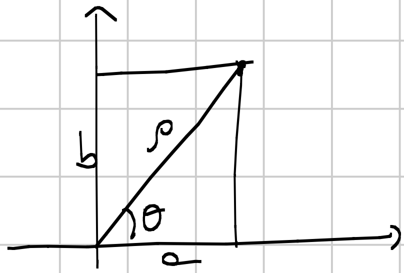
G1 TRIGONOMETRIA - MEDIO

Titolo nota 07/09/2009

$\cos n\alpha \qquad \sum_{k=0}^{n} \sin^2 kx$

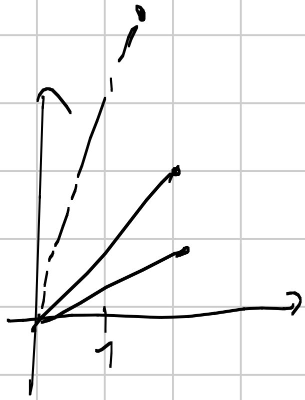


$$e^{a+ib} = e^{a}(\cos b + i\sin b)$$

$$a+ib = \rho e^{i\theta} = \rho\cos\theta + i\rho\sin\theta$$

$$\rho_1 e^{i\theta_1} \cdot \rho_2 e^{i\theta_2} = \rho_1 \rho_2 \, e^{i(\theta_1+\theta_2)}$$

$$e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$$



$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) =$$

$$\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin(\theta_1+\theta_2))$$

$$\cos n\alpha = \mathrm{Re}(\cos n\alpha + i\sin n\alpha) =$$

$$= \mathrm{Re}(e^{in\alpha})$$

$$= \mathrm{Re}((\cos\alpha + i\sin\alpha)^n)$$

$$= \mathrm{Re} \sum_{k=0}^{n} \binom{n}{k} \cos^k\alpha \, (i\sin\alpha)^{n-k}$$

$n=4$

$$= \cos^4\alpha - 6\cos^2\alpha\sin^2\alpha + \sin^4\alpha$$

$$\sum_{k=0}^{n} \sin^2 kx =$$

$$\cancel{\sum \sin kx = \sum \mathrm{Im}\, e^{ikx} = \mathrm{Im} \sum e^{ixk} = \mathrm{Im}\, \frac{e^{i(n+1)x}-1}{e^{ix}-1}}$$

$$\mathrm{Im}\, zw \neq \cancel{\mathrm{Im} z}\, \mathrm{Im}\, w$$

$$\sum_{k=0}^{n} \frac{1-\cos 2kx}{2} = \frac{n+1}{2} - \frac{1}{2} \sum \cos 2kx$$

$$\sum \cos 2kx = \mathrm{Re} \sum e^{i2kx} = \mathrm{Re}\, \frac{e^{i2x(n+1)}-1}{e^{i2x}-1}$$

$$= \mathrm{Re}\, \frac{(e^{i2x(n+1)}-1)(e^{-i2x}-1)}{(e^{i2x}-1)(e^{-i2x}-1)} =$$

$$= \mathrm{Re}\,\frac{e^{i2xm} - e^{-i2x} - e^{i2x(m+1)} + 1}{2 - 2\cos 2x} =$$

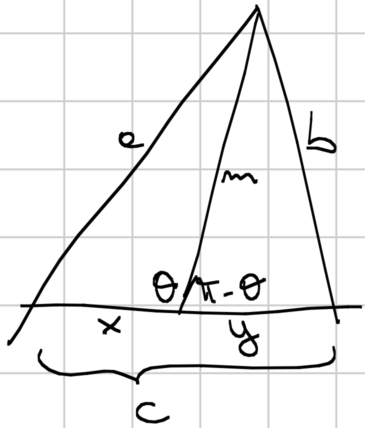
$$= \frac{\cos 2mx - \cos 2(m+1)x + 1 - \cos 2x}{4\sin^2 x}$$

$$= \frac{+\cancel{2}\sin(2m+1)x \cdot \sin x + \cancel{2}\sin^{\cancel{2}} x}{\cancel{4}_2 \sin^{\cancel{2}} x}$$

$$= \frac{\sin(2m+1)x + \sin x}{2\sin x}$$

$\cos x - \cos y =$
$= -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$

TEOREMA DI STEWART



$$a^2 y + b^2 x = c\,(m^2 + xy)$$

$$y\left(a^2 = m^2 + x^2 - 2mx\cos\theta\right)$$
$$x\left(b^2 = m^2 + y^2 + 2my\cos\theta\right)$$
$$a^2 y + b^2 x = (x+y)m^2 + (x+y)xy$$
$$= c\,(m^2 + xy)$$

