A2 basic (teppic)

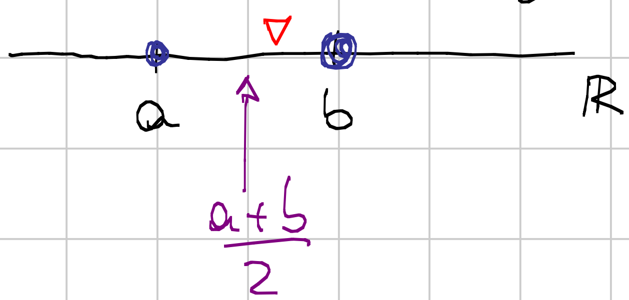
Titolo nota 09/09/2009

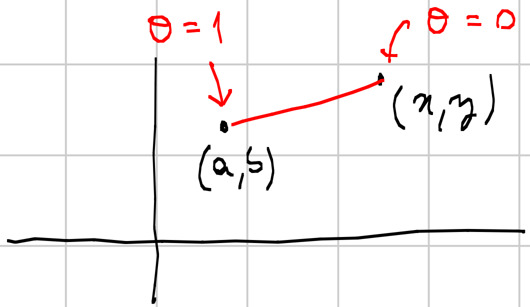
Disuguaglianze

MEDIE

$$\frac{a_1 + a_2 + \dots + a_n}{n} = AM\{a_1, a_2, \dots, a_n\} \qquad a_i \in \mathbb{R}$$

media pesata $\quad \frac{1}{3}a + \frac{2}{3}b \qquad \frac{1}{3} + \frac{2}{3} = 1$





$$\big(\theta a + (1-\theta)x , \theta b + (1-\theta) y\big) \qquad \theta \in [0,1]$$

$$\sqrt{\frac{a_1^2 + a_2^2 + \dots + a_n^2}{n}} = QM\{a_1 \dots a_n\}$$

$$\sqrt[n]{a_1 a_2 \cdot \ldots \cdot a_n} = GM\{a_1, \ldots, a_n\} \qquad a_i > 0$$

$$\left(\frac{\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}}{n}\right)^{-1} = HM\{a_1, \ldots, a_n\}$$

- Generalizzazione : media p-esima

$$\left(\frac{a_1^p + a_2^p + \ldots + a_n^p}{n}\right)^{\frac{1}{p}} = M_p\{a_1, \ldots, a_n\} \qquad p \in \mathbb{R}$$

$$HM = M_{-1} \qquad GM \approx M_0 \qquad AM = M_1 \qquad QM = M_2$$

- Disuguaglianze delle medie : $a_i > 0$ allora

$$p < q \Rightarrow M_p \leq M_q \quad \text{con} = \text{solo se} \quad a_1 = a_2 = \ldots = a_n$$

In particolare : $HM \leq GM \quad GM \leq AM \quad AM \leq QM$

Dim la disug delle medie per $n=2$ $a,b>0$

GM - AM : $\sqrt{ab} \overset{?}{\leq} \frac{a+b}{2}$

$\Nearrow$

$ab \overset{?}{\leq} \frac{(a+b)^2}{4} \Leftarrow 4ab \overset{?}{\leq} a^2+2ab+b^2 \Leftarrow 0 \overset{!}{\leq} a^2-2ab+b^2=(a-b)^2$

AM - QM : $\frac{a+b}{2} \overset{?}{\leq} \sqrt{\frac{a^2+b^2}{2}}$

$\frac{a^2+2ab+b^2}{4} \overset{?}{\leq} \frac{a^2+b^2}{2}$ $a^2+2ab+b^2 \overset{?}{\leq} 2a^2+2b^2$ $0 \overset{!}{\leq} a^2-2ab+b^2=(a-b)^2$

HM - GM : $\left(\frac{\frac{1}{a}+\frac{1}{b}}{2}\right)^{-1} = \frac{2ab}{a+b} \overset{?}{\leq} \sqrt{ab}$

$\frac{4a^2b^2}{(a+b)^2} \overset{?}{\leq} ab$ $4ab \overset{?}{\leq} (a+b)^2$ $0 \overset{!}{\leq} a^2-2ab+b^2=(a-b)^2$

