SENIOR 2010 – DAN (COMBI + LINEAR ALGEBRA)

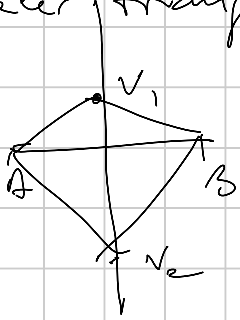
Titolo nota 08/09/2010

For starters IMO 1989? (double counting)
Given n, k. Given n points in the plane, no 3 collinear (in general position); for each point — there are $\geq k$ other points situated at a same distance from that point.
How large (find a bound) can k be?

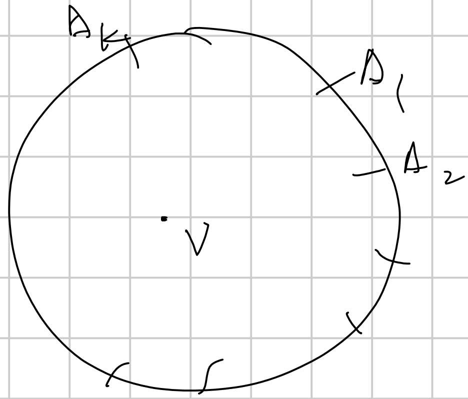
<u>Solution</u> Count all isosceles triangles.

$\{\underbrace{V}_{2^\circ}, \underbrace{\{A, B\}}_{1^\circ}\}$

1° $\binom{n}{2}$; $N \leq n(n-1)$



2° n;



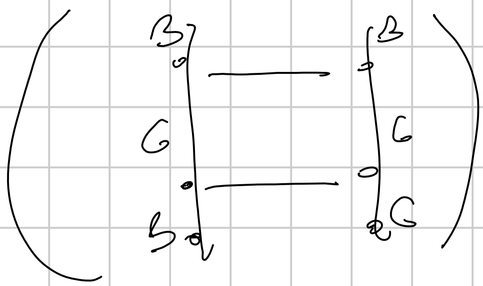
Any A_i, A_j

$N \geq n\frac{k(k-1)}{2}$

□

<u>Problem 2</u> (China 2005?).

We have a $2n \times n$ array. Boys & girls.



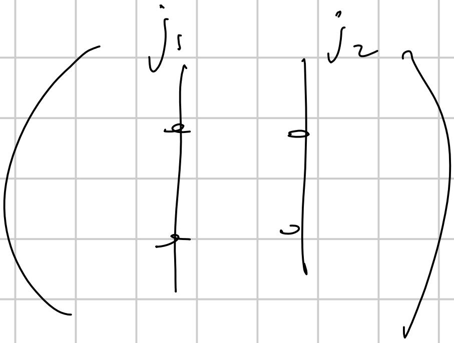
Any two columns, There are at least as many differences in gender (by row) as there are coincidences of gender.

To establish a bound for B boys!

$i \left(\quad b_i \qquad g_i = n - b_i \quad \right)$

N = total # of differences of gender

$$N = \sum_{i=1}^{2m} b_i g_i \quad \sum_{i=1}^{2m} d_i(m-d_i) \quad \ldots$$



$$N \geq n\binom{m}{2} = n\frac{m(m-1)}{2}$$

$$\sum_{i=1}^{2m} d_i = B \qquad f(x) = x(m-x)$$

$$\sum_{i=1}^{2m} d_i(m-d_i) \leq 2m\,\frac{\sum_{i=1}^{2m} d_i}{2m}\left(m - \frac{\sum d_i}{2m}\right) =$$

$$= \cancel{2m}\,\frac{B}{\cancel{2m}}\left(\frac{2mn - B}{2m}\right)$$

$$n\frac{m(m-1)}{\cancel{2}} \leq N \leq B\cdot\frac{2mn-B}{\cancel{2m}}$$

$$B^2 - 2mnB + n^2m(m-1) \leq 0 \qquad \Delta = \cancel{m^2n^2} - \cancel{m^2n^2} + n^2m$$

$$mn - n\sqrt{m} \leq \underline{B \leq mn + n\sqrt{m}}$$

$2n = 22$, $m = 75$ $\qquad \underline{mn - n\sqrt{m}} \approx$

ELEMENTS OF LINEAR ALGEBRA
(VECTOR)

K "body" of numbers, $+$ (0), $\cdot$ (1)

field ($\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}_p$).
of scalars.