LUNGHEZZA MEDIANA

$x = y = \frac{c}{2}$

$$\frac{a^2+b^2}{2} = m^2 + \frac{c^2}{4}$$

$$\frac{2a^2+2b^2-c^2}{4} = m^2$$

LUNGHEZZA BISETTRICE

$x + y = c$

$$\begin{cases} \frac{x}{a} = \frac{y}{b} \\ x+y = c \end{cases}$$

$$\begin{cases} x = \frac{a}{a+b}c \\ y = \frac{b}{a+b}c \end{cases}$$

$$\frac{x}{\sin\theta_1} = \frac{a}{\sin\varphi_2}$$

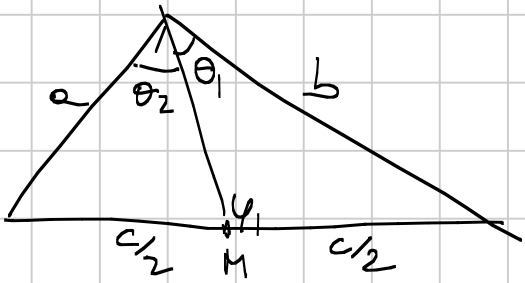
$$\frac{y}{\sin\theta_2} = \frac{b}{\sin\varphi_2}$$

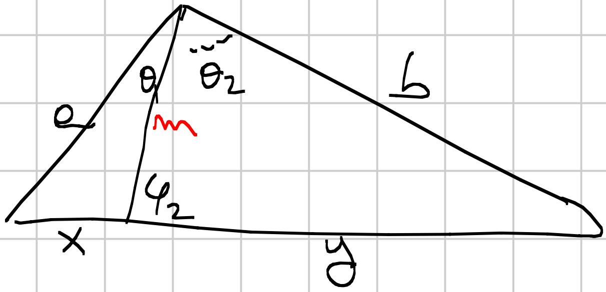
$$\frac{x}{y} = \frac{a}{b}\cdot\frac{\sin\theta_1}{\sin\theta_2}$$

$$\frac{c}{2\sin\theta_2} = \frac{a}{\sin\varphi_1}$$

$$\frac{c}{2\sin\theta_1} = \frac{b}{\sin\varphi_1}$$

$$\boxed{\frac{\sin\theta_1}{\sin\theta_2} = \frac{a}{b}}$$





$$\frac{x}{y} = \frac{a^2}{b^2}$$

Esercizio! caratterizzare $x, y, z > 0$ che soddisfano
$$x^2 + y^2 + z^2 + 2xyz = 1$$

$x = \sin\frac{A}{2}$ $\quad y = \sin\frac{B}{2}$ $\quad z = \sin\frac{C}{2}$ con ABC triangolo

$$A + B + C = \pi$$

Dim

$$\sin^2\frac{A}{2} + \sin^2\frac{B}{2} + \sin^2\frac{C}{2} + 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = 1$$

$$\sin\frac{C}{2} = \sin\frac{\pi}{2} - \frac{A}{2} - \frac{B}{2} = \cos\frac{A}{2} + \frac{B}{2}$$

$$\sin^2\frac{A}{2} + \sin^2\frac{B}{2} + \cos^2\frac{A}{2}\cos^2\frac{B}{2} + \sin^2\frac{A}{2}\sin^2\frac{B}{2} +$$
$$\cancel{-2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{A}{2}\sin\frac{B}{2}} + 2\sin\frac{A}{2}\sin\frac{B}{2}$$
$$\left(\cancel{\cos\frac{A}{2}\cos\frac{B}{2}} - \sin\frac{A}{2}\sin\frac{B}{2}\right) = 1$$

$$\sin^2\frac{A}{2} + \sin^2\frac{B}{2} + \left(1 - \sin^2\frac{A}{2}\right)\left(1 - \sin^2\frac{B}{2}\right) - \sin^2\frac{A}{2}\sin^2\frac{B}{2} = 1$$

Supponiamo x, y, z soddisfa.

