \dfrac{AB}{BC} = \cos\beta$

$\sin\left(\dfrac{\pi}{2} - \beta\right) = \cos(\beta)$

$\sin^2\gamma + \cos^2\gamma = 1$

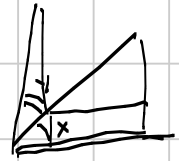
F. di addizione

$$\sin(x+y) = \sin x \cdot \cos y + \sin y \cdot \cos x$$

$$\cos(x+y) = \cos x \cdot \cos y - \sin x \sin y$$

$$\sin(x-y) = \sin x \cdot \cos y - \sin y \cos x$$

$$\cos(x-y) = \cos x \cdot \cos y + \sin x \sin y$$



$$e^{ix} = u$$

$$\begin{cases} e^{ix} = \cos x + i \sin x \\ e^{-ix} = \cos x - i \sin x \\ \cos x = \dfrac{e^{ix} + e^{-ix}}{2} \qquad \sin x = \dfrac{e^{ix} - e^{-ix}}{2i} \end{cases}$$

Werner - prostaferesi

- $\sin(x+y) + \sin(x-y) = 2 \sin x \cos y$
- $\cos(x+y) + \cos(x-y) = 2 \cos x \cos y$

$\sin(x+y) - \sin(x-y) = 2 \sin y \cos x$

$\cos(x+y) - \cos(x-y) = -2 \sin x \sin y$

$x+y = u \qquad x = (u+v)/2$

$x-y = v \qquad y = (u-v)/2$

..

$$\sin u + \sin v = 2 \sin \frac{u+v}{2} \cos \frac{u-v}{2}$$

$\sin(2x) = \sin(x+x) = 2\sin x\cos x$

$\cos(2x) = \cos(x+x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

$\tan(2x) = \dfrac{2\sin x\cos x}{\cos^2 x - \sin^2 x} = \dfrac{2\tan x}{1-\tan^2 x}$ | $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x\tan y}$

$\cos(n\arccos x) \overset{?}{\doteq}$ polinomio! di Chebyshev del 2° tipo

$\cos\theta = u$

$\cos n\theta \overset{?}{=}$

$T_n(x)$

$\partial T_n = n$

$\begin{cases} T_0(x) = 1 \\ T_1(x) = x \\ T_2(x) = 2x^2 - 1 \end{cases}$

$$\boxed{T_{n+2}(x) = 2x\,T_{n+1}(x) - T_n(x)}$$

$x = \cos y$

$\cos((n+2)y) = 2\cos y\,\cos((n+1)y) - \cos(ny)$ Werner!

$$U_n(x) = \frac{\sin((n+1)\arccos x)}{\sin(\arccos x)}$$

$U_0(x) = 1$

$U_1(x) = 2x$

$U_2(x) = 4x^2-1$

$\sin(\arccos x) = \sqrt{1-x^2}$

$$U_1(x) = \frac{2\cancel{\sin(\arccos x)}\cos(\arccos x)}{\cancel{\sin(\arccos x)}} = 2x$$

$\sin(3\theta) = \sin(\theta+2\theta) = \sin\theta\cos 2\theta + \cos\theta\sin 2\theta = \sin\theta\cos 2\theta + 2\sin\theta\cos^2\theta$

$\sin\theta\left(2\cos^2\theta - 1 + 2\cos^2\theta\right) = \sin\theta\left(4\cos^2\theta - 1\right)$

$$U_2(x) = \frac{\sin(3\arccos x)}{\sin(\arccos x)} = \frac{\cancel{\sin(\arccos x)}\,(4x^2-1)}{\cancel{\sin(\arccos x)}}$$

$$U_{n+2}(x) = 2x\,U_{n+1}(x) - U_n(x)$$ Werner!

Quali sono le radici di $T_n(x)$?