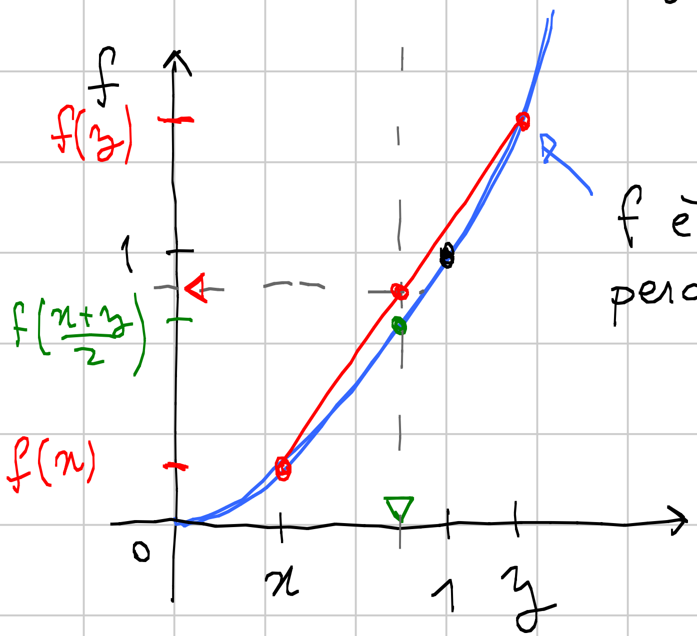
<u>Dim</u> $M_p \le M_q$ (con un po' di fiducia) $\quad p<q$

$$\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}} \overset{?}{\le} \left(\frac{a^q+b^q}{2}\right)^{\frac{1}{q}} \qquad f(x) = x^{q/p}$$

$$f\left(\frac{a^p+b^p}{2}\right) \overset{?}{\le} \frac{a^q+b^q}{2} \qquad a = x^{\frac{1}{p}} \quad b = y^{\frac{1}{p}}$$

$$f\left(\frac{x+y}{2}\right) \overset{?}{\le} \frac{f(x)+f(y)}{2}$$

!



f è convessa perchè $\frac{q}{p} > 1$

T.1.3 $\quad x^\alpha + y^2 + z^2 - xy\sqrt{z} \ge M \qquad M \in \mathbb{R} \qquad \forall\, x,y,z>0$

$x = \sqrt{t}$

$$t^{\alpha/2} + y^2 + z^2 - y\sqrt{tz} \ge M$$

omogenea se $\alpha = 4$

$\longrightarrow \quad \alpha < 4 \qquad\qquad x = t^{\frac{2}{\alpha}} > t^{\frac{1}{2}}$

$$t^2 + y^2 + z^2 - \underbrace{y\sqrt{z}\, t^{2/\alpha}}_{\text{grado} > 2} \qquad y = t \quad z = t$$

$$3t^2 - t^{2+\varepsilon} = t^2(3 - t^\varepsilon) \longrightarrow -\infty \quad \text{per } t \to \infty$$

$\longrightarrow \quad \alpha \geq 4$

voglio dimostrare che $\quad t^{\alpha/2} + y^2 + z^2 \overset{?}{\geq} M + y\sqrt{tz}$

$$t^{\alpha/2} + y^2 + z^2 \geq t^2 + y^2 + z^2 \qquad t^2 + y^2 + z^2 \overset{?}{\geq} M + y\sqrt{tz}$$

QM - GM $\quad \{t, z\} \qquad \dfrac{t^2 + z^2}{2} \geq tz$

$t^2 + y^2 + z^2 \geq y^2 + 2tz \qquad$ QM - GM $\quad \{y, \sqrt{tz}\} \qquad \dfrac{y^2 + tz}{2} \geq y\sqrt{tz}$

$$\geq \frac{y^2}{2} + \frac{tz}{2} \geq y\sqrt{tz}$$

Cauchy-Schwartz

$a_1, a_2, \ldots, a_n \qquad b_1, b_2, \ldots, b_n \qquad a_i, b_i \in \mathbb{R}$

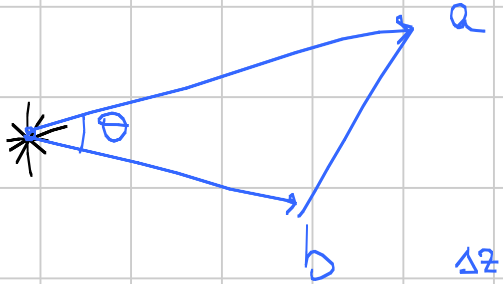
$$\left|\sum_{i=1}^{n} a_i b_i\right| \leq \sqrt{\sum_{i=1}^{n} a_i^2} \cdot \sqrt{\sum_{j=1}^{n} b_j^2}$$