$x \le 1 \quad y \le 1 \quad \Rightarrow \quad \exists\, A, B \in [0, \pi]$ tali che

$x = \sin\frac{A}{2} \qquad y = \sin\frac{B}{2}$

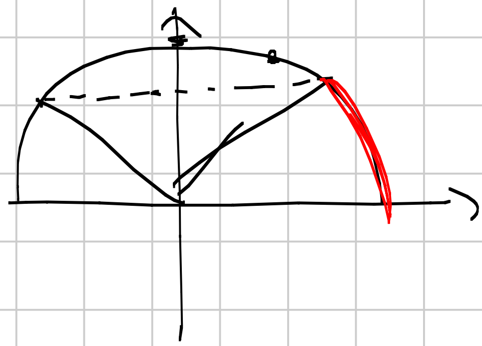
$$z = -xy \pm \sqrt{x^2y^2 - (x^2+y^2-1)}$$

$$= -xy \pm \sqrt{(1-x^2)(1-y^2)}$$

$\exists!$ soluzione $\quad z = \sin\frac{\pi - A - B}{2}$

$A, B, \pi - A - B$ possono essere angoli di un triangolo?

$A + B < \pi$



$$\sin^2\frac{A}{2} + \sin^2\frac{B}{2} < 1$$

$$\frac{1-\cos A + 1 - \cos B}{2} < 1$$

$$0 < \cos A + \cos B$$

$A \qquad B < \pi - A$

ABC triangolo acutangolo

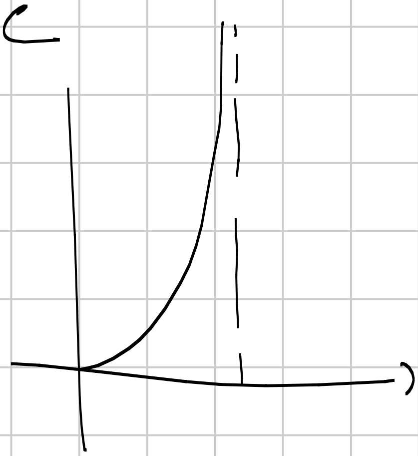
$$\tan A + \tan B + \tan C = \tan A \tan B \tan C \ge 3\sqrt{3}$$

$$(\sin A \cos B \cos C + \sin B \cos A \cos C) + \sin C \cos A \cos B = \sin A \sin B \sin C$$

$A + B + C = \pi$

$$\cos C\,(\underbrace{\sin A+B}_{=\,\sin C}) = \sin C\,(\underbrace{-\cos A+B}_{=\,\cos C})$$

$$\frac{\tan A + \tan B + \tan C}{3} \ge \tan\frac{180}{3} = \tan 60$$

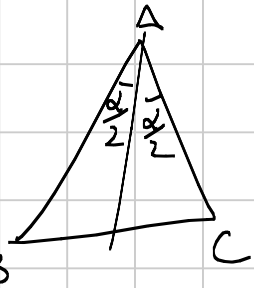


$$\cos A + \cos B + \cos C = 1 + \frac{r}{R}$$

$$\frac{r}{R} = \frac{\dfrac{\text{Area } \Delta ABC}{p}}{\dfrac{abc}{4 \text{ Area } \Delta ABC}} = \frac{4 \text{ Area } \Delta ABC^2}{p\, abc}$$

$$= \frac{4 \cancel{p}(p-a)(p-b)(p-c)}{\cancel{p}\, abc}$$

Lemmino $\quad \sin^2 \frac{A}{2} = \frac{(p-b)(p-c)}{bc}$



$$\frac{1-\cos A}{2} \overset{?}{=} \frac{(p-b)(p-c)}{bc}$$

$$2bc - 2bc\cos A = (a+c-b)(a+b-c)$$

$$= a^2 - (b-c)^2$$

$$b^2 + c^2 - 2bc\cos A = a^2$$

$\cos \frac{\alpha}{2} \qquad \frac{1}{2} bc \sin\alpha = \text{area}$

$$\frac{r}{R} = 4 \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}$$

$$\cos A + \cos B + \cos C - 1 = 4 \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}$$

sum → product

$$\cos A + \cos B = 2 \cos\frac{A+B}{2} \cos\frac{A-B}{2}$$

$$\cos\left(\frac{\pi}{2} - \frac{C}{2}\right)$$

$$\cos\frac{A+B}{2} = \sin\frac{C}{2}$$

$$\cos C - 1 = -2\sin^2\frac{C}{2}$$

$$\cancel{2} \cancel{\sin\frac{C}{2}} \cos\frac{A-B}{2} - \cancel{2} \sin^{\cancel{2}}\frac{C}{2} \overset{?}{=} \cancel{4}\,2 \sin\frac{A}{2} \sin\frac{B}{2} \cancel{\sin\frac{C}{2}}$$

