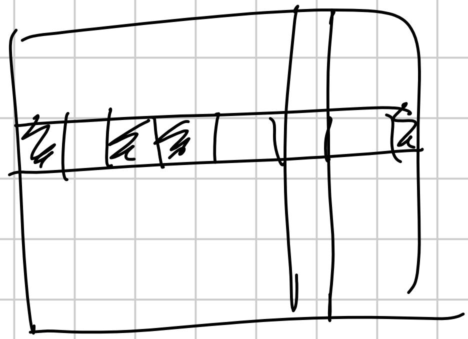
ADVANCED - 2010 - COMBINATORICS MISCELLANEA

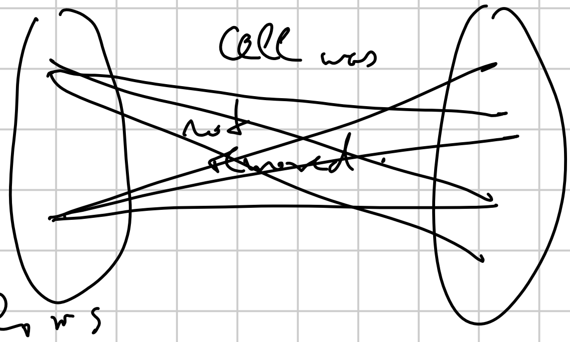
Titolo nota 10/09/2010

Problem 1. A $2n \times 2n$ chessboard; remove at will cells, but not more than n on any row or column.

Prove that one can still place $2n$ rooks so that they do not attack each other.



Solution. If no cells removed $(2n)!$ ways.



Answer = perfect matching

Marriage Theorem. Bipartite graph $|A|=|B|$, if for any $S \subseteq A$, $|S| \leq |\Gamma(S)|$.

$\Rightarrow$ yes, there is a perfect matching.

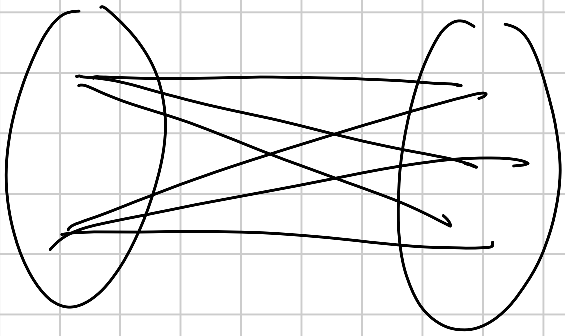
If $|S| \leq n$, ok: even one row $R \in S$
$|\Gamma(R)| \geq n$

If $|S| > n$, $\underset{\substack{\shortparallel\\ B}}{|\Gamma(S)|} = 2n$

By M.T. $\longrightarrow$ Perfect matching can be done.

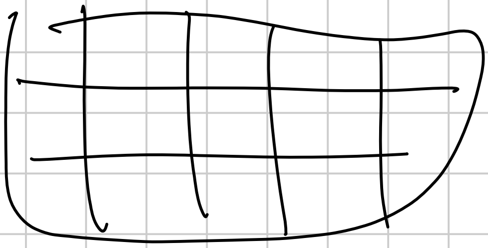
Particular case

K-regular



A

B



Set (finite) = mk elements.

π partition into m classes of k elements each.

π' —— " ——

Find m elements which represent both partitions.

G group, H subgroup

$$G = \bigsqcup_{i=1}^{\text{index}} \underbrace{Hx_i}_{z_i} = \bigsqcup_{i=1}^{\text{index}} \underbrace{y_i H}_{z_i}$$

Problem 2. (AMM) Island A, n families.

Ministry Agriculture : partitioned A into n equal

—"— Hunting : —— " ——

Ministry Family : we should allocate to each family one each so that they intersect

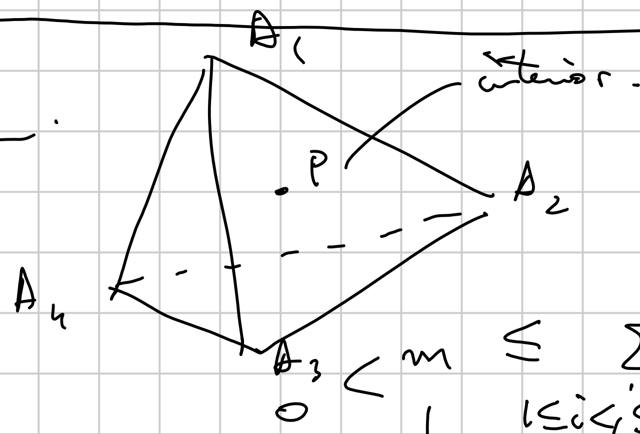
They succeeded!

