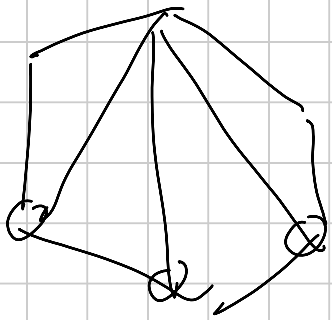
COMBINATORICS.

Titolo nota 06/09/2010

1. Induction.

1.1. Given a convex polygon n, triangulated by diagonals, such that the number of segments meeting at each vertex is even.



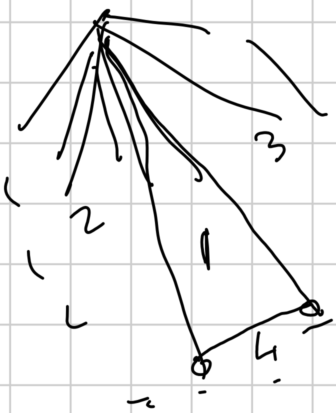
Question? When?

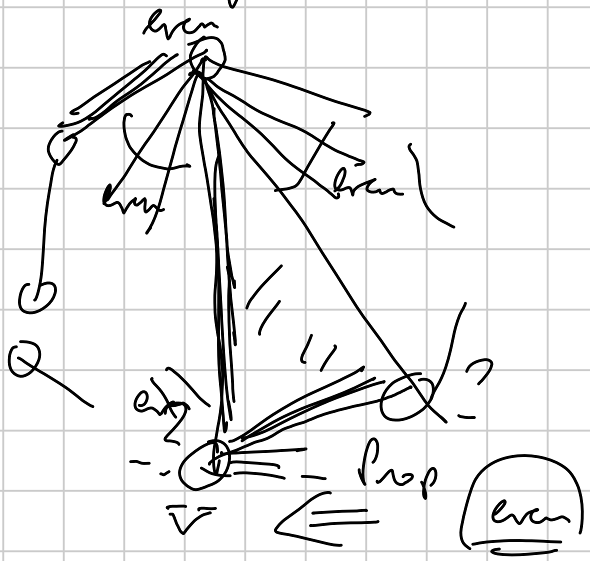
$n = 3$ ✓

$n = 4$ —

$n = 5$ —

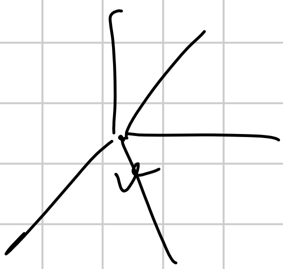
$n = 6$ ✓





Claim v also has even # segments

Polygon + sides diagonals = Graph $G = (V, E)$



$\deg v$ = # edges incident at v.

$$\sum_{v \in V} \deg v = 2|E| \quad \text{even}$$

#(vertices of odd degree) —— $\sum_{\deg v = \text{odd}} \deg v$ = even

$\Rightarrow$ even : **Proposition**. The number of vertices of odd degree in a finite graph is even.

[Start again. "Define" the induction predicate.]

Induction on # of sides have to be ≥ 3.

Sufficient condition — show how (exhibit example) Necessary Condition

1) to just describe example

2) to inductively build it.

Find some "special value" — "threshold" value

Example. $\{1, 2, \dots, 2n\}$

What is minimum cardinality of a subset so ~~that~~ there must exist 2 coprime numbers in that subset.

Ⓓ prove that $|S| \geq t$ ... value

$\Downarrow$

Ⓔ ~~prove~~ that $|S| = t-1$ not true.

example.