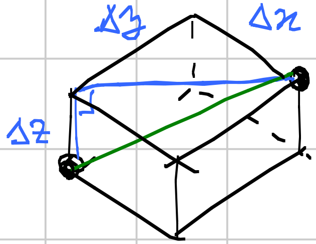
$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \sum_{i=1}^{n} a_i^2 \cdot \sum_{j=1}^{n} b_j^2$$

$\mathbb{R}^3$ ma anche $\mathbb{R}^n$ $\qquad \vec{a} = (a_1, \ldots, a_n) \qquad \vec{b} = (b_1, \ldots, b_n)$

$$\boxed{|\vec{a}||\vec{b}|\cos\theta = \vec{a} \cdot \vec{b} = \sum_{i=1}^{n} a_i b_i}$$

$$|\vec{a} - \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2\boxed{|\vec{a}||\vec{b}|\cos\theta}$$





$^2 = \Delta x^2 + \Delta y^2$

$^2 = \Delta z^2 + {}^2$

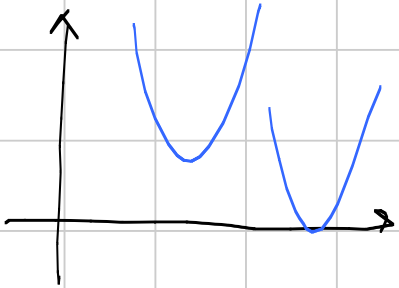
$$|\vec{a}-\vec{b}|^2=(a_1-b_1)^2+(a_2-b_2)^2+\dots+(a_n+b_n)^2$$

$$=\sum a_i^2+\sum b_i^2-2\sum a_ib_i=|\vec{a}|^2+|\vec{b}|^2-2\boxed{\sum_{i=1}^{n} a_ib_i}$$

C-S : $|\vec{a}\cdot\vec{b}|\leq|\vec{a}||\vec{b}|$

<u>Dim</u>: $|\vec{a}\cdot\vec{b}|=|\vec{a}||\vec{b}||\cos\theta|\leq|\vec{a}||\vec{b}|$

<u>Dim2</u> : $0\leq\sum_{i=1}^{n}(a_i-\lambda b_i)^2=\sum a_i^2-2\lambda\sum a_ib_i+\lambda^2\sum b_i^2=p(\lambda)$



p non può avere due radici reali distinte

quindi $\Delta\leq 0$

$0\geq\frac{\Delta}{4}=\left(\sum a_ib_i\right)^2-\sum a_i^2\sum b_i^2$ fine.

- L'uguaglianza si ha quando $\vec{a} \parallel \vec{b}$, ovvero solo se $\exists \lambda \in \mathbb{R}$ t.c. $a_i = \lambda b_i \quad \forall i = 1, \dots, n$.

T.1.2 $\sqrt{a+4b+9c+16d+5e}\,\sqrt{a^3+b^3+c^3+d^3+5e^3} \leq a^2+2b^2+3c^2+4d^2+5e^2$

Proviamo C-S: $\left(a^{\frac{1}{2}}, 2b^{\frac{1}{2}}, 3c^{\frac{1}{2}}, 4d^{\frac{1}{2}}, \sqrt{5}e^{\frac{1}{2}}\right)$, $\left(a^{\frac{3}{2}}, b^{\frac{3}{2}}, c^{\frac{3}{2}}, d^{\frac{3}{2}}, \sqrt{5}e^{\frac{3}{2}}\right)$

$$LHS \geq |a^2+2b^2+3c^2+4d^2+5e^2| = RHS$$

il contrario dell'ipotesi!!!

$\Rightarrow LHS = RHS \Rightarrow$ uguaglianza in C-S $\Rightarrow \vec{a} \parallel \vec{b}$

$$a^{3/2} = \lambda a^{\frac{1}{2}} \qquad b^{\frac{3}{2}} = \lambda 2 b^{\frac{1}{2}} \qquad \dots$$

$$\lambda = \frac{a^{3/2}}{a^{1/2}} = \frac{b^{3/2}}{2b^{1/2}} = \frac{c^{3/2}}{3c^{1/2}} = \frac{d^{3/2}}{4d^{1/2}} = \frac{e^{3/2}}{e^{1/2}} \qquad \lambda = a = \frac{b}{2} = \frac{c}{3} = \frac{d}{4} = e$$