Ministry Religion : declared a miracle!

a) Maths. Institute : $\longrightarrow$ refuted: it could allways be done!

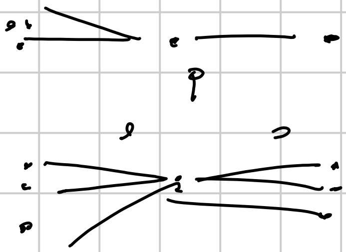
b) Find the fraction λ of area A inland, so that by a clever allocation, each family receives a $|C_i \cap H_i| \geq \lambda$.

GEOMETRY



$$m \le \sum_{1 \le i < j \le 4} \angle A_i P A_j \le M$$

Claim: $m = 3\pi$, $M = 4\pi$ (6π)



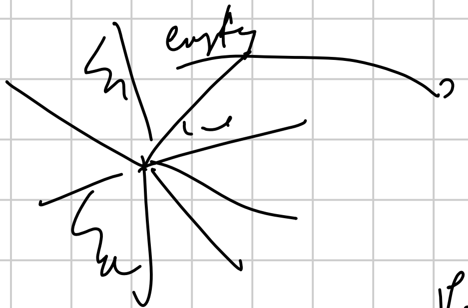
$\Sigma \approx 3\pi + 3 \cdot 0 = 3\pi$

$\Sigma \approx 2 \cdot 0 + 4 \cdot \pi = 4\pi.$

Convex polyhedron

Σ angles $< 2\pi$

"Cut" along edges — put on a plane



$2\pi - \Sigma$ angles $=$ "defect"

Σ defects $=$? $\boxed{4\pi}$.

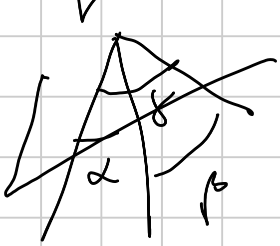
Polyhedron

$\Uparrow$

Euler: $V + F = E + 2$.

Proof. $\sum_{\text{around a vertex}}$ angles $< 2\pi$

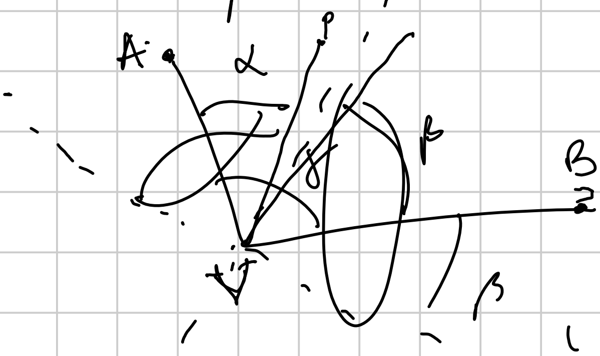
show **Lemma**. Vertex of a ~~tetr~~tetrahedron.



α, β, γ have the "triangular" ineq. property

$$\begin{cases} \alpha+\beta \geq \gamma \\ \beta+\gamma \geq \alpha \\ \gamma+\alpha \geq \beta \end{cases}$$

In the plane draw the layout of γ:

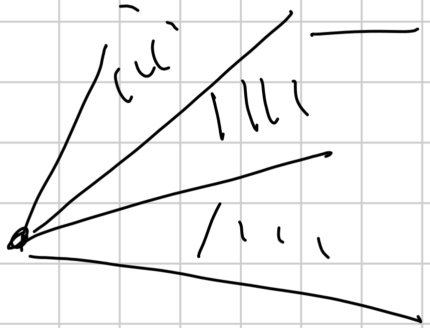


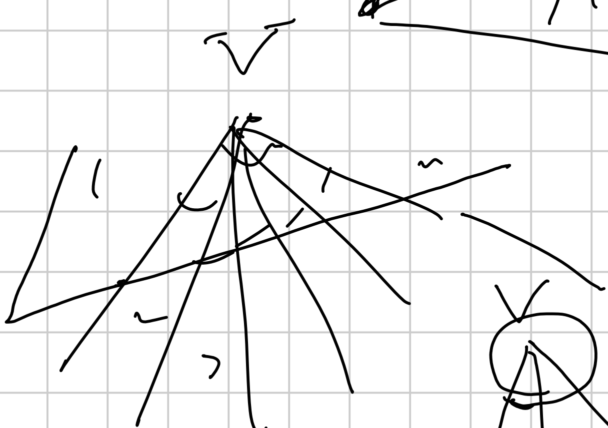
Assume $\alpha+\beta < \gamma$.

Locus in $\mathbb{R}^3$ of points P s.t. $\measuredangle PVA = \alpha$?

Q

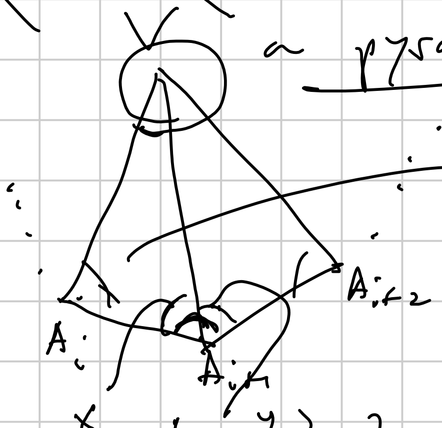
$= \beta$





$\sum_{i=1}^{n} \alpha_i$

a pyramid.



π

$\sum_{\text{along all lateral faces}} = n\pi$

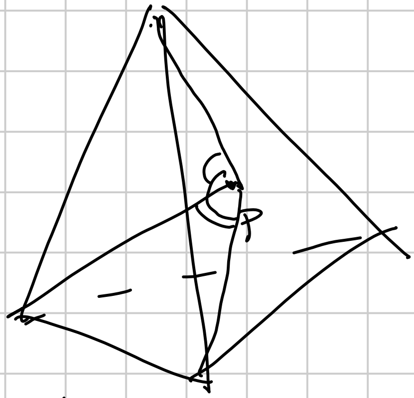
$x + y \geq z$

$\sum z = (n-2)\pi$

$$\sum^{6} \measuredangle A_i P A_j \leq 4\pi$$

$$\sum^{3} \leq 2\pi$$

for all groups of 3 out of the 4 vertices



Four times $\cancel{2\sum^{6}} \leq \frac{\cancel{8}\pi}{4}$

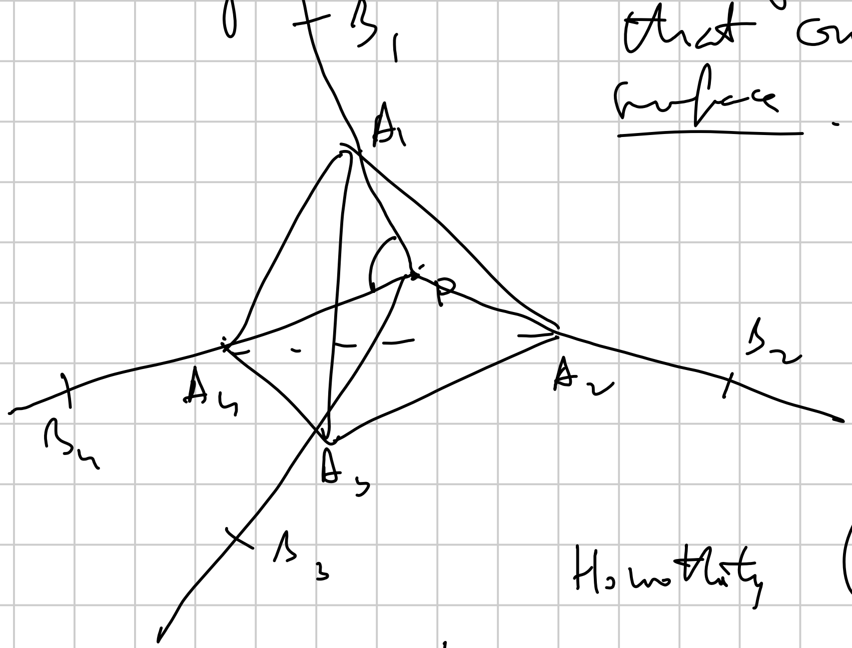
$$3 \leq \sum^{6} \leq \frac{\widehat{A_1PA_3} + \widehat{A_3PA_4} + \widehat{A_4PA_2} + \widehat{A_2PA_1}}{2\pi}$$

$(1,3,4,2)$

Lateral. Why is this happening (is it?).