$$\cos\frac{A-B}{2} - \cos\frac{A+B}{2} = 2 \sin\frac{A}{2} \sin\frac{B}{2}$$

x, y, z reali > 1 $\quad \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$

$$\sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1} \le \sqrt{x+y+z}$$

$$(x, y, z) = (a+1, b+1, c+1)$$

$$\sqrt{a} + \sqrt{b} + \sqrt{c} \le \sqrt{a+b+c+3}$$

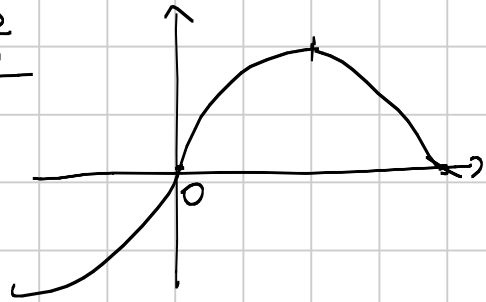
$$ab + bc + ca + 2abc = 1$$

$$(\sqrt{ab})^2 + (\sqrt{bc})^2 + (\sqrt{ca})^2 + 2\sqrt{ab}\sqrt{bc}\sqrt{ca} = 1$$

$$\sqrt{ab} + \sqrt{bc} + \sqrt{ca} \le \frac{3}{2}$$

$$\sqrt{ab} = \sin\frac{A}{2} \qquad \sqrt{bc} = \sin\frac{B}{2} \qquad \sqrt{ac} = \sin\frac{C}{2}$$

$$\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} \le \frac{3}{2}$$



$$\frac{1}{3}\sin\frac{A}{2} + \frac{1}{3}\sin\frac{B}{2} + \frac{1}{3}\sin\frac{C}{2}$$

$$\le \sin\frac{A+B+C}{2 \cdot 3} = \frac{1}{2}$$

Fatto: siano $\alpha, \beta, \gamma \in (0, \pi)$

α, β, γ angoli di un triangolo $\iff$

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2} + \tan\frac{\beta}{2}\tan\frac{\gamma}{2} + \tan\frac{\gamma}{2}\tan\frac{\alpha}{2} = 1$$

$$\Rightarrow \quad \tan\frac{\gamma}{2} \overset{?}{=} \frac{1 - \tan\frac{\alpha}{2}\tan\frac{\beta}{2}}{\tan\frac{\alpha}{2} + \tan\beta/_2}$$

$$\| \qquad \frac{1}{\tan\frac{\alpha+\beta}{2}}$$

$$\tan x = \frac{1}{\tan y} \qquad x + y = \frac{\pi}{2} + k\pi$$

$$\tan x = \tan\frac{\pi}{2} - y = \frac{\cos y}{\sin y} = \frac{1}{\tan y}$$

Fissiamo α e β e γ che soddisfano -

$$\delta = \pi - \alpha - \beta$$

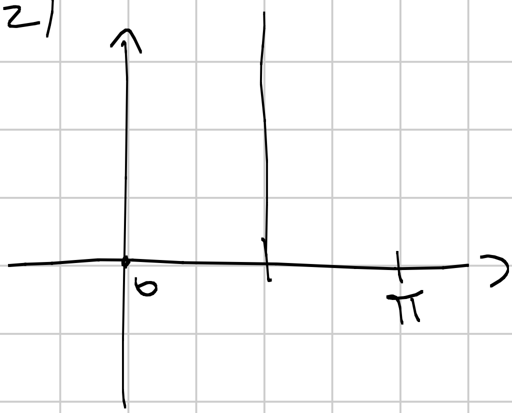
$$\tan\frac{\delta}{2} = \tan\frac{\pi-\alpha-\beta}{2} = \frac{1}{\tan\frac{\alpha+\beta}{2}}$$

(α,β,γ) verificano $\Rightarrow$ (α,β,δ) verificano

$$\gamma = \delta + 2k\pi$$

$$\left(\tan\frac{\beta}{2}+\tan\frac{\alpha}{2}\right)\left(\tan\frac{\gamma}{2}-\tan\frac{\delta}{2}\right) = 0$$

$$\frac{\gamma}{2} = \frac{\delta}{2} + k\pi$$



Esercizio:

$a,b,c \in (0,1)$ tali che $ab+bc+ca=1$

$$\frac{a}{1-a^2}+\frac{b}{1-b^2}+\frac{c}{1-c^2} \geq \frac{3}{4}\left(\frac{1-a^2}{a}+\frac{1-b^2}{b}+\frac{1-c^2}{c}\right)$$

$$A < \frac{\pi}{2}$$

$$a = \tan\frac{A}{2} \qquad b = \tan\frac{B}{2} \qquad c = \tan\frac{C}{2}$$

$$\frac{\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}} = \frac{\sin\frac{A}{2}\cdot\cos\frac{A}{2}}{\cos^2\frac{A}{2}-\sin^2\frac{A}{2}} = \frac{\sin A}{2\cos A} = \frac{\tan A}{2}$$

$$\sum \tan A \geq 3\sum\frac{1}{\tan A}$$

$$= 3\,\frac{\tan A\tan B+\tan B\tan C+\tan C\tan A}{\tan A\tan B\tan C}$$

$$\tan A\tan B\tan C = \tan A+\tan B+\tan C$$

$$A+B+C=\pi$$

$$\left(\sum\tan A\right)^2 \geq 3\sum\tan A\tan B$$

$$\tan^2A+\tan^2B+\tan^2C-\tan A\tan B-\tan B\tan C-\tan C\tan A > 0$$

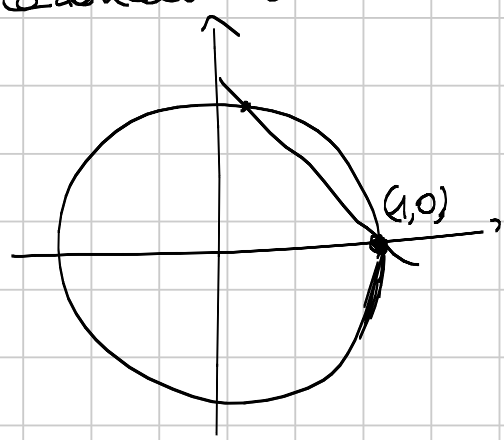
$$\sum_{cycl} \frac{1}{2}(\tan A - \tan B)^2 > 0$$

IMO 75/5

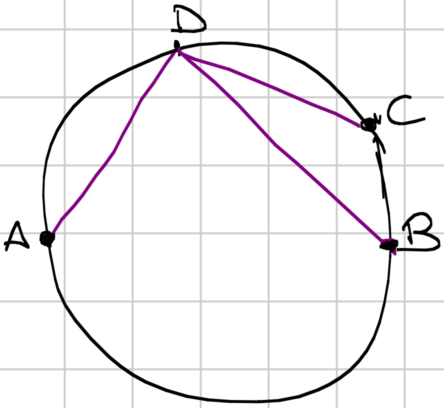
Dimostrare che esistono ∞ pti sulla circonferenza unitaria tali che a 2 a 2 abbiano distanza razionale.

Lemma: $\exists$ ∞ pti a coordinate razionali sulla circo unitaria.

$$\begin{cases} y = m(x-1) \\ y^2 + x^2 = 1 \end{cases}$$



$m^2(x-1)^2 + x^2 = 1$ $\qquad x = 1$

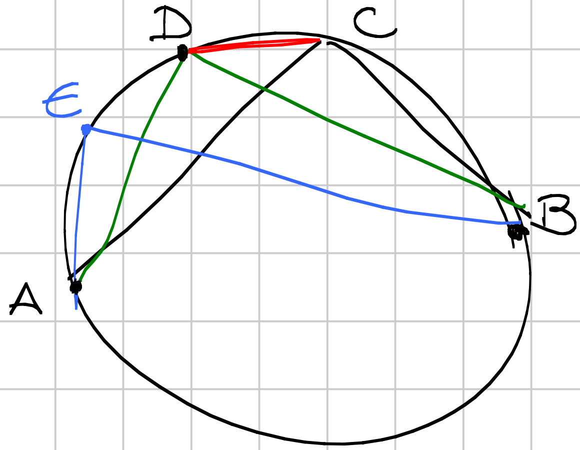


AD DB AC BC AB siano razionali.

