C1 medium (Teppic)

Titolo nota 08/09/2009

$$\sum_{k=1}^{n} k \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

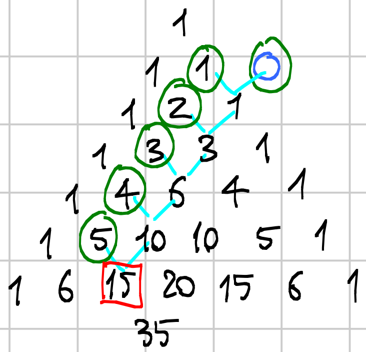
$$\sum_{k=1}^{n} k^4 = ?$$

$$n^5 = \sum_{k=0}^{n-1}\left[(k+1)^5 - k^5\right] = \sum_{k=0}^{n-1}\left[5k^4 + 10k^3 + 10k^2 + 5k + 1\right]$$

$$5\sum_{0}^{n-1} k^4 = n^5 - \dots$$

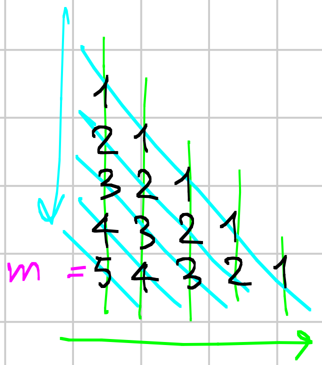
pol di deg $= n$

$$\boxed{\sum_{k=0}^{m}\binom{n+k}{n} = \binom{n+m+1}{n+1}}$$



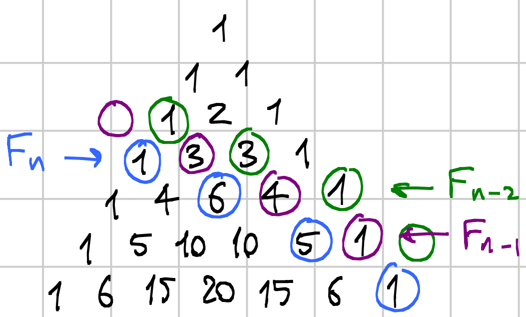
$$\sum_{k=0}^{n}\binom{k+2}{2} = \binom{n+3}{3}$$

$$\sum_{0}^{n}\frac{(k+2)(k+1)}{2} = \frac{(n+3)(n+2)(n+1)}{6}$$



double counting

$$\sum_{n=1}^{m+1}\frac{n(n-1)}{2} = \sum_{n=1}^{m} n(m-n) \iff \sum_{n=1}^{m} n^2 = \dots$$



$$\cancel{\sum_{k}\binom{n+k}{2k} = \sum_{k}\binom{n+k}{n-k}} = F_n$$

$$F_n = F_{n-1} + F_{n-2}$$

$$F_0 = 0 \quad F_1 = 1 \quad F_2 = 1 \quad F_3 = 2$$

$$F_4 = 3 \quad F_5 = 5$$

$$F_{2n+1} = \sum_{k}\binom{n+k}{n-k}$$

$$F_{2n} = \sum_{k}\binom{n+k}{n-k-1}$$

$$\longrightarrow F_n = \sum_{k}\binom{n-1-k}{2k} \quad \text{(verificare)}$$

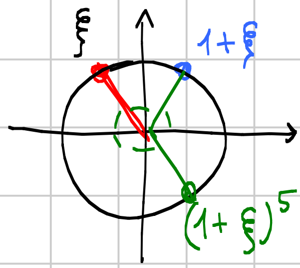
FUNZIONI GENERATRICI E PARENTI

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \qquad (x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

$$x = 1 \qquad x = -1$$

$$\sum \binom{n}{k} = 2^n \qquad \sum_{2|k} \binom{n}{k} = 2^{n-1} \qquad \sum_{3|k} \binom{n}{k} =$$