V – set of vectors. $<$ add, $\mathbf{0}$, $-v$; multiplication: scalar $\cdot$ vector.

$$\lambda(u+v) = \lambda u + \lambda v \qquad 1\cdot v = v$$
$$\lambda(\mu u) = (\lambda\mu)u \qquad 0\cdot v = \mathbf{0}$$
$$\lambda\cdot\mathbf{0} = \mathbf{0}$$

$$\sum_{i=1}^{m} \lambda_i v_i = \text{linear combination}.$$

Can it be equal to $\mathbf{0}$? without all λ_i's
all v_i's
being null.

$l = 0$ but not trivial = relation

Consider those set of vectors where there are no relations. Such a set S linearly independent vectors. $0 \notin S$.

Take any l.c. $\sum_{i=1}^{n} \lambda_i v_i = u$, $v_i \in S$

unique

$\cancel{\sum_{j=1}^{m} \mu_j w_j, \; w_j \in S}$

Basis. V = lin. ind set B such that any $u \in V$ can be represented as $\sum_i^n \lambda_i v_i$, $v_i \in B$.

Finite $B = \{b_1, b_2, \dots b_d\}$

$$K^d = \underbrace{K \times K \times \dots \times K}_{d} = \{(x_1, x_2, \dots x_d) \mid x_i \in K\}$$

k-plets

$0 = (0, 0, \dots, 0)$

$-v = (-x_1, -x_2, \dots -x_d)$

$\lambda v = (\lambda x_1, \lambda x_2, \dots, \lambda x_d)$

Canonical basis: $e_1 = (1, 0, 0, \dots 0)$
$e_2 = (0, 1, 0, \dots 0)$
$e_d = (0, 0, \dots 0, 1)$

$$\boxed{v = \sum_{i=1}^{d} x_i e_i}$$

$V \simeq K^d$
isomorphic.

$b_i \longleftrightarrow e_i$

Theorem. If there exists a finite basis B with d elements, then any other basis has also d elements.

Lemma (The Exchange Lemma).

If I have two bases B, B', and chose $b' \in B'$, there exists some $b \in B$, so that $(B \setminus \{b\}) \cup \{b'\}$ is a basis.

Proof. Because B basis, $b' = \sum_{i=1}^{m} \lambda_i b_i$; there must be $\lambda_i \neq 0$

$\lambda_i b_i = \sum_{j \neq i} \lambda_j b_j - b'$; divide by λ_i

$b_i = \ldots - - - -$

∎

B least elem

B' larger

made of vectors B'

j-th

$B' \setminus (\quad)$

$\dim V = \dim (K^d) = d = |B|$.

$u = (u_1, u_2, \ldots u_d)$
$v = (v_1, v_2, \ldots, v_d)$
$u \cdot v = \langle u, v \rangle \overset{def}{=} \sum_{i=1}^{d} u_i v_i \in K$

Example: All polynomials of degree $\leq d-1$ coefficients in K

Basis $= \{1, x, x^2, \ldots x^{d-1}\}$

$\mathbb{R}^2 =$ vectors

$\mathbb{C}$

$\mathbb{R}[x]$

$$\langle u_1+u_2, v\rangle = \langle u_1, v\rangle + \langle u_2, v\rangle$$

$$\langle \lambda u, v\rangle = \lambda \langle u, v\rangle$$

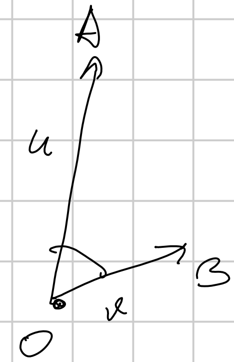
Dot product — called for a bilinear form

$$\langle u, u\rangle = \sum_{i=1}^{d} u_i^2 = \text{square of the Euclidean distance from } 0 \text{ to}$$

$$\|u\|^2$$

$K = \mathbb{R}$

What is $\dfrac{\langle u, v\rangle}{\|u\|\cdot\|v\|} = \cos(u, v)$



Prove Cauchy–Schwarz

(variational methods)

$$0 \le \|u + \lambda v\|^2 = \langle u+\lambda v, u+\lambda v\rangle =$$

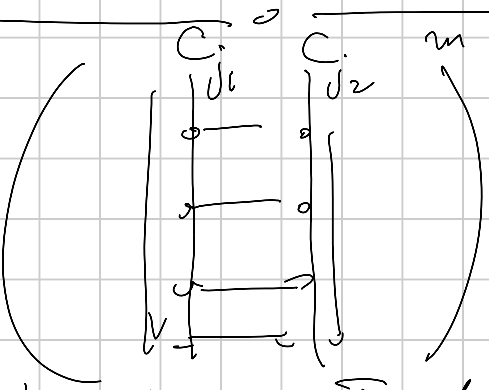
$$= \langle u,u\rangle + 2\lambda\langle u,v\rangle + \lambda^2\langle v,v\rangle =$$

$$= \|u\|^2 + 2\langle u,v\rangle\lambda + \|v\|^2\cdot\lambda^2$$

$\Delta \le 0$ $\quad$ $\langle u,v\rangle^2 - \|u\|^2\cdot\|v\|^2 \le 0$ $\quad$ C–S