$\cos(n\arccos x) = 0$

$n\arccos x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

$\arccos x = \frac{\pi}{2n}, \frac{3\pi}{2n}, \frac{5\pi}{2n}, \ldots$

$x = \cos\left(\frac{\pi}{2n}\right), \cos\left(\frac{3\pi}{2n}\right), \ldots$

Quali sono le radici di $U_n(x)$? $\quad \sin\left(\frac{\pi}{2n}\right), \dots$

$$\sum_{n=1}^{10^3} \cos\Big(\underbrace{\frac{n\pi}{15\cdot 7^8}}\Big) = \qquad \cos x = \frac{e^{ix}+e^{-ix}}{2}$$

$$\sum_{n=1}^{10^3} e^{i\frac{n\pi}{15\cdot 7^8}} = \sum_{n=1}^{10^3} (u)^n$$

$$u \doteq e^{i\frac{\pi}{15\cdot 7^8}} \longrightarrow = \left| \frac{u^{10^3+1}-1}{u-1} \right|$$

$$\underbrace{\sum_{n=1}^{5} \cos\left(\frac{(2n-1)\pi}{10}\right)}_{\text{le radici di } T_5} \overset{?}{=}$$

la somma delle radici $\xrightarrow{\text{Viète}}$ i coeff. del polinomio

$\Updownarrow$

f. sym delle radici

$\parallel$

$$\frac{T_5(0)}{2^5} = 0$$

x

$2x^2-1$

$4x^3-3x$

$8x^4-8x^2+1$

$16x^5-16x^3+2x-4x^3+6x$

$$\sum_{n=1}^{5} \cos^2\left(\frac{(2n-1)\pi}{10}\right)$$

Bisezione

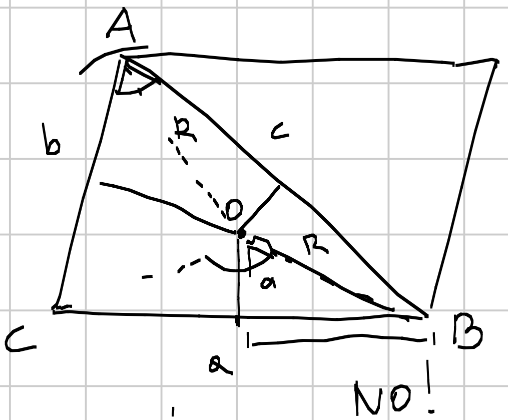
$$\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

$$\cos^2 x = \frac{\cos 2x + 1}{2} \qquad \sin^2 x = \frac{1-\cos 2x}{2}$$

$$\cos x = \sqrt{\frac{1+\cos 2x}{2}} \qquad \sin x = \sqrt{\frac{1-\cos 2x}{2}}$$

$$\cos\frac{x}{2} = \sqrt{\frac{1+\cos x}{2}} \qquad \sin\frac{x}{2} = \sqrt{\frac{1-\cos x}{2}}$$

$$\tan\frac{x}{2} = \sqrt{\frac{1-\cos x}{1+\cos x}} = \sqrt{\frac{1-\cos^2 x}{(1+\cos x)^2}} = \frac{\sin x}{1+\cos x}$$



T. del seno

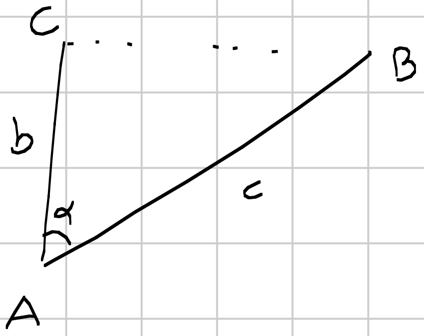
$$\Delta = \frac{1}{2} ab \sin\gamma = \frac{1}{2} ac \sin\beta$$

$$\frac{\sin\gamma}{c} = \frac{\sin\beta}{b} = \frac{\sin\alpha}{a} = \frac{1}{2R}$$