Problem. Given is $n \times n$ matrix, entries are symbols a_{ij}. Each row i seen

$$(a_{i1}, a_{i2}, \dots, a_{in}) \in (\text{space of symbols})^n$$

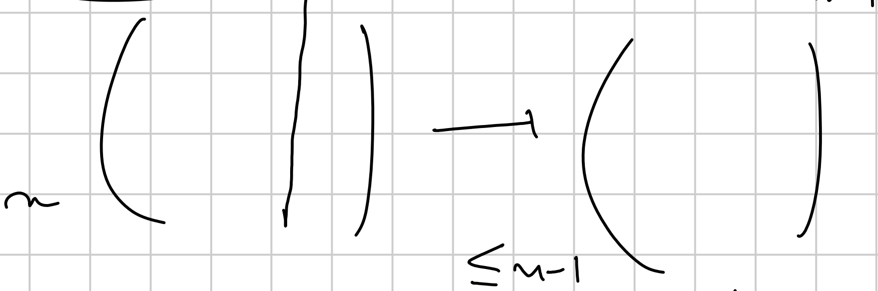
Given is the fact $R_i \neq R_j$ as vectors.
Show that we may choose to remove some column, so that in the new $n \times (n-1)$ matrix, the rows are still distinct (as vectors).

$m = n+1$

$$
n \quad \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}
$$

Square $k \times k$

Solution 1



n $\quad$ n $\quad$ $n-1$

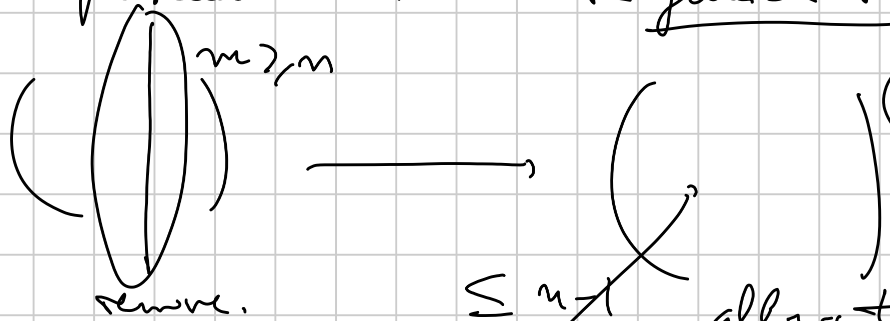
$\leq n-1$

$=$ ___ ✓

$<$ ___ not allow induction

Matrix not square anymore ←

"Patch" to this obstacle: replace our problem with a more general one:



$m \geq n$ $\quad$ n $\quad$ remove.

$(m-1) \geq n-1$

$\leq n-1$

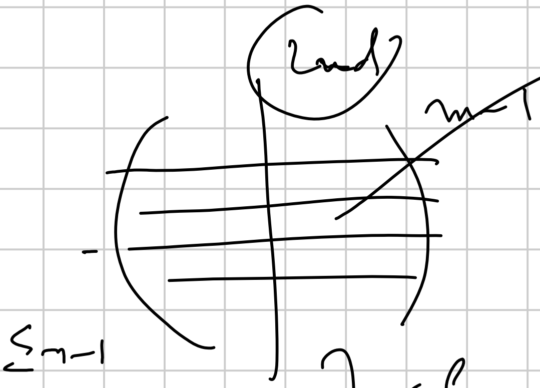
allows to apply induction.

(Check the truth thesis for "small" values. $n=1,2$)

Ind. Hyp → YES, I can remove a column, s.t. remaining rows, distinct.

n = m = 1, remove the single column → original statement true ?

by a vacuous argument — YES



∃ column may be removed, rows to remain distinct.

Now → we go back, remove a column from $n \times m$ matrix. □.

Solution 2. (Story) Ross Honsberger.