Esercizio : $\displaystyle \sum_{3|k} \binom{2009}{k} = ?$



$$(1+\xi)^{2009} = \sum_k \binom{n}{k} \xi^k$$

$$(1+\xi)^{2009} = 1 \cdot (1+\xi)^5 = \frac{1}{2} - \frac{\sqrt{3}}{2} i$$

$$\sum_k \binom{n}{k} \xi^k = \underbrace{\sum_{\substack{k \equiv 0 \\ (\mathrm{mod}\ 3)}} \binom{n}{k}}_{A} \cdot 1 + \underbrace{\sum_{k \equiv 1} \binom{n}{k}}_{B} \xi + \underbrace{\sum_{k \equiv 2} \binom{n}{k}}_{C} \bar{\xi}$$

$$A + B\xi + C\bar{\xi} = \frac{1}{2} - \frac{\sqrt{3}}{2} i \qquad \xi = -\frac{1}{2} + \frac{\sqrt{3}}{2} i \qquad \bar{\xi} = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$$

$$\begin{cases} \mathrm{Re} : \ 1 \cdot A - \frac{1}{2} B - \frac{1}{2} C = \frac{1}{2} & 2A - B - C = 1 \\ \mathrm{Im} : \ \frac{\sqrt{3}}{2} \left(B + C(-1) \right) = \frac{\sqrt{3}}{2} (-1) & C - B = 1 \\ \sum : \ A + B + C = \sum_k \binom{n}{k} = 2^n & A = \dfrac{2^{2009}+1}{3} \end{cases}$$

$$\begin{array}{lcl} (1+\xi)^{2009} & = & A + B\xi + C\bar{\xi} \\ (1+\bar{\xi})^{2009} & = & A + B\bar{\xi} + C\xi \\ (1+1)^{2009} & = & A + B + C \\ \hline 1 + 2^{2009} & = & 3A + 0 + 0 \end{array}$$

$$\sum_{k=0}^{n+1} \binom{n+1}{k} x^k = (1+x)^{n+1} = (1+x)(1+x)^n = (1+x) \sum_k \binom{n}{k} x^k$$

$$= \sum_{k=0}^{n}\binom{n}{k}x^k + \sum_{k=1}^{n+1}\binom{n}{k-1}x^k = \sum_{k=0}^{n+1} x^k\left[\binom{n}{k}+\binom{n}{k-1}\right]$$

Più in generale:

$$\sum_{k=0}^{m+n}\binom{m+n}{k}x^k = (1+x)^m(1+x)^n = \sum_{i=0}^{m}\binom{m}{i}x^i\sum_{j=0}^{n}\binom{n}{j}x^j$$

$$= \sum_{i=0}^{m}\sum_{j=0}^{n}\binom{n}{j}x^j\binom{m}{i}x^i = \sum_{i=0}^{m}\sum_{j=0}^{n}\binom{n}{j}\binom{m}{i}x^{i+j} = \sum_{k=0}^{m+n}\sum_{i=0}^{k}\binom{m}{i}\binom{n}{k-i}x^k$$

$i+j$

$k \wedge m$

$$\binom{m+n}{k} = \sum_{i=0}^{k}\binom{m}{i}\binom{n}{k-i}$$

$0 \vee k-n$

Superenalotto: prob di fare $k = 0,1,\dots,6$ è $\dfrac{\binom{6}{k}\binom{84}{6-k}}{\binom{90}{6}}$

(distribuzione ipergeometrica)

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

$$20+60+36+4+0 = 120 = \binom{10}{3}$$

$$\sum_{k=0}^{n}\binom{n}{k}k = \sum \frac{n!}{\cancel{k!}\,(n-k)!}\cdot\cancel{k} = \dots = n\,2^{n-1}$$

$(k-1)!$

$\binom{n}{k}$ squadre di k persone su n a disposizione

$\binom{n}{k}k$ squadre di k persone con un portiere

$$\sum_{k=0}^{n}\binom{n}{k}k = n\,2^{n-1}$$

$$\sum_{k=0}^{n}\binom{n}{k}k\,x^k = x\sum_{k=0}^{n}\binom{n}{k}k\,x^{k-1}$$

1 $\quad$ a_k

$$\boxed{\frac{d}{dx}\sum_{k=0}^{n}a_k x^k = \sum_{k=0}^{n}k\,a_k x^{k-1}}$$

$$= x\frac{d}{dx}\sum_{k}\binom{n}{k}x^k = x\frac{d}{dx}(1+x)^n$$

$$= x\,n\,(1+x)^{n-1}\cdot 1 = n\,2^{n-1}$$

1

$$\boxed{\frac{d}{dx}(f(x))^n = n(f(x))^{n-1}\cdot f'(x)}$$

$$\sum_{k=0}^{n}\binom{n}{k}k^2 = \sum_k \binom{n}{k}k(k-1) + \sum_k \binom{n}{k}k = n(n-1)2^{n-2} + n2^{n-1}$$