2

Back Chinese problem



Boys — $+1$

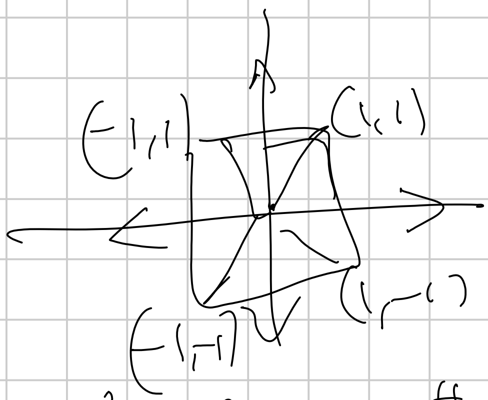
Girls — -1

$$\langle c_{j_1}, c_{j_2}\rangle = \sum_{i=1}^{2m} \ell(e_{ij_1})\cdot\ell(e_{ij_2}) \le 0.$$

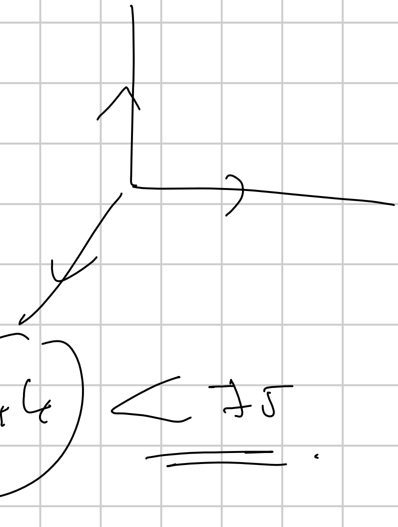
Translates the given problem.

$\mathbb{R}^{2m}$

Result. The max. number of vectors, which 2 by 2 make $\ge \pi/2$ angle is twice the dimension of the space ($4m$)



$2n = 22$, $\# \leq 44 < 75$



<u>Definition</u>. $\langle u, v \rangle = 0$, orthogonal $u \perp v$

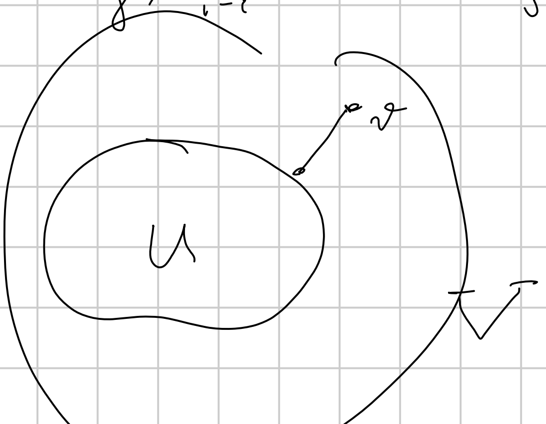
Assuming $\|u\| = 0 \iff u = 0$

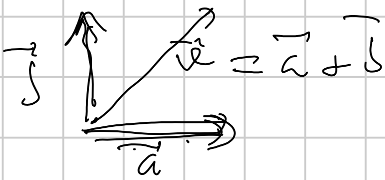
<u>Proposition</u>. A set of pairwise orthogonal vectors is linearly independent.

<u>Proof</u>. Assume $\sum_{i=1} \lambda_i v_i = 0 \quad \Big| \cdot v_j$

$$\langle v_j, \sum_{i=1} \lambda_i v_i \rangle = \lambda_j \|v_j\|^2 + \sum_{i \neq j} \lambda_i \underbrace{\langle v_j, v_i \rangle}_{0} = \lambda_j \|v_j\|^2$$