Assume, that any C_k column we remove, there appear duplicate rows : R_i and R_j.
For $k' \neq k$ → could they be the same ? NO.
Build graph G : vertices R_i
edges between $R_i \overset{k}{—} R_j$

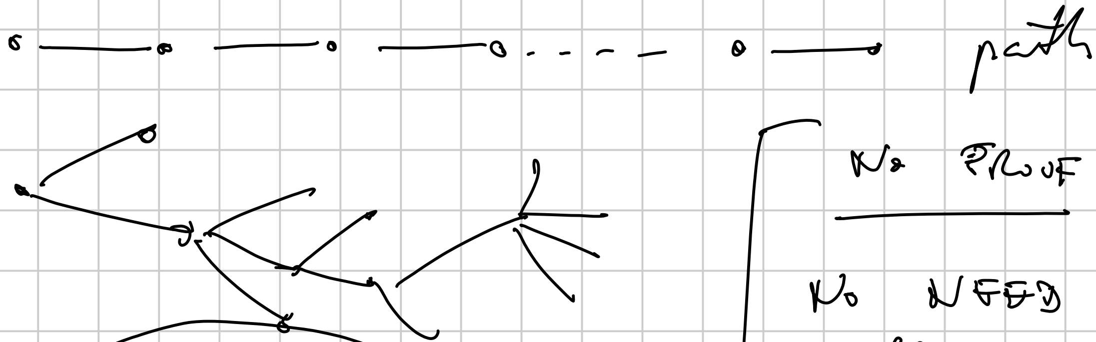
$|G| = n$, $\|G\| = n$.

What do we know about such graphs ?

There must be a cycle ! (YES)

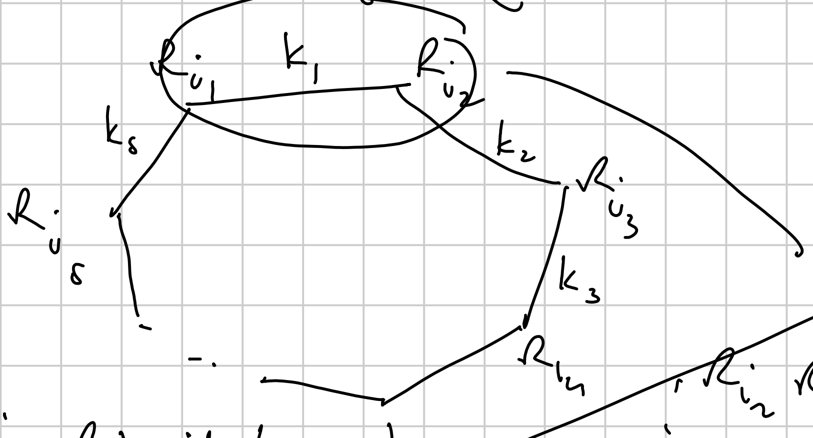
Because the largest number of edges graph n vertices may have, so that no cycles are present — is $\boxed{n-1}$

$n-1$ the maximal # edges of an acyclic graph. Are called trees:



No PROOF

NO NEED PROOF.