ora 2

$X=\{1,2,\dots,n\}$

$$\sum_{\substack{A\subseteq X\\ A\neq\emptyset}} \max A + \sum_{\substack{A\subseteq X\\ A\neq\emptyset}} \underbrace{\min A^s}_{\substack{n+1-\max A\\ \parallel}} = \sum_{\substack{A\subseteq X\\ A\neq\emptyset}} (n+1) = (2^n-1)(n+1)$$

$$\sum_{\substack{A\subseteq X\\ A\neq\emptyset}} \max A = \sum_{\underset{\max A}{k}=1}^{n} k\,2^{k-1} \overset{(1)}{=} \sum_{k=1}^{n} k\,\underset{2}{x}^{k-1} = \frac{d}{dx}\sum_{k=0}^{n} x^k = \frac{d}{dx}\frac{1-x^{n+1}}{1-x}$$

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x)-g'(x)f(x)}{g^2(x)}$$

$$= \frac{-(n+1)x^n(1-x)-(-1)(1-x^{n+1})}{(1-x)^2} \underset{x=2}{=} -(n+1)2^n(-1)-(-1)(1-2^{n+1})$$

$$=(n+1)2^n+1-2^{n+1}$$

$$=(n-1)2^n+1$$

SERIE GEOMETRICHE & Co

$$\sum_{k=0}^{n-1} x^k = \frac{1-x^n}{1-x} \qquad \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

formalmente

$+\ |x|<1$

$$1+x+x^2+\dots = \frac{1}{1-x}$$

(2) $$\sum_{k=0}^{\infty} kx^{k-1} = \sum_{k=1}^{\infty} x^{k-1}\sum_{i=1}^{k} 1 = \sum_{i=1}^{\infty}\sum_{k=i}^{\infty} x^{k-1} = \sum_{i=1}^{\infty}\left(x^{i-1}+x^i+x^{i+1}+\dots\right)$$

$$=\sum_{i=1}^{\infty} x^{i-1}\left(1+x+x^2+\dots\right) = \sum_{i=1}^{\infty} x^{i-1}\frac{1}{1-x} = \frac{1}{1-x}\sum_{i=1}^{\infty} x^{i-1} = \frac{1}{1-x}\left(1+x+x^2+\dots\right)$$

$$=\left(\frac{1}{1-x}\right)^2$$

$$\boxed{\sum_{k=0}^{\infty}\binom{n+k}{n}x^k = \left(\frac{1}{1-x}\right)^{n+1}}$$

si dimostra per induzione sostituendo $\binom{n+k}{n} = \sum_{i=0}^{k}\binom{n-1+i}{n-1}$

(3) $\underbrace{0 + 1 + 2x + 3x^2 + 4x^3 + \dots}_{Y} = (1 + x + x^2 + \dots) + x + 2x^2 + 3x^3 + \dots$

$$= \frac{1}{1-x} + x(1 + 2x + 3x^2 + \dots)$$

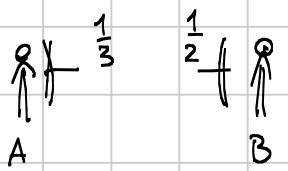
$$Y = \frac{1}{1-x} + xY \qquad Y(1-x) = \frac{1}{1-x} \qquad Y = \left(\frac{1}{1-x}\right)^2$$

questo può avere anche la soluzione $Y = +\infty$

Se si vuole usare questo metodo bisogna anche dimostrare che la serie converge (per x abbastanza piccolo)