$$R\sin\alpha = \frac{a}{2}$$

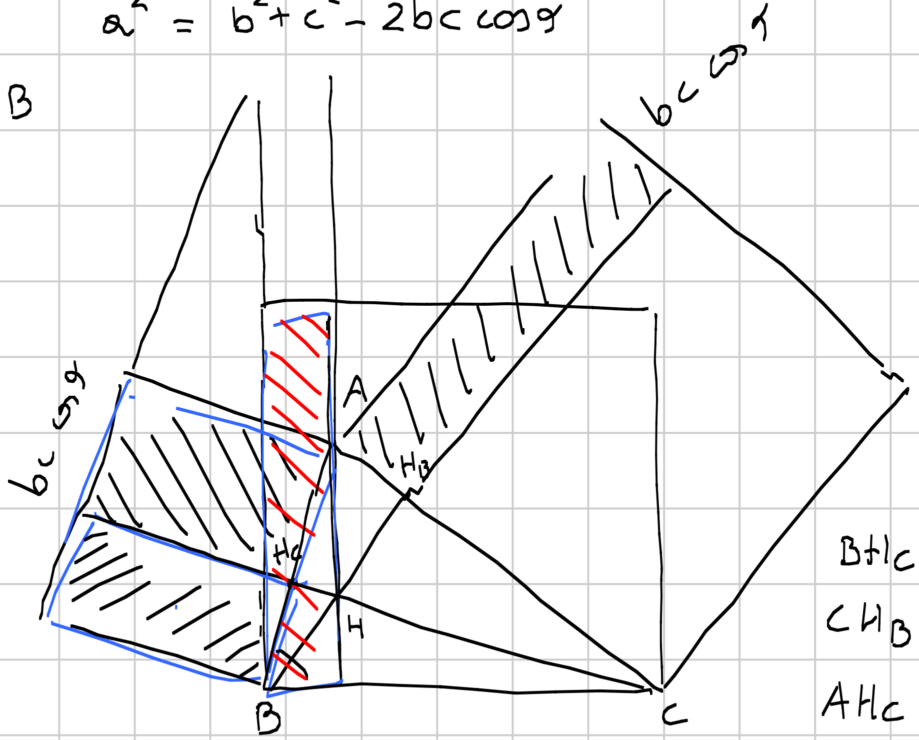
$$a = 2R\sin\alpha \qquad \frac{a}{\sin\alpha} = 2R$$