$$\lambda_j = 0, \quad \|v_j\| \neq 0$$

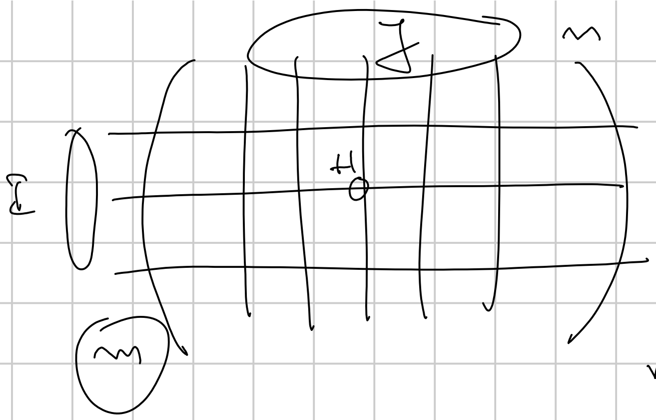




$\Big| \cdot$ prod

<u>Problem</u> (PUTNAM 2007?)

Given a $m \times n$ array whose entries are ± 1.
Look for the $+1$. Find some rows, columns I J
= <u>subarray made of just +1's</u>.



Prove $\boxed{|I|\cdot|J| \le \underline{n}}$

We know that for any two rows R_i, R_j: $\underline{\sum_{k=1}^{n} x_{ik}\cdot x_{jk} = 0}$

$\langle R_i, R_j \rangle = 0$; $\boxed{\underline{R_i \perp R_j}}$.

$\underline{n=1}$ $(1, 1, \text{— —}, 1)$

$n = 2m'$

$$n=2 \quad \begin{pmatrix} +1 & \cdots & +1 & +1 & \cdots & +1 \\ & & & & & \\ +1 & \cdots & +1 & -1 & \cdots & -1 \end{pmatrix}$$

<u>Solution</u>. My rows = vectors, n-dimensional.

Assume I, J, $R = \sum_{i\in I} R_i$;

$$\|R\|^2 = \langle R, R\rangle = \left\langle \sum_{i\in I} R_i, \sum_{i\in I} R_i \right\rangle =$$

$$= \sum_{i\in I} \|R_i\|^2 = \underline{n|I|}.$$

$$\|R\|^2 = \sum_{1}^{n} (\text{coordinate})^2 = \sum_{j\in J} (\ )^2 + \sum_{j\notin J} (\ )^2$$

$$\ge \sum_{j\in J} (\ )^2 = |I|^2\cdot|J|$$

$$n\cancel{|I|} \ge |I|^{\cancel{2}}\cdot|J| \to \boxed{\underline{|I|\cdot|J| \le n}}$$ ∎

<u>Problem</u>. Set $|A| = n$, $A_1, A_2, \ldots, A_m \subseteq A$

$|A_i| = \text{odd}$, $|A_i \cap A_j| = \text{even}$, $i \ne j$.

Prove $\underline{m \le n}$. Trivial example if we take all <u>singletons</u>.

$B \subseteq A$, $\chi_B : A \longrightarrow \{0,1\} = \mathbb{Z}_2$

$|A| = n$

$$\chi_B(a) = \begin{cases} 0 & \text{if } a \notin B \\ 1 & \text{if } a \in B \end{cases}$$

$$\chi_B : (\underset{1^{st}}{0/1}, \underset{2^{nd}}{0/1}, \ldots, \underset{n^{th}}{0/1}) \longrightarrow$$

$$B \longrightarrow \chi_B = (\quad) \in \mathbb{Z}_2^n$$

C

$B \Delta C = (B \cup C) \setminus (B \cap C)$

$(B \setminus C) \cup (C \setminus B)$

$$\chi_B + \chi_C = \chi_{\boxed{B \Delta C}}$$

A, $(\mathcal{P}(A), \Delta, \cap)$ Boolean ring.

$A_1, A_2, \ldots, A_m$ —— $\chi_{A_1}, \chi_{A_2}, \ldots, \chi_{A_m}$

$|A_i| = \text{odd} \iff \|\chi_{A_i}\|^2 = 1$

$|A_i \cap A_j| = \text{even} \iff \langle \chi_{A_i}, \chi_{A_j} \rangle = 0$

$\{\chi_{A_i}\}$ $\boxed{\text{linearly independent}}$ $\subset \mathbb{Z}_2^n$ — n-dimensional

$m \leq n$.

Problem. ("baby" Lindstrom). (China)

Again $|A| = n$, $A_1, A_2, \ldots, A_{n+1} \subseteq A$. non-empty

Prove there exist I, J, $I \cap J = \emptyset$, $\{1, 2, \ldots, n+1\}$

such that $\bigcup_{i \in I} A_i = \bigcup_{j \in J} A_j$.

Counterexample for n subsets: all singletons.

Solution. $A_i \to X_{A_i}$ vectors. $\boxed{n+1 \text{ such}}$.