$$\begin{matrix} 1 & x & x^2 \\ x & x^2 & \\ x^2 & & \end{matrix}$$

è il double counting



$$P_A = \frac{1}{3} + \frac{2}{3} \cdot \frac{1}{2} \cdot P_A$$

$$\frac{2}{3} P_A = \frac{1}{3} \qquad P_A = \frac{1}{2}$$

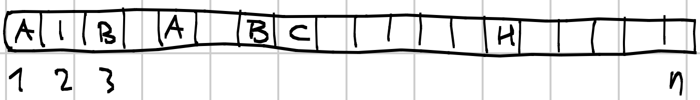
$$P_A = \frac{1}{3} + \frac{2}{3} \cdot \frac{1}{2} \cdot \frac{1}{3} + \left(\frac{2}{3} \cdot \frac{1}{2}\right)^2 \cdot \frac{1}{3} + \dots$$

A, B, C, ..., H, I 9 persone

$\{1, 2, \dots, n\}$ n carte

→ A dovrà avere tante carte quante B, C, D, E assieme.

Q: In quanti modi si possono distribuire le carte?



k_A # di A

k_B # di B

....

k_I # di I

$$\binom{n}{k_A\ k_B\ \dots\ k_I} = \frac{n!}{k_A!\,k_B! \dots k_I!}$$

$$\binom{n}{k_A} = \frac{n!}{k_A!(n-k_A)!} \times \frac{(n-k_A)!}{k_B!(n-k_A-k_B)!} \times \dots$$

$$k_A = k_B + k_C + k_D + k_E$$

$$(x_A + x_B + \dots + x_I)^n = \sum_{\substack{k_A, \dots, k_I \\ \sum k_i = n}} \binom{n}{k_A\ \dots\ k_I} x_A^{k_A} x_B^{k_B} \cdots x_I^{k_I}$$

$$(x+y+y+y+y+1+1+1+1)^n = \sum_{k\dots} \binom{n}{k_A \dots k_I} x^{k_A} y^{k_B+k_C+k_D+k_E}$$

$y = \frac{1}{x}$

$$\left(x+\frac{4}{x}+4\right)^n \overset{!}{=} \sum \binom{n}{\dots} x^{k_A} x^{-k_B-\dots-k_E}$$

i modi di dare le carte che voglio contare sono tutti e soli quelli che corrispondono a x^0 nell'espressione $\left(x+\frac{4}{x}+4\right)^n = \dots$

$$= \frac{1}{x^n}\left(x^2+4x+4\right)^n = \frac{1}{x^n}(x+2)^{2n}$$ il termine di grado 0 è: il termine di grado n di $(x+2)^{2n}$

$$(x+2)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} x^k 2^{2n-k} \qquad \dots \text{ è } \binom{2n}{n} 2^n$$

PRINCIPIO INCLUSIONE - ESCLUSIONE

$A_1, A_2, \dots, A_n$ insiemi finiti $\subseteq \{1,2,\dots,N\}$ $\quad |H| = \#$ di elem di H

$$\left|\bigcup_{i=1}^n A_i\right| = \sum_{1\le i\le n} |A_i| - \sum_{1\le i<j\le n} |A_i \cap A_j| + \sum_{1\le i<j<k\le n} |A_i\cap A_j\cap A_k| - \dots \pm \left|\bigcap_{i=1}^n A_i\right|$$

__Dim__ $$\text{RHS} = \sum_{1\le i\le n} \sum_{h=1}^N \mathbb{1}_{h\in A_i} - \sum_{1\le i<j\le n} \sum_{h=1}^N \mathbb{1}_{h\in A_i\cap A_j} + \dots$$

$$\mathbb{1}_{\text{qualcosa}} = \begin{cases} 1 & \text{qualcosa è vero} \\ 0 & \text{altrimenti} \end{cases}$$

$$= \sum_{h=1}^N \left\{ \overbrace{\sum_{1\le i\le n} \mathbb{1}_{h\in A_i}}^{n_h} - \overbrace{\sum_{i,j} \mathbb{1}_{h\in A_i\cap A_j}}^{\binom{n_h}{2}} + \dots \right\}$$