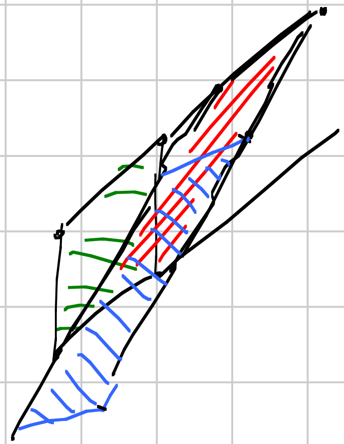
T. del coseno



$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

$$\cos A = \frac{b^2+c^2-a^2}{2bc}$$





$BH_C = a\cos\beta$

$CH_B = a\cos\gamma$

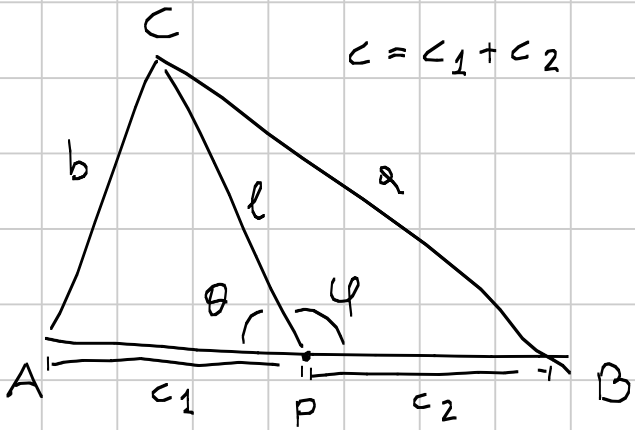
$AH_C = b\,\boxed{\cos\alpha}$

$AH_B = c\,\boxed{\cos\alpha}$

$$(c^2 - bc\cos\alpha) + (b^2 - bc\cos\alpha)$$

$$= a^2$$

T. Stewart.



$c = c_1 + c_2$

$$\cos\theta = \frac{c_1^2 + \ell^2 - b^2}{2\ell c_1}$$

$$\cos\varphi = \frac{c_2^2 + \ell^2 - a^2}{2\ell c_2}$$

$$c_2(c_1^2 + \ell^2 - b^2) + c_1(c_2^2 + \ell^2 - a^2) = 0$$

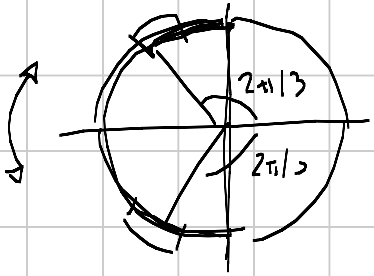
$$c_2 c_1 c + c\ell^2 = c_2 b^2 + c_1 a^2$$

$$c(c_2 c_1 + \ell^2) = c_2 b^2 + c_1 a^2$$

$$c_2 c_1 + \ell^2 = \frac{c_2 b^2 + c_1 a^2}{c}$$

$$\boxed{\ell^2 = \frac{c_2 b^2 + c_1 a^2}{c} - c_1 c_2}$$

2) $\cos(2^n\theta) < 0 \quad \forall n \in \mathbb{N}$



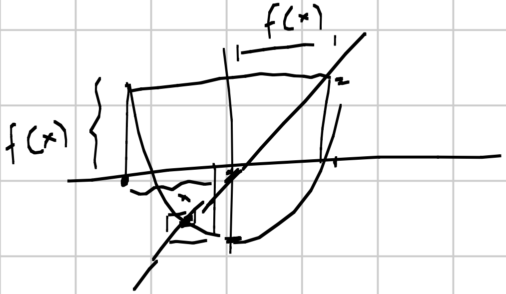
$$\theta = \underbrace{\pi\left(1 + \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \dots\right)}_{} =$$

$$\pi \sum_{n=0}^{+\infty} 2^{-n} = \pi \cdot \frac{1}{1+\frac{1}{2}} = \frac{2\pi}{3}$$

$$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$$

$$a_0 = \cos\theta = x \qquad a_1 = \cos 2\theta = 2x^2 - 1$$

$$a_{n+1} = 2a_n^2 - 1$$



il pt fisso di
$$2x^2 - 1 = x$$
é iperbolico
(repulsivo)

$$f(x) = ax^2 + bx + c$$

$f^{(n)}(x)$ si scrive in termini di f. elementari se e solo se $\exists\, \gamma, \beta$

$$f(x) = \frac{2(\gamma x + \beta)^2 - 1 - \beta}{\gamma}$$

Cor. $x^2 + 2$, $x^2 - 1/2$

$$g(x) = 2x^2 - 1 = T_2(\cos\theta) \qquad x \doteq \cos\theta$$

$$g^{(n)}(x) = T_{2^n}(\cos\theta) = \cos(2^n \arccos\theta)$$

1- Dividiamo per $4R^2$

$$\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$$

$$1-\cos^2\alpha + 1-\cos^2\beta + 1-\cos^2\gamma = 2$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$$\frac{\cos 2\alpha+1}{2} + \frac{\cos 2\beta+1}{2} + \frac{\cos 2\gamma+1}{2} = 1$$

$$\sum_{cyc} \cos 2\alpha = -1$$

$$\cos 2\alpha + \cos 2\beta = \cos\big((\alpha+\beta)+(\alpha-\beta)\big) + \cos\big((\alpha+\beta)-(\alpha-\beta)\big)$$