R_{i_1} R_{i_2} identical except on k_1

R_{i_2} R_{i_3} iden. $\neq$ on k_1

R_{i_3} R_{i_4} ident $\neq$ on k_1

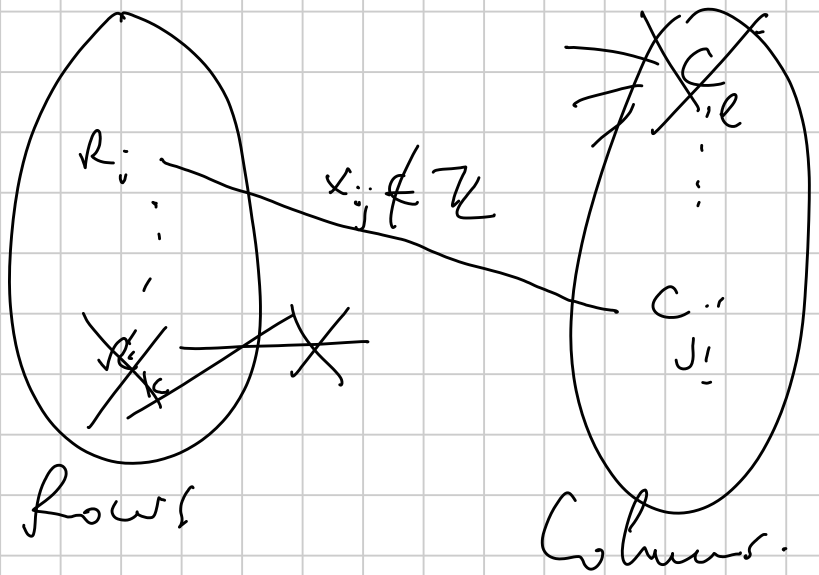
R_{i_s}, R_{i_1} ident. on k_1

$\boxed{\text{CONTRADICTION}}$. $\square$.

Generalization. Given $n \times m$ matrix. $m \geq n$ $\to$ we may remove $\boxed{m-n+1}$ columns $\longrightarrow$ rows to remain distinct.

Problem. We have some rectangular array $m \times n$, whose entries real numbers x_{ij}.
We know that sums by rows and by columns are integer numbers.
Prove that we can replace (if need is) x_{ij} with y_{ij} integer: $y_{ij} \in \{\lfloor x_{ij} \rfloor, \lceil x_{ij} \rceil\}$, such that the sums by rows and columns stay the same.

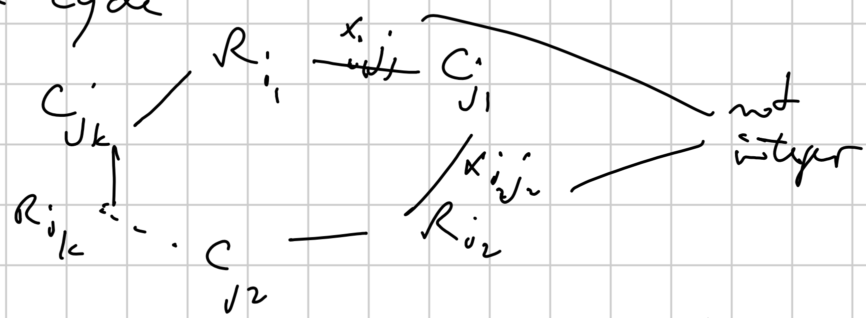
Solution. Those $x_{ij} \in \mathbb{Z}$ do not matter.



Graph, where vertices partitioned 2 classes, and edges are only between classes. is called bipartite

$\boxed{\text{Results about bipartite graphs}}$ $\iff$ Cycles are of even length.

Let's take a cycle



Let us denote by ε the least distance from any of the $x_{i_\ell j_\ell}$ to their neighboring integers.

$x_{i_1j_1} \xleftarrow{\varepsilon}$; $x_{i_2j_2} \xrightarrow{\varepsilon}$, $x_{i_3j_3} \xleftarrow{\varepsilon}$, and so on.

$x_{i_kj_k} \xrightarrow{\varepsilon}$

End $\longrightarrow$ some (≥ 1) x_{ij}'s $\to$ become integer and all the conditions are obeyed.

Reverse induction $\longleftrightarrow$ finite algorithm (loop)
Build a new bipartite graph (strictly less edges).

Problem. Noga Alon (Combinatorial Nullstellensatz).

Consider $\{1, 2, \dots, n\}$. Consider two permutations $\sigma, \tau \in S_n$.

$\sigma(1), \sigma(2), \dots, \sigma(n)$ || rows of ± 1's.
$\tau(1), \tau(2), \dots, \tau(n)$

We look at functions $f : \{1, 2, \dots, n\} \to \{-1, +1\}$.

We look at $\sum_{k=i}^{k=j} f(\sigma(k))$ (sums of consecutive terms in the σ-sequence)

Question is? Can we find such an f so that all these "partial sums", for both σ and τ, they all are at most $\underline{\underline{2}}$ in absolute value?

Solution. What if only σ was given?