$e = a < b < c < d$ $\qquad$ $d \leq e$ falso

RIARRANGIAMENTO

$a_1, a_2, \dots, a_n \qquad b_1, b_2, \dots, b_n \qquad a_i, b_j \in \mathbb{R}$

$a_1 \geq a_2 \geq \dots \geq a_n \qquad b_1 \geq b_2 \geq \dots \geq b_n$

$$a_1 b_1 + a_2 b_2 + \dots + a_n b_n = \sum_{i=1}^{n} a_i b_i \geq \sum_{i=1}^{n} a_i b_{\sigma(i)} \geq \sum_{i=1}^{n} a_i b_{n+1-i} = a_1 b_n + a_2 b_{n-1} + \dots + a_n b_1$$

$\forall \sigma : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$ biunivoca $\quad$ (permutazione)

Dim

7 2 5 4 3 $\boxed{1}$ 6 $\quad$ 7 2 5 4 1 3 6 $\quad$ ... $\quad$ 1 7 $\boxed{2}$ 5 4 3 6

... 1 2 3 4 5 6 7

$a_1 b_7 + a_2 b_2 + a_3 b_5 + \dots + a_6 b_1 + a_7 b_6 \quad \dots \quad a_1 b_1 + \dots + a_7 b_7$

È possibile da σ ricondursi all'identità usando solo trasposizioni in cui i due elementi che scambio, prima di farlo erano in ordine decrescente

τ scambia $j, j+1$ perché $\sigma(j) > \sigma(j+1)$ e $b_{\sigma(j)} \leq b_{\sigma(j+1)}$

$$b_{\tau\sigma(j)} = b_{\sigma(j+1)} \geq b_{\sigma(j)} = b_{\tau\sigma(j+1)}$$

$$\sum_{i=1}^{n} a_i b_{\sigma(i)} - \sum_{i=1}^{n} a_i b_{\tau\sigma(i)} = \sum_{\substack{i \neq j \\ i \neq j+1}} \underbrace{\left(a_i b_{\sigma(i)} - a_i b_{\tau\sigma(i)}\right)}_{= 0} +$$

$$+ a_j b_{\sigma(j)} + a_{j+1} b_{\sigma(j+1)} - a_j b_{\tau\sigma(j)} - a_{j+1} b_{\tau\sigma(j+1)}$$

$$= a_j b_{\sigma(j)} + a_{j+1} b_{\sigma(j+1)} - a_j b_{\sigma(j+1)} - a_{j+1} b_{\sigma(j)}$$

$$= \underbrace{(a_j - a_{j+1})}_{\leq 0} \underbrace{(b_{\sigma(j)} - b_{\sigma(j+1)})}_{\leq 0} \Rightarrow \sum_i a_i b_{\tau\sigma(i)} \geq \sum_i a_i b_{\sigma(i)}$$

Chebyshev

$a_1, a_2, \dots, a_n \qquad b_1, b_2, \dots, b_n \qquad a_i, b_i \in \mathbb{R} \qquad a_1 \geq a_2 \geq \dots \geq a_n$

$\circledast = \overline{ab} - \bar{a}\,\bar{b}$ è positivo se a, b concordi

e negativo se a, b discordi

$$\circledast = \frac{1}{n}\sum_{i=1}^{n} a_i b_i - \frac{1}{n}\sum_{i=1}^{n} a_i \;\frac{1}{n}\sum_{i=1}^{n} b_i \qquad \geq 0 \quad \text{se} \quad b_1 \geq b_2 \geq \dots \geq b_n$$

$$n\sum_{i=1}^{n} a_i b_i - \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i \qquad \leq 0 \quad \text{se} \quad b_1 \leq b_2 \leq \dots \leq b_n$$

<u>Dim</u> (concordi): $a_1 \geq \dots \geq a_n \qquad b_1 \geq \dots \geq b_n$

$$\frac{1}{n}\sum (a_i - \bar{a})(b_i - \bar{b}) = \circledast \qquad \text{(claim)}$$

$$\frac{1}{n}\sum_i (a_i - \bar{a})(b_i - \bar{b}) = \frac{1}{n}\sum a_i b_i - \frac{1}{n}\bar{a}\sum b_i - \frac{1}{n}\bar{b}\sum a_i + \frac{1}{n}\sum_i \bar{a}\,\bar{b}$$

$$= \frac{1}{n}\sum a_i b_i - \bar{a}\,\bar{b} - \bar{b}\,\bar{a} + \bar{a}\,\bar{b}$$

$$\boxed{\frac{1}{n}\sum_i (-\bar{a})\, b_i = -\bar{a}\,\frac{1}{n}\sum b_i}$$

$$= \overline{ab} - \bar{a}\,\bar{b} = \circledast$$

Per induzione su n:

$n=1 \qquad a_1 b_1 - a_1 b_1 \geq 0 \qquad$ ok

$$\overline{a'} = 0 = \overline{b'} \qquad \sum a_i' = 0 = \sum b_i'$$

$\overline{ab} - \bar{a}\,\bar{b} \overset{?}{\geq} 0 \qquad$ BDC. $\overline{a'b'} \geq 0 \qquad$ dove $a_i' = a_i - \bar{a} \qquad b_i' = b_i - \bar{b}$

$$n\,\overline{a'b'} = \sum_1^n a_i' b_i' = \sum_1^{n-1} a_i' b_i' + a_n' b_n' \geq \frac{1}{n}\sum^{n-1} a_i' \sum^{n-1} b_i' + a_n' b_n'$$

$$= \frac{1}{n}(0 - a_n')(0 - b_n') + a_n' b_n' = \underset{0}{\overset{}{a_n'}}\,\underset{0}{b_n'}\ \frac{n+1}{n} \geq 0 \qquad \blacksquare$$

■ Bernoulli: $\quad \forall n = 1, 2, \dots \qquad \forall x > -1$

$(1+x)^n \geq 1 + nx \qquad$ (dim facile) $\qquad$ (vale anche se $n \geq 1$ è reale)

☒ ESERCIZI

A2.9
$$\sum_{i=1}^{n} \frac{x_i}{\sqrt{1-x_i}} \geq \sqrt{\frac{n}{n-1}} \qquad x_i > 0 \text{ t.c. } \sum_i x_i = 1$$

Ⓒ Convessità

Disug di Jensen "istantanea"

Se f è convessa $(f'' > 0)$
$$f\left(\frac{x_1+x_2+\ldots+x_n}{n}\right) \leq \frac{f(x_1)+\ldots+f(x_n)}{n}$$

$$\sum_i \frac{x_i}{\sqrt{1-x_i}} =: \sum f(x_i) \qquad f(z) = \frac{z}{\sqrt{1-z}} = z(1-z)^{-\frac{1}{2}}$$

$$f'(z) = 1(1-z)^{-\frac{1}{2}} + z\left(-\frac{1}{2}\right)\cdot(1-z)^{-\frac{1}{2}-1}\cdot(-1)$$
$$= (1-z)^{-3/2}\left(1-z+\frac{z}{2}\right) = \frac{1}{2}(1-z)^{-3/2}(2-z)$$

$$f''(z) = \frac{1}{2}(1-z)^{-5/2}\left(\frac{3}{2}-(1-z)\right) = \frac{1}{2}(1-z)^{-5/2}\left(\frac{1}{2}+z\right) > 0$$

$$\frac{1}{n}\sum\frac{x_i}{\sqrt{1-x_i}}=\frac{\sum f(x_i)}{n}\overset{!}{\geqslant} f\left(\frac{\sum x_i}{n}\right)=f\left(\frac{1}{n}\right)=\frac{\frac{1}{n}}{\sqrt{1-\frac{1}{n}}}=\frac{1}{n}\sqrt{\frac{n}{n-1}}$$

$$\sum\frac{x_i}{\sqrt{1-x_i}}\overset{!}{\geqslant}\sqrt{\frac{n}{n-1}}$$

(2) $$\frac{x}{\sqrt{1-x}}=\frac{1}{\sqrt{1-x}}-\sqrt{1-x}$$

$$\sum_i\frac{x_i}{\sqrt{1-x_i}}=\sum_i\frac{1}{\sqrt{1-x_i}}-\sum\sqrt{1-x_i}=\sum(1-x_i)^{-\frac{1}{2}}-\sum(1-x_i)^{\frac{1}{2}}$$

$$M_{-\frac{1}{2}}\{1-x_i\}=\left(\frac{1}{n}\sum(1-x_i)^{-\frac{1}{2}}\right)^{-2}\leqslant M_1\{1-x_i\}=\frac{1}{n}\sum(1-x_i)=\frac{1}{n}(n-1)$$

elevo a $-\frac{1}{2}$ $$\frac{1}{n}\sum(1-x_i)^{-\frac{1}{2}}\geqslant\sqrt{n}\,\frac{1}{\sqrt{n-1}}$$ $$\boxed{\sum(1-x_i)^{-\frac{1}{2}}\geqslant n\sqrt{\frac{n}{n-1}}}$$