$h \in \bigcup_1^n A_i \qquad h \in A_{i_1}, A_{i_2}, \dots, A_{i_{n_h}} \qquad n_h$ insiemi ≥ 1

$$= \sum_{h\in \bigcup A_i} \left\{ \binom{n_h}{1} - \binom{n_h}{2} + \binom{n_h}{3} - \dots \right\} = \sum_{h\in\bigcup A_i} 1 = \left|\bigcup_i A_i\right|$$

- funzioni surgettive $f: A \to B \qquad a = |A| \ge |B| = b$

Quante sono le funzioni surgettive

$G_i := \{ f \text{ che non hanno } i \text{ nell'immagine} \} \qquad i = 1,2,\dots,b$

$$b^a - \left|\bigcup_{i=1}^{b} G_i\right| = b^a - \sum_i |G_i| + \sum_{i,j} |G_i \cap G_j| - \sum_{i,j,k} |G_i \cap G_j \cap G_k| + \dots$$

$$= b^a - (b-1)^a b + (b-2)^a \binom{b}{2} - \dots = \sum_{k=0}^{b} (b-k)^a \binom{b}{k} (-1)^k$$

$a = b = n \qquad n! = \sum_{k=0}^{n} (n-k)^n \binom{n}{k} (-1)^k$

più in generale : $\quad n! = \sum_{k=0}^{n} (t-k)^n \binom{n}{k} (-1)^k \qquad \forall t \in \mathbb{R}$

$C \subseteq B \qquad c = |C|$

$f : A \to B$ che coprono c

$$b^a - (b-1)^a c + (b-2)^a \binom{c}{2} - \dots = \sum_{k=0}^{c} (b-k)^a \binom{c}{k} (-1)^k$$

$n = a = c < b$ esce la formula generale per $n!$

- $f : A \to A$ permutazioni senza punti fissi $\quad n! \sum_{0}^{n} (-1)^k \frac{1}{k!}$

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ora 3

Una follia da diffondere: $\quad \sum_{k=1}^{n} \binom{n}{k} (-1)^{k+1} \frac{1}{k} = \sum_{k=1}^{n} \frac{1}{k}$

Serve : $\quad \frac{d}{dt} e^{-t} = -e^{-t}$

$\qquad \frac{d}{dt} e^{-kt} = -k e^{-kt}$

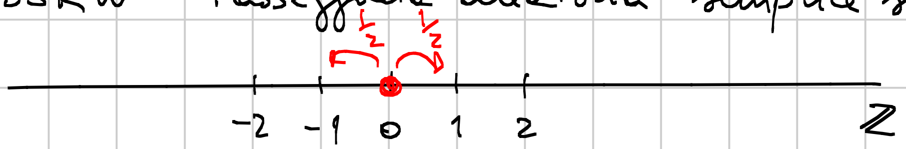
$$LHS = \sum_{k=1}^{n} \binom{n}{k} (-1)^{k+1} \overbrace{\int_0^\infty e^{-kt} dt}^{\frac{1}{k}} = \int_0^\infty \sum_{k=1}^{n} \binom{n}{k} (-1)^{k+1} e^{-kt} dt$$

$$= \int_0^\infty \left[1 - (1-e^{-t})^n\right] dt = \int_0^\infty [1-(1-e^{-t})][1 + (1-e^{-t}) + (\quad)^2 + \dots + (\quad)^{n-1}]$$

$$= \int_0^\infty e^{-t} \sum_{k=0}^{n-1} (1-e^{-t})^k dt = \sum_{k=0}^{n-1} \int_0^\infty e^{-t} (1-e^{-t})^k dt = \sum_{k=0}^{n-1} \int_0^\infty \frac{d}{dt}\left\{(1-e^{-t})^{k+1}\right\} \frac{1}{k+1} dt$$

$$= \sum_{k=0}^{n-1} \frac{1}{k+1} \left[(1-e^{-t})^{k+1}\right]_0^\infty = \sum_{k=1}^{n} \frac{1}{k}$$