$$\|$$

$$2\cos(\alpha+\beta)\cos(\alpha-\beta) = -2\cos\gamma\cos(\alpha-\beta)$$

$$\cos 2\gamma = 2\cos^2\gamma - 1$$

$$-\cos\gamma\big(\cos\gamma - \cos(\alpha-\beta)\big) = 0$$

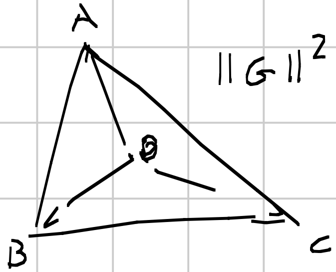
$$\cos\gamma\big(\cos(\alpha+\beta) + \cos(\alpha-\beta)\big) = 0$$

$$\prod_{cyc} \cos\alpha = 0$$

$\sin u \, \sin v \, \sin w$

$$= \frac{1}{8}\left(\sin(u+v-w) + \sin(u-v+w) + \sin(-u+v+w) - \sin\left(\overbrace{u+v+w}\right)\right)$$

$$OH^2 = 9R^2 - (a^2+b^2+c^2)$$

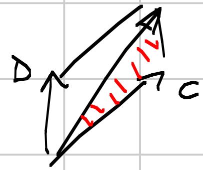


$\hat{G} = \dfrac{\hat{A}+\hat{B}+\hat{C}}{3}$

$\|G\|^2$

$\|\overbrace{A+B}^{D}+C\|^2$

$\|D+C\|^2$



$$\left.\begin{array}{l} H \in \Gamma_{ABC} \\ \sigma_{AB}(H) \in \Gamma_{ABC} \end{array}\right\} \quad H \in \text{lati} \Downarrow H \in \text{vertici} \Downarrow ABC \perp$$

$O \equiv H$ assi $\equiv$ altezze in ogni lato ABC è isoscele $\rightarrow$ ABC equilatero

1.

$\sin\alpha \cdot \sin\alpha + \sin\beta \cdot \sin\beta = \sin\alpha\cos\beta + \sin\beta\cos\alpha \qquad 0 < \alpha, \beta < \pi/2$

$\sin\alpha \sin\left(\frac{\pi}{2} - \beta\right) + \sin\beta \sin\left(\frac{\pi}{2} - \alpha\right)$

$$\left|\begin{array}{l} \alpha + \beta > \pi/2 \\ \alpha > \pi/2 - \beta \\ \beta > \pi/2 - \alpha \end{array}\right|$$

$\alpha + \beta = \pi/2$.

Erone

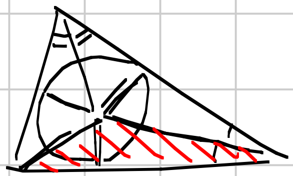
$\Delta = \frac{1}{2} ab \sin\gamma \qquad \Delta^2 = \frac{1}{4} a^2 b^2 (1 + \cos\gamma)(1 - \cos\gamma)$

$\cos\gamma = \frac{a^2 + b^2 - c^2}{2ab} \qquad 1 + \cos\gamma = \frac{(a+b)^2 - c^2}{2ab} = \frac{(a+b+c)(a+b-c)}{2ab} \qquad p = \frac{a+b+c}{2}$

$$\Delta^2 = p(p-a)(p-b)(p-c)$$

$$\Delta = \frac{abc}{4R} = \frac{1}{2}ab\left(\frac{c}{2R}\right) = \frac{1}{2}ab\sin\gamma$$

$$\Delta = pr$$



$$1+\cos\alpha = 2\cos^2\frac{\alpha}{2}$$

Dalle f. di bisezione segue

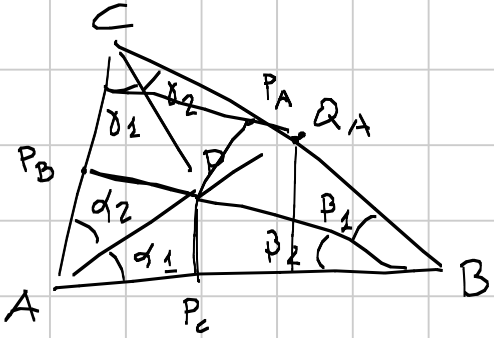
$$\sin\frac{\alpha}{2} = \sqrt{\frac{(p-b)(p-c)}{bc}} \qquad \cos\frac{\alpha}{2} = \sqrt{\frac{p(p-a)}{bc}} \qquad \text{(Briggs)}$$