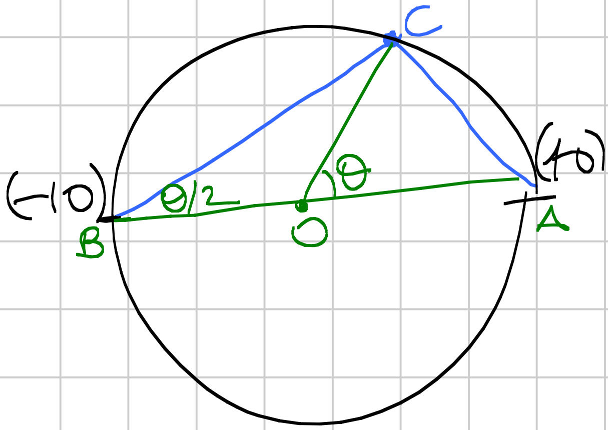
CD razionale

Tolomeo

$$AD\cdot BC + AB\cdot CD \overset{?}{=} AC\cdot BD$$



Dobbiamo dim che scelti A e B con $|AB|$ razionale $\Rightarrow \exists\ \infty$ pti con dist razionale da A e B.



$$\begin{cases} CA = 2\sin\frac{\theta}{2} \\ CB = 2\cos\frac{\theta}{2} \end{cases}$$

α $\qquad\theta = 2\alpha$

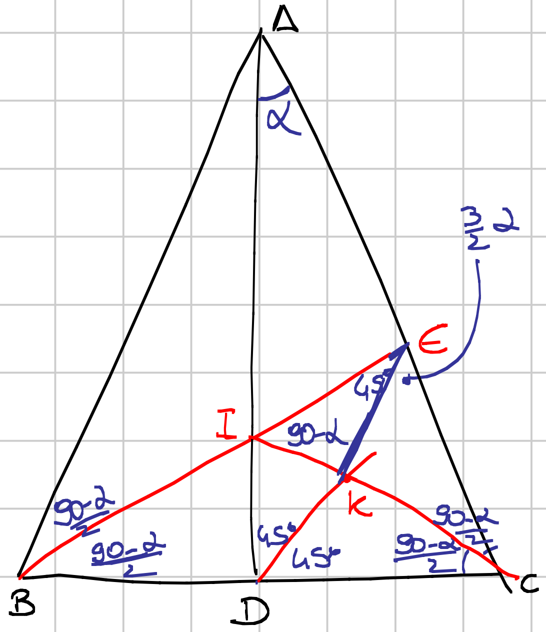
$\begin{cases} x = \cos\alpha \\ y = \sin\alpha \end{cases}$ erano razionali

IMO 09/4

ΔABC isoscele

$K\hat{E}I = 45°$

I, C, K sono allineati perché stanno sulla bisettrice in $\hat{C}$.



$$180-90+\alpha-45-45+\frac{\alpha}{2}=\frac{3}{2}\alpha$$

$$\frac{IK}{KC}=\frac{ID}{DC}=\operatorname{tg}\frac{90-\alpha}{2}$$

$$\begin{cases} \dfrac{IK}{\sin 45}=\dfrac{KE}{\sin 90-\alpha} \\ \dfrac{KC}{\sin\frac{3}{2}\alpha}=\dfrac{KE}{\sin\frac{90-\alpha}{2}} \end{cases} \Rightarrow \frac{IK}{KC}=\frac{\sin\frac{90-\alpha}{2}}{\cos\alpha}\;\frac{\sin 45}{\sin\frac{3}{2}\alpha}$$

$$\frac{\cancel{\sin\frac{90-\alpha}{2}}}{\cos\frac{90-\alpha}{2}}=\frac{\cancel{\sin\frac{90-\alpha}{2}}}{\cos\alpha}\cdot\frac{\sin 45}{\sin\frac{3}{2}\alpha}$$