$+1 \quad -1 \qquad\qquad +1 \quad -1$

$\sigma(1), \sigma(2) \quad \dots \quad \sigma(n)$

choose to start $+1/-1$ ← $\quad 1 \quad -1 \; + \; -1 \qquad \pm 1$

$\tau(1) \quad \tau(2) \qquad\qquad \tau(n)$

$-1 \quad +1$

Subcase $\boxed{n = 2m}$ (even)

$\sigma(1) \overset{\text{red}}{—} \sigma(2) \quad \sigma(3) \overset{\text{red}}{—} \sigma(4) \quad \dots\dots \quad \sigma(2m-1) \overset{\text{red}}{—} \sigma(2m)$

$\tau(1) \underset{\text{blue}}{—} \tau(2) \quad \tau(3) \underset{\text{blue}}{—} \tau(4) \qquad\qquad \tau(2m-1) \underset{\text{blue}}{—} \tau(2m)$

The vertices of my graph G: $\{1, 2, \dots, n = 2m\}$

edges? → some red, some blue.

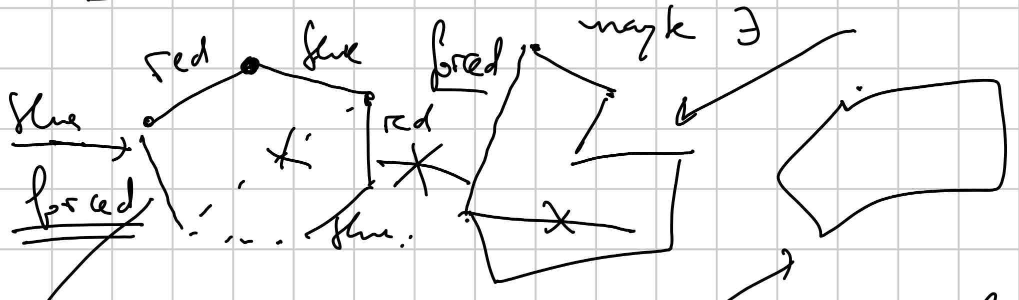
Structure of G?. $\deg(v) = 2$ (1 red + 1 blue).

G is a 2-regular graph.

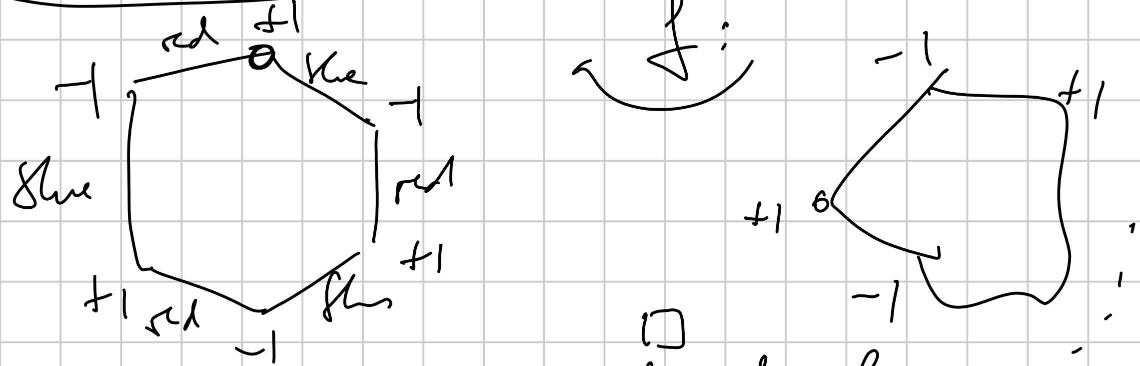
What do we know about 2-regular graphs ?

$\Updownarrow$

A union of disjoint cycles



Colors come into play $\Longrightarrow$ length of the cycle is $\boxed{\text{even}}$.



$\square$ Proved for n even.

Kill the subcase n odd !