■ SSRW Passeggiata aleatoria semplice simmetrica



$n = 1, 2, \dots$ i tempi

$S_n \in \mathbb{Z}$ le posizioni

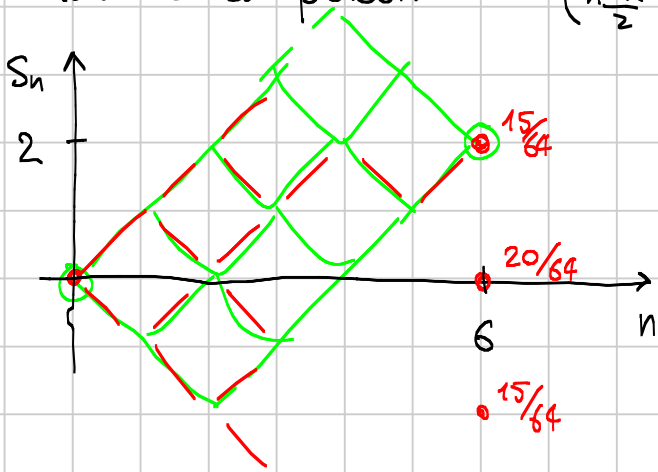
$P(S_0 = 0) = 1$ $S_0 = 0$ sempre

$P(S_1 = 1) = \frac{1}{2} = P(S_1 = -1)$

$$P(S_n = k) = \begin{cases} 0 & \text{se } k+n \text{ dispari} \\ \frac{1}{2^n}\binom{n}{\frac{n-k}{2}} & \text{se } k+n \text{ pari} \end{cases}$$

n passi, posiz k $\Rightarrow$ $\frac{n+k}{2}$ a DX $\frac{n-k}{2}$ a SX

Il numero di percorsi è $\binom{n}{\frac{n-k}{2}}$; ciascuno ha prob $\frac{1}{2^n}$



6 mosse posiz 2

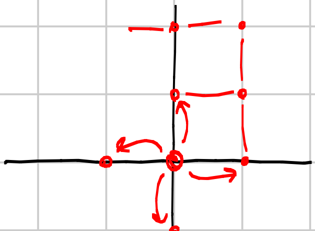
per forza 4 ↑ e 2 ↓

$\binom{6}{2} = \binom{6}{4} = 15$

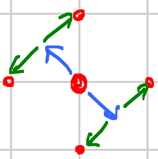
$P(S_6 = 4) = \frac{15}{64}$

- In due dimensioni : (X_n, Y_n) le coordinate al tempo n

$\mathbb{Z}^2$



$P\big((X_n, Y_n) = (j, k)\big)$



La mossa blu lascia costante $X_n + Y_n$ e modifica $X_n - Y_n$

La mossa verde lascia costante $X_n - Y_n$ e modifica $X_n + Y_n$

$X_n + Y_n = S_n$ $j + k = s$ n mosse qui

$X_n - Y_n = T_n$ $j - k = t$ n mosse qui (indipendenti)

$$P((X_n,Y_n)=(j,k)) = \binom{n}{\frac{n-s}{2}}\frac{1}{2^n}\cdot\binom{n}{\frac{n-t}{2}}\frac{1}{2^n} = \binom{n}{\frac{n+j+k}{2}}\binom{n}{\frac{n+j-k}{2}}\cdot\frac{1}{4^n}$$

		2		
	2	1	2	
2	1	02	1	2
	2	1	2	
		2		

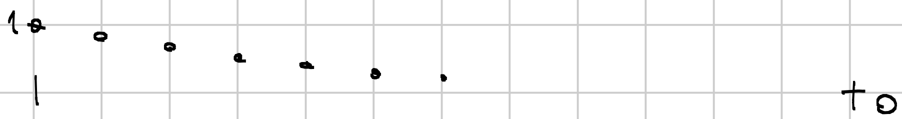
- hitting probability & hitting time

$P_A(k)$ prob che finisca in A prima che in B, partendo da k
a noi interessa $P_A(0)$

$$P_A(k) = \frac{1}{2}P_A(k-1) + \frac{1}{2}P_A(k+1) = \frac{P_A(k-1)+P_A(k+1)}{2}$$
$$P_A(-A) = 1$$
$$P_A(B) = 0$$