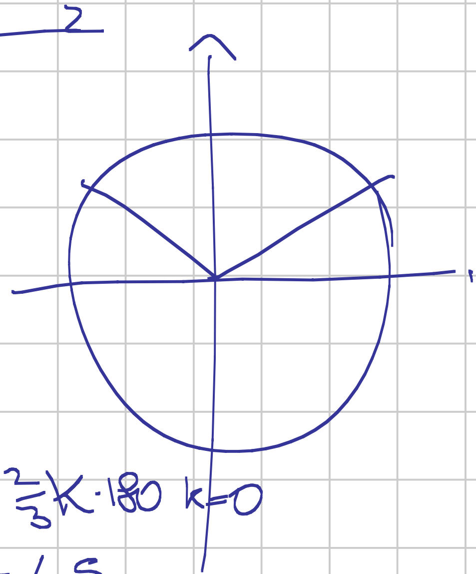
$$\sin\frac{3\alpha}{2}\cdot\cos\alpha=\sin 45\cdot\cos 45-\frac{\alpha}{2}$$

$$\theta=\frac{\alpha}{2} \qquad \sin\theta=\frac{t}{1+t^2} \qquad t=\operatorname{tg}\frac{\theta}{2}$$

$$\sin x\cdot\cos x=\frac{\sin x+y+\sin x-y}{2}$$

$$\sin\frac{3\alpha}{2}\cdot\cos\alpha=\frac{\sin\frac{5}{2}\alpha+\sin\frac{\alpha}{2}}{2}$$

$$\sin 45\;\cos 45-\frac{\alpha}{2}=\frac{\sin 90-\frac{\alpha}{2}+\sin\frac{\alpha}{2}}{2}$$



$$\sin\frac{5}{2}\alpha=\sin 90-\frac{\alpha}{2}$$

$$\frac{5}{2}\alpha=90-\frac{\alpha}{2}+2\pi\cdot k$$

$$\pi-\frac{5}{2}\alpha=90-\frac{\alpha}{2}+2\pi\cdot k$$

$$3\alpha=90+2k\cdot 180 \Rightarrow \alpha=30+\frac{2}{3}k\cdot 180 \quad k=0$$

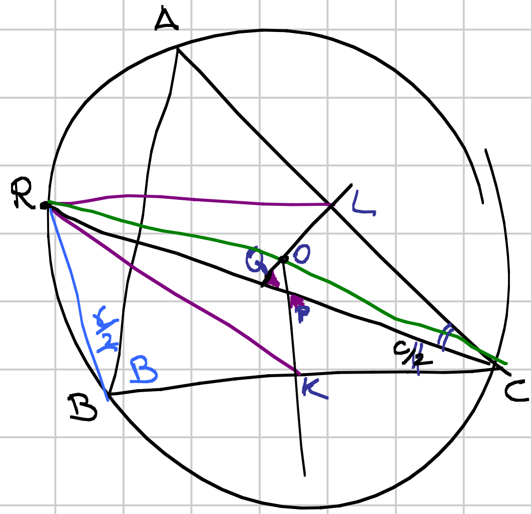
$$-2\alpha=-90+2\cdot k\cdot 180 \qquad \alpha=45$$

IMO 07/4

P pto medio $\widehat{AB}$

OPQ è isoscele.

Tesi: RLQ e RPK hanno la stessa area.



$\text{Area } P\hat{K}R = \frac{1}{2} PK \cdot PR \cdot \sin 90 - \frac{C}{2}$

$= \frac{1}{2} PK \cdot PR \cos\frac{C}{2}$

$\frac{PK}{KC} = tg\frac{C}{2}$

$PK = \frac{a}{2} tg\frac{C}{2}$

$PR = CR - PC = 2R \sin\frac{C}{2}+B - \frac{a}{2\cos C/2}$

$\frac{1}{2} \frac{a}{2} tg\frac{C}{2} \cdot \left(2R\sin\frac{C}{2}+B - \frac{a}{2\cos C/2}\right) \cdot \cos\frac{C}{2}$

$= \frac{1}{4} tg\frac{C}{2} \cdot \frac{a}{2} \underbrace{\left(4R\sin\frac{C}{2}+B \cdot \cos\frac{C}{2} - a\right)}_{b \cdot f(a,b,\hat{A},\hat{B},\hat{C})}$