Nepero

$$\frac{b+c}{b-c} = \frac{\tan\frac{\beta+\gamma}{2}}{\tan\frac{\beta-\gamma}{2}} = \frac{\sin(\beta+\gamma)}{\sin(\beta-\gamma)}\left(\frac{\cos\frac{\beta-\gamma}{2}}{\cos\frac{\beta+\gamma}{2}}\right)^2 = \frac{\sin\beta\cos\gamma+\sin\gamma\cos\beta}{\sin\beta\cos\gamma-\sin\gamma\cos\beta}\left(\frac{2\cos\frac{\beta-\gamma}{2}\sin\frac{\beta+\gamma}{2}}{2\cos\frac{\beta+\gamma}{2}\sin\frac{\beta+\gamma}{2}}\right)^2$$

$$= \frac{(a^2+b^2-c^2)+(a^2-b^2+c^2)}{(a^2+b^2-c^2)-(a^2-b^2+c^2)}\left(\frac{\sin\beta+\sin\gamma}{\sin\alpha}\right)^2 = \frac{a^2}{b^2-c^2}\left(\frac{b+c}{a}\right)^2$$

T. di Ceva

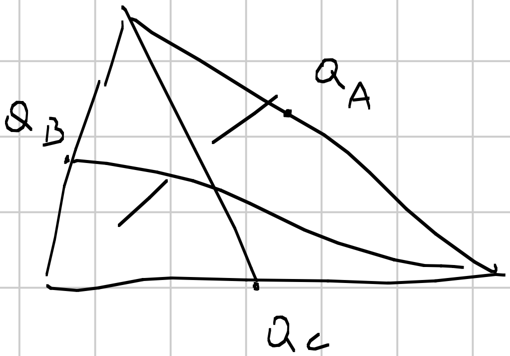


$$\frac{AP \cdot BP \cdot CP}{BP \cdot CP \cdot AP} = 1$$

$$PAB: \frac{AP}{BP} = \frac{\sin \beta_2}{\sin \alpha_1}$$

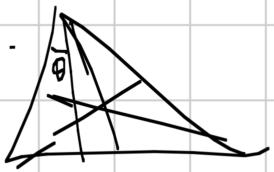
$$\prod_{cyc} \sin \alpha_1 = \prod_{cyc} \sin \alpha_2$$

$$\frac{PP_A \cdot PP_B \cdot PP_C}{PP_B \cdot PP_C \cdot PP_A} = 1$$

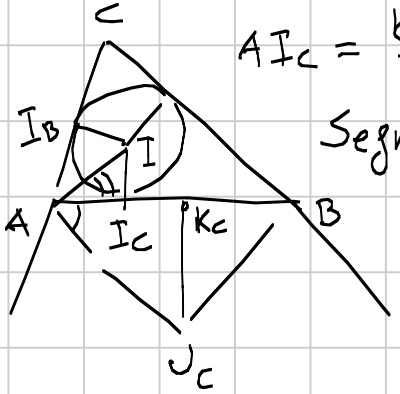


$$\boxed{\frac{BQ_A \cdot CQ_B \cdot AQ_C}{CQ_A \cdot AQ_B \cdot BQ_C}} = 1$$

CQ_C, BQ_B, AQ_A concorrono se e solo se vale una tre



4. (T. Carnot)



$AI_C = \frac{b+c-a}{2} = p-a$ $\qquad \widehat{K_C A J_C} = \frac{\pi - \alpha}{2}$

Segm. di Soddy $\qquad r_C = J_C K_C \qquad \frac{J_C K_C}{A K_C} = \frac{1}{\tan \frac{\alpha}{2}} = \frac{1+\cos\alpha}{\sin\alpha}$