$$P_A(k+1) - P_A(k) = P_A(k) - P_A(k-1)$$

$$\boxed{P_A(k) = \frac{B-k}{B+A}}$$



$$P_A(0) = \frac{B}{A+B} \qquad P_B(0) = \frac{A}{A+B}$$

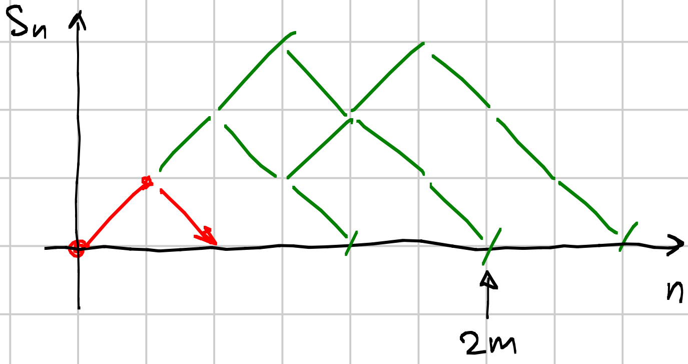
→ $t(k)$ il numero medio di passi necessari per finire su una barriera:

$$t(k) = \frac{1}{2}(t(k-1)+1) + \frac{1}{2}(t(k+1)+1) = 1 + \frac{1}{2}t(k+1) + \frac{1}{2}t(k-1)$$
$$t(-A) = t(B) = 0$$
$$t(k+1) - t(k) = 2 + t(k) - t(k-1) \Rightarrow t(k) = ak^2 + bk + c$$

sostituendo si trova:

$$t(k) = (B-k)(A+k) \qquad t(0) = AB$$

Escursioni della SSRW

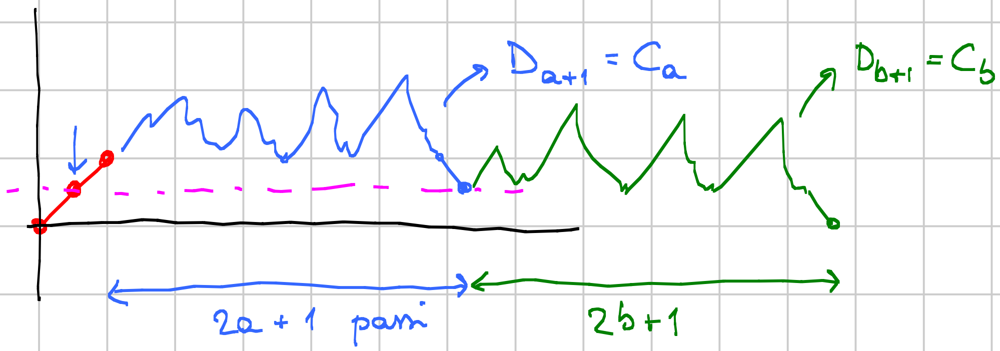


Qual è la prob o quanti sono i percorsi che fanno tornare S_n in zero per la prima volta al tempo $2m$?

$$P(S_{2m}=0) = \binom{2m}{m}\frac{1}{2^{2m}}$$ prob che sia in zero

Conto i percorsi: $m = 1, 2, \dots$ $D_m = 1, 1, 2, 5, 14, 42, 429$

$m = 0, 1, 2, \dots$ $C_m = 1, 1, 2, 5, 14, 42, 429, \dots$ numeri di Catalan



$$2 + 2a+1 + 2b+1 = 2(a+b+2) = 2m$$ $C_a \cdot C_b$ i modi

Al variare di a e b:

$$C_{m-1} = D_m = \sum_{\substack{a+b=m-2\\ a,b\geq 0}} C_a C_b = \sum_{a=0}^{m-2} C_a C_{m-2-a}$$

$$\boxed{C_{m+1} = \sum_{k=0}^{m} C_k C_{m-k}}$$ $1, 1, 2, 5, 14, \dots$

$$C_{m+1}x^{m+1} = \sum_{k=0}^{m} C_k C_{m-k} x^{m+1} = x\sum_{k=0}^{m} C_k x^k C_{m-k} x^{m-k}$$