$= 2m+1$ ~~reduce to $\boxed{2m}$ (ambiguous)~~ true

"enlarge"

true $\boxed{2m+2}$

$\downarrow$

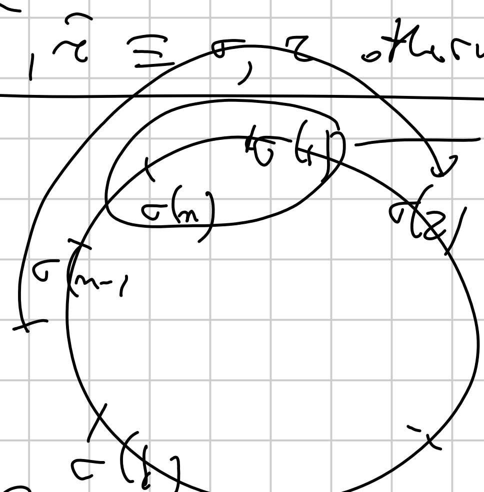
f found

Need $\tilde{\sigma}, \tilde{\tau} \in \mathcal{S}_{2m+2}$

$\tilde{\sigma}(2m+1) = \tilde{\tau}(2m+1) = 2m+2$ (fix $2m+2$)

$\tilde{\sigma}, \tilde{\tau} \equiv \sigma, \tau$ otherwise

One last word.



Become neighbours

Both σ, τ

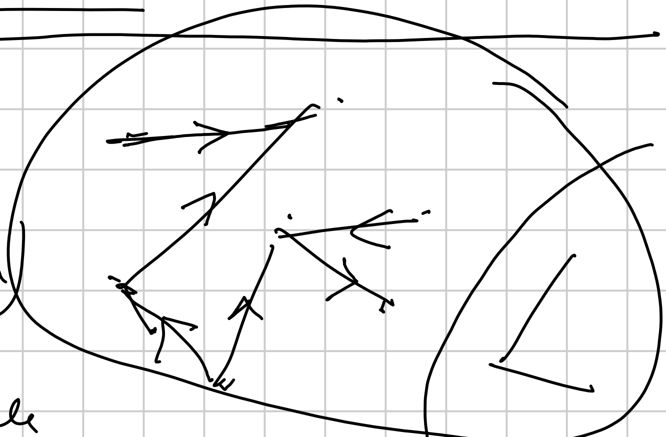
Q: Does the result stay true?

For n even — YES.

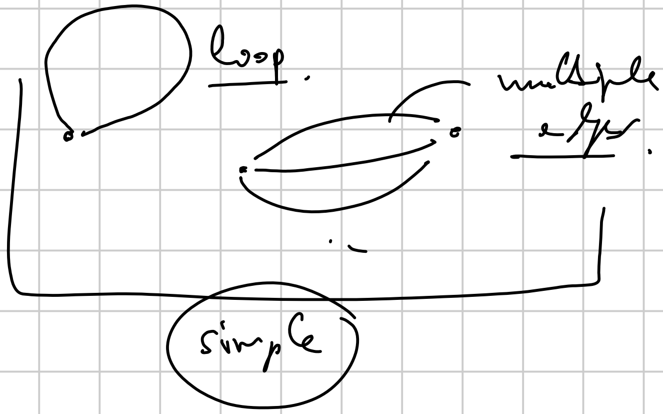
n odd — $\boxed{\text{No}}$ — Need a model.