One way – you do a nice drawing on a sphere.

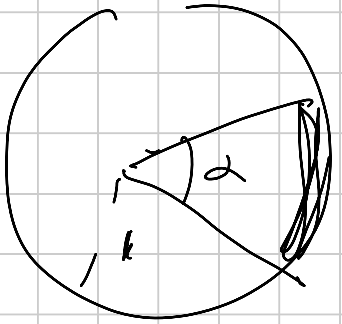
"geodesic" = "arc" of minimal length that connects 2 points on a surface.

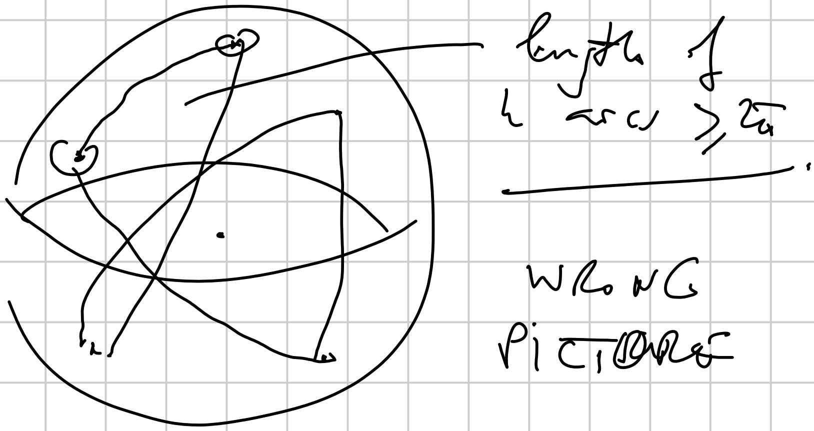


Can take B_i such that PB_i = equal. P center of the sphere $\triangle$ $B_1B_2B_3B_4$ inscribed.

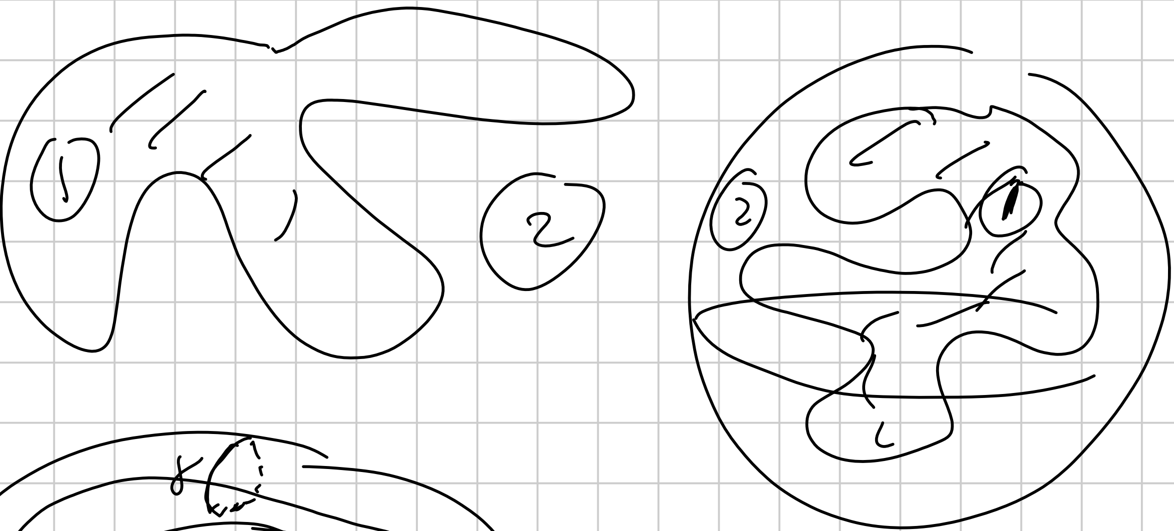
Homothety $= 1$.