$m = j+k$

$$c(x) = \sum_{m=0}^{\infty} C_m x^m = C_0 + \sum_{m=0}^{\infty} C_{m+1}x^{m+1} = \overset{1}{\overset{\shortparallel}{C_0}} + x\sum_{m=0}^{\infty}\sum_{k=0}^{m} C_k x^k C_{m-k} x^{m-k}$$

$$= 1 + x\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} C_k x^k C_j x^j = 1 + x\,c(x)^2$$

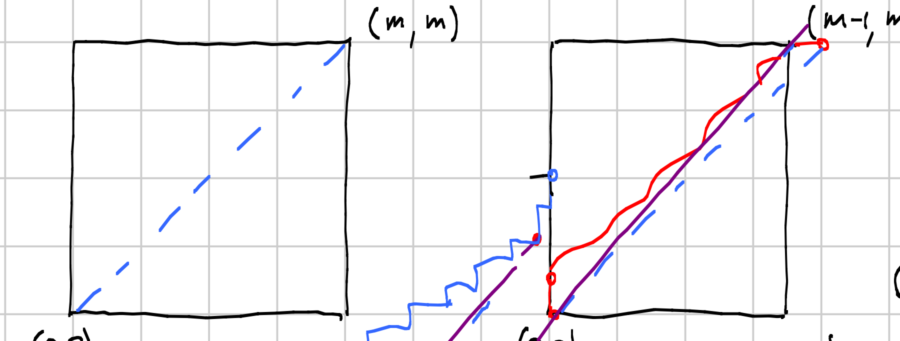
$$c(x) = 1 + x\,c(x)^2 \qquad\qquad x Y^2 - Y + 1 = 0$$

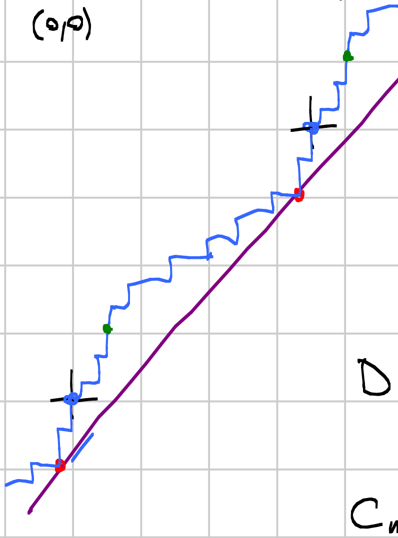
$$Y = \frac{1 \pm \sqrt{1-4x}}{2x}$$

$$c(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$$

1) $c(x) < \infty$ per x piccolo abbastanza
2) con il $+$ viene un asintoto in 0
3) per continuità quella con il $+$ è sempre esclusa

$$c(x) = \frac{1 - \sqrt{1-4x}}{2x}$$





Due candidati: nell'urna ci sono m voti per A e $m-1$ voti per B.
Qual è la prob che A resti in vantaggio per tutto il tempo

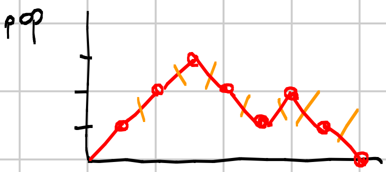
Per ogni percorso sopra la diagonale ci sono esattamente $2m-1$ percorsi qualsiasi

$$D_m = \frac{1}{2m-1}\binom{2m-1}{m}$$

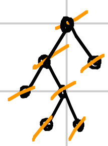
$$C_m = D_{m+1} = \binom{2m}{m}\frac{1}{m+1} \qquad \text{NB è intero!!}$$

- Altre caratterizzazioni

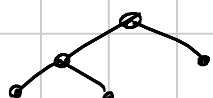
escursioni aleatorie $\leftrightarrow$ alberi binari $\leftrightarrow$ prod non associativi



2m passi



m foglie
2m-1 nodi



$((a \times (b \times c)) \times d)$

m fattori

escursioni aleatorie $\longleftrightarrow$ alberi con radice