$A K_C = \frac{a+c-b}{2}$

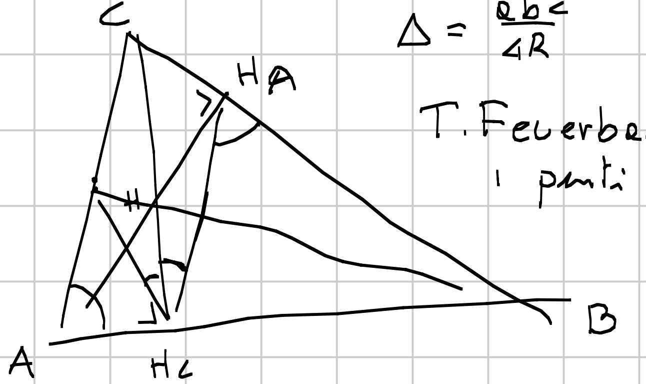
$\frac{abc}{4R} = \Delta \qquad R = \frac{abc}{4\Delta}$

$$abc = \left(\sum_{cyc} p(p-b)(p-c)\right) - \prod_{cyc}(p-a) = g(a,b,c)$$

$$\begin{cases} (a \mid g(a,b,c) \longleftrightarrow g(0,b,c) = 0)_{cyc} \\ g(1,1,1) = 1 \end{cases}$$

$$p^2(p-c) + p^2(p-b) = p^2(2p-b-c) = ap^2 = 0.$$

5. (Dim. Eulero $R \geq 2r$)



$$\Delta = \frac{abc}{4R}$$

T. Feuerbach

i punti medi dei lati e i piedi delle alt stanno sulla stessa Γ

Il raggio della circ. circoscritta a $H_A H_B H_C$ $= R/2$

$$BH_A \cdot BC = BH_C \cdot BA$$

BH_AH_C e BCA sono simili

$$\frac{H_C H_A}{b} = \frac{BH_A}{c} = \frac{c\cos\beta}{c} = \cos\beta$$

$$H_A H_C = b\cos\beta$$

$$[H_A H_B H_C] = \frac{abc \prod_{cyc} \cos\alpha}{2R} \qquad [ABC] = \frac{abc}{4R}$$

$$\prod_{cyc} \cos\alpha \le \frac{1}{8}$$

$$\Updownarrow$$

$$\sum \cos 2\alpha \ge k$$

$$\Updownarrow$$

$$\sum \cos^2\alpha \ge k$$

$$\Updownarrow$$

$$\sum \sin^2\alpha \ge k \xrightarrow{\cdot R^2} \boxed{9R^2 - \underbrace{(a^2+b^2+c^2)}_{= OH^2} \ge 0}$$

$$\boxed{\begin{array}{c} ABC \\ [I_A I_B I_c] = 2p \cdot R \end{array}}$$

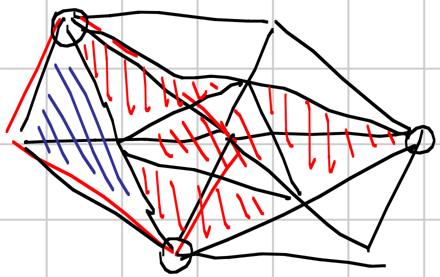
$$R \ge 2r$$

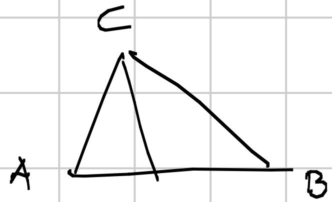
6. $a, b, c \in \mathbb{R}^+$ $\quad (a+b > c)_{cyc} \quad a+b+c > 2\,\mathrm{Max}(a,b,c)$

$A, B, C \in \mathbb{R}^+$

$$\begin{cases} a = B + C \\ b = A + C \\ c = A + B \end{cases} \qquad \left(\sqrt{\frac{1}{2}a^2 + \frac{1}{2}b^2 - \frac{1}{4}c^2}\right)_{cyc}$$

Le mediane formano un triangolo !





$$4M + 3P = 6P$$

$$M = \frac{3}{4}P.$$