$\cap$ $\mathbb{R}^n$ (same $\mathbb{Z}_2^n$)

Applying Lin. Alg. n-dimensional vector space

$\Longrightarrow$ There must exist a nontrivial relation between $n+1$ vectors: $\sum_{i=1}^{n+1} \lambda_i X_{A_i} = \underline{0}$, without all λ_i's being null. all not null

$\to$ (there must be ≥ 2 such not-null scalar λ_i's)

$$\Longrightarrow \sum_{i \in U} \lambda_i X_{A_i} = \sum_{j \in V} (-\lambda_j) X_{A_j}$$

$U, V \subset \{1, 2, \ldots, n+1\}$ disjoint.

positive λ_i negative λ_j

I J

$$\boxed{\bigcup_U A_i \stackrel{?}{=} \bigcup_V A_j}$$

D

Problem (Lindström). $A_1, A_2, \ldots, A_{n+2}$

one can find I, J sets of indexes, disjoint, s.t.

$$\bigcup_{i \in I} A_i = \bigcup_{j \in J} A_j, \text{ but also } \bigcap_{i \in I} A_i = \bigcap_{j \in J} A_j.$$

Solution. Counterexample: all singletons, and A.

2 approaches. 1st $A_i \to (X_{A_i}, X_{\overline{A_i}})$ 2n-vector

$$(\underbrace{x_1, x_2, \ldots x_n}, 1-x_1, 1-x_2, \ldots, 1-x_n) \in \mathbb{R}^{2n}$$

(set of vectors $S \subset V$; consider all $\sum_{i=1}^{t} \lambda_i x_i$, $x_i \in S$

$\langle S \rangle \subseteq V$ a linear space generated by S.)

$|A| = n$, all $B \subseteq A$ $\quad \left(\chi_B \mid \chi_{\bar{B}} \right)_{2n} \subset \mathbb{R}^{2n}$

generate a copy of $\mathbb{R}^n$

$\dim \langle S \rangle \geq n$
$= \ ?$

$\boxed{n+1}$

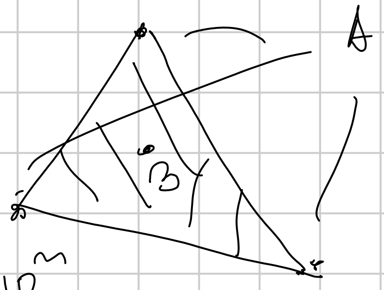
There must be a non-trivial relation. $\boxed{\sum_{i=1}^{n+2} \lambda_i (\chi_{A_i} \chi_{\bar{A}_i}) = 0}$

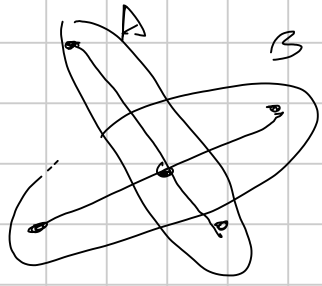
We separate $\underbrace{\text{positive } \lambda\text{'s}}_{U} = \underbrace{\text{negative } \lambda\text{'s}}_{\bar{U}}$

2nd — Helly's Theorem. If a finite collection of convex sets intersect 3 by 3, then they all have a common point.

$\mathbb{R}^n$, need to intersect in groups of $n+1$
$\Uparrow$
Radon's Theorem. If $n+2$ points in $\mathbb{R}^n$ then there is at least one way to partition them into two ~~non~~empty sets A, B such that $\boxed{\operatorname{conv}(A) \cap \operatorname{conv}(B) \neq \emptyset}$

$\mathbb{R}^2$: $\boxed{n+2 = 4}$:





$\in \mathbb{R}^n$

χ_{A_i} $\quad 1, 2, \dots, n+2$ can be partitioned. A, B

$x \in \operatorname{conv}(A) \cap \operatorname{conv}(B) \neq \emptyset$.

$$\sum_{i \in A} \lambda_i x_{A_i} = \sum_{j \in B} \mu_j x_{A_j}$$

I $\qquad\qquad\qquad\qquad\qquad$ J

x writes in conv(A) $\qquad\qquad$ x writes conv.(B)

affine combination.
(sum scalars is 1)

→ not just linear combinations.
they are convex combinations

scalars $\begin{cases} \geq 0 \\ \text{their sum} = 1 \end{cases}$

Memento → How many points (N) (no 3 collinear), can one have in the plane, such that for any three → at least one distance equal to 1

$N \geq 6$

$\boxed{N = 7}$ → find model
→ more & cannot (B)