$$M_{\frac{1}{2}}\{1-x_i\}=\left(\frac{1}{n}\sum(1-x_i)^{\frac{1}{2}}\right)^2\leqslant M_1\{1-x_i\}=\frac{n-1}{n}$$

elevo a $\frac{1}{2}$ $$\boxed{\sum(1-x_i)^{\frac{1}{2}}\leqslant n\sqrt{\frac{n-1}{n}}}$$

$$LHS \geq n\sqrt{\frac{n}{n-1}} - n\sqrt{\frac{n-1}{n}} = n\frac{n-(n-1)}{\sqrt{n}\sqrt{n-1}} = \frac{n}{\sqrt{n}\sqrt{n-1}} = \sqrt{\frac{n}{n-1}}$$

ora 3

(1) a_i, b_i concordi $\quad \overline{ab} \geq \bar{a}\bar{b}$

$$\sum \frac{x_i}{\sqrt{1-x_i}}$$

$$\begin{array}{cccc} a_1 & a_2 & & a_n \\ x_1 \geq & x_2 \geq & \dots \geq & x_n \end{array}$$

$$1-x_1 \leq 1-x_2 \leq \dots \leq 1-x_n$$

$$\begin{array}{cccc} \frac{1}{\sqrt{1-x_1}} \geq & \frac{1}{\sqrt{1-x_2}} \geq & \dots \geq & \frac{1}{\sqrt{1-x_n}} \\ b_1 & b_2 & & b_n \end{array}$$

Cheb. $\Rightarrow \displaystyle\sum \frac{x_i}{\sqrt{1-x_i}} \overset{!}{\geq} \frac{1}{n}\sum a_i \sum b_i = \frac{1}{n}\underbrace{\sum x_i}_{\overset{1}{=}}\sum_i \frac{1}{\sqrt{1-x_i}} \overset{?}{\geq} \sqrt{\frac{n}{n-1}}$

$$\frac{1}{n}\sum (1-x_i)^{-\frac{1}{2}} \geq \sqrt{n}\frac{1}{\sqrt{n-1}} \quad (\text{dalla } (2))$$

fine.

☒ ESEMPIO : NESBITT

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$$

$\forall a, b, c > 0$

uguaglianza sse $a = b = c$

(1) Riarrangiamento

a, b, c $\qquad$ $a \geq b \geq c$

$\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$

$$(a+b+c) - a \leq (a+b+c) - b \leq (a+b+c) - c$$

$$\frac{1}{b+c} \geq \frac{1}{c+a} \geq \frac{1}{a+b}$$

$$LHS = \sum_{cyc} \frac{a}{b+c} \underset{\text{RIARR}}{\geq} \sum_{cyc} \frac{a}{a+b} = \frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a}$$

$$\frac{a}{a+b} + \frac{b}{c+a} + \frac{c}{b+c}$$

$$\sum_{cyc} \frac{a}{b+c} \geq \sum_{cyc} \frac{b}{a+b} = \frac{b}{a+b} + \frac{c}{b+c} + \frac{a}{c+a}$$

$$2 \sum_{cyc} \frac{a}{b+c} \geq 3$$

$\square$

② Cauchy – Schwartz $\qquad \sum a_i^2 \sum b_i^2 \geq \left(\sum a_i b_i\right)^2$

$$3+\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \overset{?}{\geq} \frac{3}{2}+3$$

$$(a+b+c)\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right) \overset{?}{\geq} \frac{9}{2}$$

$$\left[(a+b)+(b+c)+(c+a)\right]\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right) \overset{?}{\geq} 9$$

$$\text{C.S.}: \ \text{LHS} \overset{!}{\geq} \left(\frac{\sqrt{a+b}}{\sqrt{a+b}}+\frac{\sqrt{b+c}}{\sqrt{b+c}}+\frac{\sqrt{c+a}}{\sqrt{c+a}}\right)^2 = 9$$

③ Normalizzazione

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq \frac{3}{2}$$

è omogenea:

$a \leftarrow \lambda a \quad b \leftarrow \lambda b \quad c \leftarrow \lambda c$

ottengo la stessa formula

Allora posso porre $\lambda = \frac{1}{a+b+c}$ e supporre wlog che $a+b+c=1$

Rifaccio: $x = \frac{a}{a+b+c}$ e ciclici $\lambda = \frac{1}{a+b+c}$

$$\frac{\lambda a}{\lambda b+\lambda c} + \frac{\lambda b}{\lambda c+\lambda a} + \frac{\lambda c}{\lambda a+\lambda c} = \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \geq \frac{3}{2}$$