$\sigma = id$. $\tau = \begin{pmatrix} 1 & 2 & \dots & n \\ & & & \end{pmatrix}$ all σ(evens), all σ(odds) $\pi = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ a_1 & a_2 & & & a_n \end{pmatrix}$

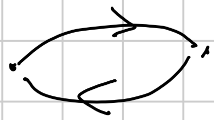
Problem. $G = (V, E)$



Oriented, directed, (digraph)

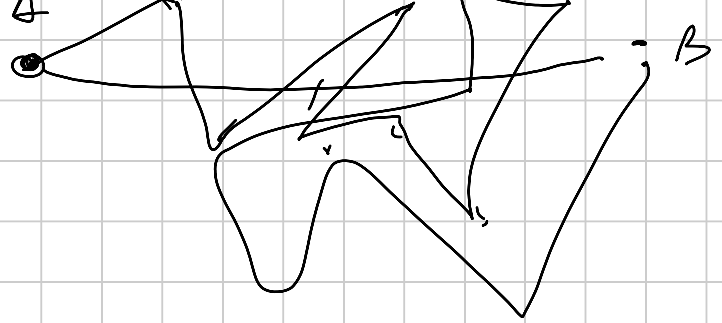


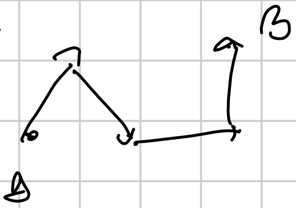
enclave



A complete digraph = tournament. (round-robin).

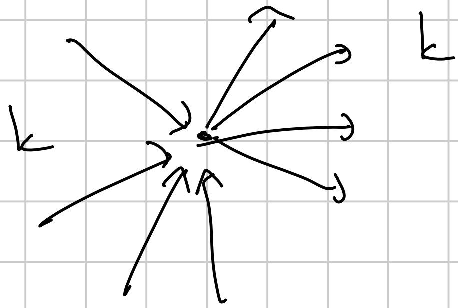
A graph G is called connected if any two vertices are linked by some path.



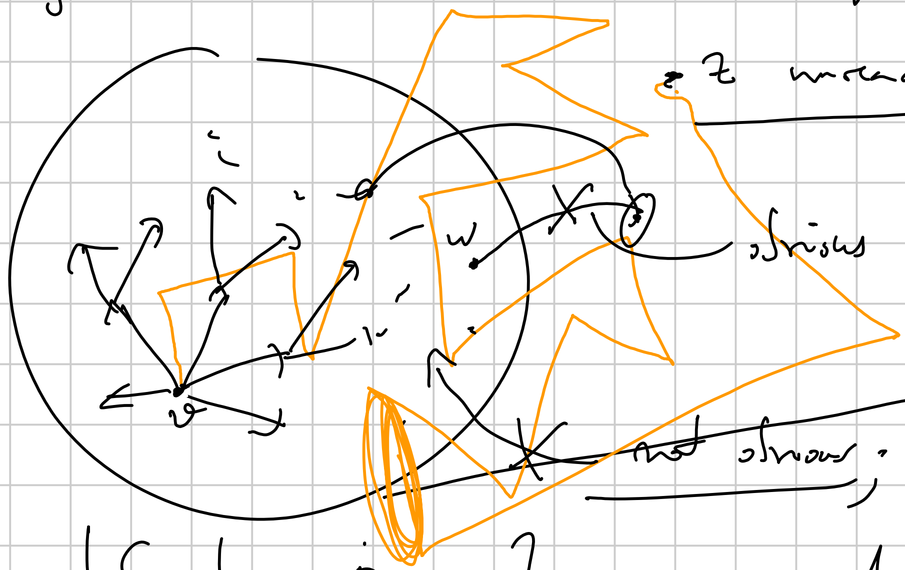


Definition For a digraph $\vec{G}$ — strongly connected if $\forall A, B \longrightarrow$ path.
— weakly connected but if when ignoring the orientation, it is

Problem. Any $\boxed{k\text{-regular digraph}}$ which is <u>weakly connected</u>, is also strongly connected.



Solution. Start v; build the set C_v, made of all w which are reachable from v.



z unreachable

obvious because def

contradiction

not obvious; (no orientation is possible).

$|C_v| = p$; ? arrows come out of $x \in C_v$? $= pk$

? arrows coming in $x \in C_v$? $\cancel{=}\ pk$ same.