$\sin x \cdot \cos y = \frac{\sin x+y + \sin x-y}{2}$

$4R\, \frac{\sin C+B + \sin B}{2} - a = 2R(\sin A + \sin B) - a$

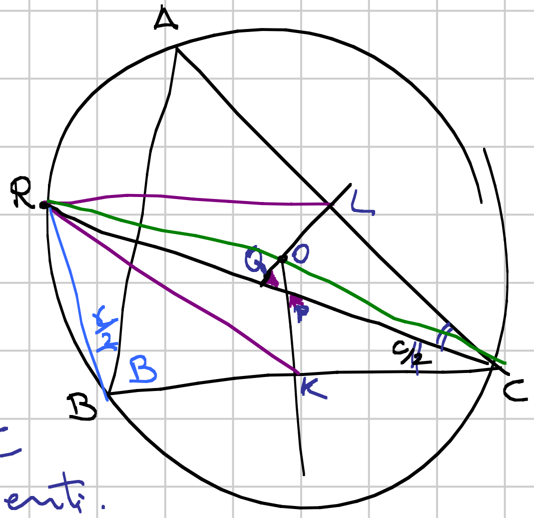
$= a + b - a = +b$

$\frac{1}{4} tg\frac{C}{2} \cdot \frac{ab}{2}$

Dim "Euclidea"

$\text{Area } PKR = \frac{1}{2} PK \cdot PR \cdot \cos\frac{C}{2}$

$PK \cdot PR \overset{?}{=} QL \cdot RQ$



ORQ e OPC

$$\begin{cases} OR = OC & \text{(raggi)} \\ O\hat{Q}R = O\hat{P}C \\ OQ = OP \end{cases}$$

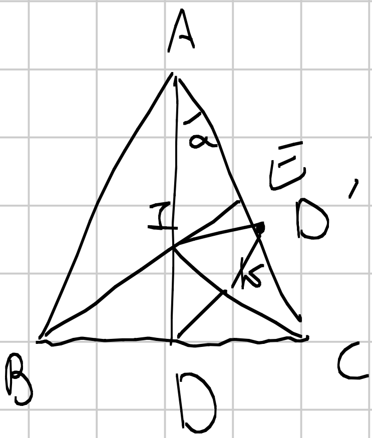
$\Rightarrow$ ORQ e OPC sono congruenti.

$PR = QC$

$QR = PC$

$PK \cdot QC = QL \cdot PC$

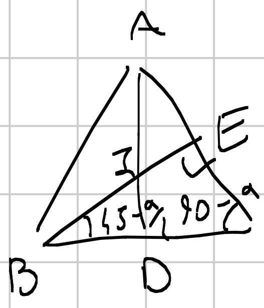
$\frac{PK}{PC} = \frac{QL}{QC}$ vero per similitudine di $P\hat{K}C$ e $C\hat{L}Q$.



$I\hat{D'}C = I\hat{D}C = 90°$

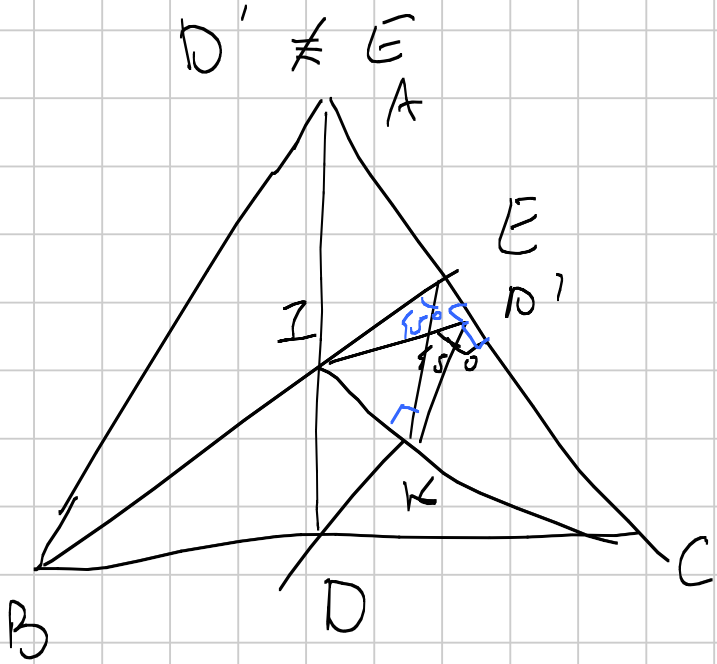
$I\hat{D'}K = I\hat{D}K = 45$

$D' \equiv E$ o $D' \neq E$



$45 - \frac{\alpha}{2} + 90 - \alpha = 90$

$\alpha = 30$



$I\hat{E}K = 45 = I\hat{D'}K$

$I\hat{D'}E = 90°$

$I\hat{K}E = 90°$

$I\hat{E}K + E\hat{I}K = 90$

$E\hat{I}K = 45$

$\quad\ \parallel$

$90 - \alpha$

$\Rightarrow \alpha = 45$