E' equivalente ($\Leftrightarrow$) lavorare su x, y, z e $x+y+z=1$

$$(\cancel{a+b+c})\left(\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b}\right) \overset{?}{\geq} \frac{9}{2}$$ vero per HM-AM

MUIRHEAD (o "bunching")

$$\sum_{sym} a^3b^2c \geq \sum_{sym} a^2b^2c^2$$ cose di questo tipo

$$a^3b^2c + b^3c^2a + c^3a^2b = \sum_{cyc} a^3b^2c$$
$$a^3c^2b + b^3a^2c + c^3b^2a = \sum_{cyc} a^3c^2b$$
$$+ \;= \sum_{sym} a^3b^2c$$

$$\sum_{sym} a^2b^2c^2 = 6a^2b^2c^2 \qquad \sum_{sym} a^4bc = 2\sum_{cyc} a^4bc$$

$$\sum_{cyc} a^2b^2c^2 = 3a^2b^2c^2$$

bunching dice che confrontando due $\sum_{sym}$ dello stesso grado vince quella con gli esponenti più concentrati

$$\begin{tikzcd}
\sum_{sym} a^3b^2c = [3,2,1] \arrow[d] & {[4,1,1]} \arrow[l] \arrow[d, phantom, "?"] & {[6,0,0]} \arrow[l] \arrow[dl] \\
\sum_{sym} a^2b^2c^2 = [2,2,2] & {[3,3,0]} \arrow[ul] &
\end{tikzcd}$$

la più piccola ($[2,2,2]$) — la più grande ($[6,0,0]$)

$a_1 \ge a_2 \ge \dots \ge a_n \qquad b_1 \ge b_2 \ge \dots \ge b_n$

in realtà questi sono uguali

a_i maggiorizzano i b_i se:

$a_1 \ge b_1, \quad a_1 + a_2 \ge b_1 + b_2, \quad \dots \quad a_1 + \dots + a_n \ge b_1 + \dots + b_n$

$\sum^n a_i = \sum^n b_i$ allora:

$$\sum_{sym} x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \ge \sum_{sym} x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$$

$\forall\ x_1, x_2, \dots, x_n > 0$ con uguaglianza solo se sono tutti $=$

Esempio : $\displaystyle\sum_{cyc} \frac{2}{b(a+b)} \overset{?}{\geq} \frac{27}{(a+b+c)^2}$

$$\sum_{cyc} \frac{2}{b(a+b)} \overset{?}{\geq} \sum_{cyc} \frac{2}{c(a+b)} \overset{?}{\geq} \frac{27}{(a+b+c)^2}$$

ok per riarr.

$$2(a+b+c)^2 \cdot \sum_{cyc} ab(b+c)(c+a) \geq 27abc(a+b)(b+c)(c+a)$$

$$2\sum_{cyc} (a+b+c)^2 ab(b+c)(c+a) \geq 27abc\left(2abc + \sum_{sym} a^2b\right)$$

$$2\sum_{cyc} ab(a^2+b^2+c^2+2ab+2bc+2ca)(c^2+bc+ac+ab)$$

$$= 2\sum_{cyc} ab(c^4+a^3b+a^3c+b^3a+b^3c+3c^3b+3c^3a+3a^2c^2+3b^2c^2+$$
$$+2a^2b^2+7abc^2+5ab^2c+5a^2bc)$$

$$= 2\sum_{cyc} \big(abc^4 + a^4b^2 + a^4bc + a^2b^4 + acb^4 + 3ab^2c^3 + 3a^2bc^3 + 3a^3bc^2 + 3ab^3c^2 +$$
$$+ 2a^3b^3 + 7a^2b^2c^2 + 5a^2b^3c + 5a^3b^2c\big)$$

$$= \sum_{sym} \left(7a^2b^2c^2 + 2a^3b^3 + 2a^4b^2 + 3a^4bc + 22a^3b^2c\right) = LHS$$

$$RHS = 54a^2b^2c^2 + 27\sum_{sym} a^3b^2c = \sum_{sym}\left(9a^2b^2c^2 + 27a^3b^2c\right)$$

$LHS \overset{?}{\geq} RHS$ equiv a:

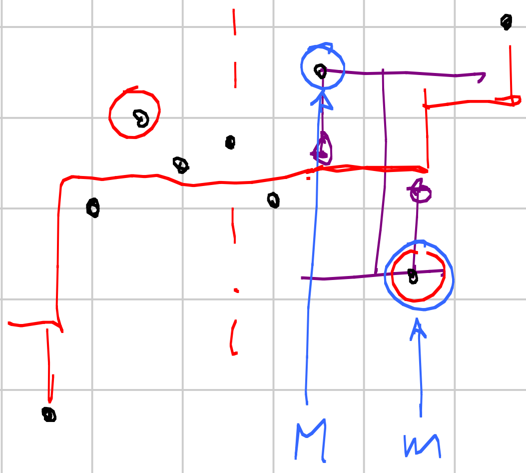
$$\sum_{sym}\left(2a^3b^3 + 2a^4b^2 + 3a^4bc\right) \overset{?}{\geq} \sum_{sym}\left(2a^2b^2c^2 + 5a^3b^2c\right)$$

$$2[3,3,0] + 2[4,2,0] + 3[4,1,1] \geq 2[2,2,2] + 5[3,2,1]$$

vera per bunching

Schur : $[3,0,0] + [1,1,1] \geq 2[2,1,0]$ (una versione)

$\boxed{\text{IMO } 2007.1}$ $\qquad a_1, a_2, \ldots, a_n \in \mathbb{R}$



$$d_i = \max_{1 \le j \le i} a_j - \min_{i \le j \le n} a_j$$

$$d = \max_i d_i$$

$$\forall x_i \in \mathbb{R}, \quad x_1 \le x_2 \le \ldots \le x_n$$

a) $\max_i |x_i - a_i| \ge \dfrac{d}{2}$ $\qquad d = a_M - a_m \qquad m \ge M$

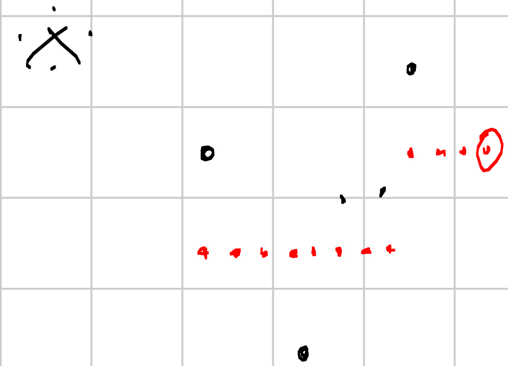
R.P.A. $\quad \forall i \quad |x_i - a_i| < \dfrac{d}{2} \qquad i = m, M$

$$|x_M - a_M| < \frac{d}{2} \qquad\qquad |x_m - a_m| < \frac{d}{2}$$

$$a_M - x_M < \frac{d}{2} \qquad\qquad x_m - a_m < \frac{d}{2}$$

зошто

$$d \le a_M - a_m + \underbrace{x_m - x_M}_{\ge 0} < d$$



b) $\exists\, x_i$ tali che $\max_i |x_i - a_i| = \frac{d}{2}$

$$x_i = \begin{cases} \dfrac{a_m + a_M}{2} = a_M - \dfrac{d}{2} = a_m + \dfrac{d}{2} & M < i < m \\ \max\limits_{M \le j \le i} a_j - \dfrac{d}{2} & i \ge m \\ \min\limits_{i \le j \le m} a_j + \dfrac{d}{2} & i \le M \end{cases}$$

BDC $|x_i - a_i| \le \frac{d}{2} \quad \forall i$

(1) $M < i < m \Rightarrow a_M - a_i \le d_i \le d \Rightarrow a_i \ge a_M - d = a_m$

$\Rightarrow a_i - a_m \le d_i \le d \Rightarrow a_i \le a_m + d = a_M$

$\Rightarrow a_m \le a_i \le a_M$ sottraggo x_i

$$-\frac{d}{2} = a_m - x_i \le a_i - x_i \le a_M - x_i = \frac{d}{2}$$

✓

(2) $x_i = \max\limits_{M \le j \le i} a_j - \frac{d}{2} \ge a_i - \frac{d}{2} \quad \Rightarrow \quad x_i - a_i \ge -\frac{d}{2}$ ✓

$$a_i \ge \min_{i \le j \le n} a_j = -d_i + \max_{1 \le j \le i} a_j \ge -d + x_i + \frac{d}{2} = x_i - \frac{d}{2}$$

$\Rightarrow \quad x_i - a_i \le \frac{d}{2}$ ✓

(3) è analogo.