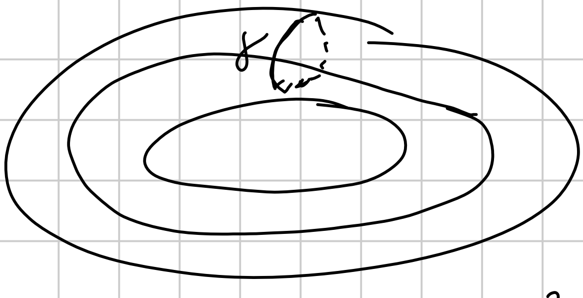
The advantage: measure of $\measuredangle$ (radians) = length of the arc great circle is "subtended".





Combi fact 1. If on surface of sphere of radius 1, curve γ simple (does not autointersect), closed, rectifiable (its length can be calculated). And if such a curve, divides surface of sphere into 2 regions, equal area, Then length $\lambda(\gamma) \geq 2\pi$.





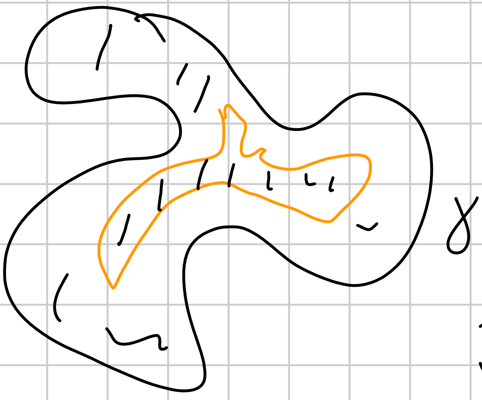
Torus (doughnut)

Proof. $\varphi : S^2 \longrightarrow S^2$

$p \longrightarrow p' =$ antipode

$\varphi(\gamma) =$ curve. γ blue

$\varphi(\gamma)$ red



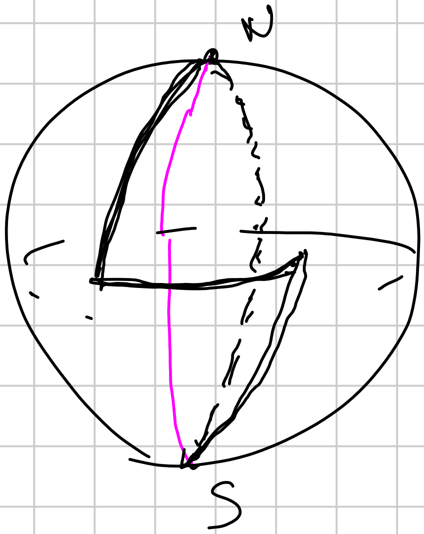
$|\gamma \cap \varphi(\gamma)| \geq \underline{2 \text{ (pairs)}}$.
P

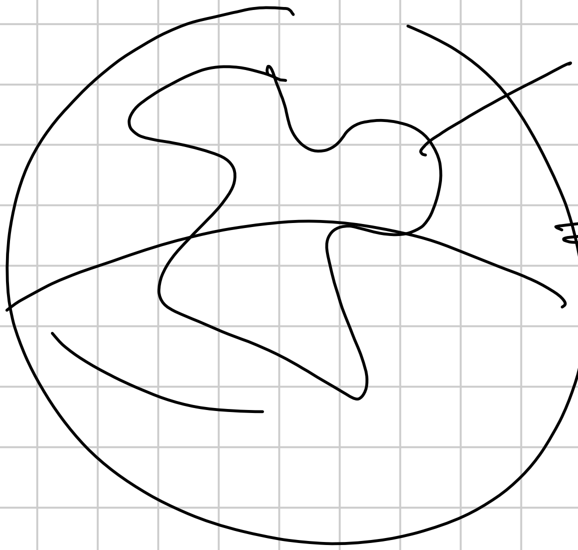
$p, p' \in \gamma$ $\quad (\varphi(\gamma))$

$\lambda(\gamma) = \widehat{pp'} + \widehat{p'p} \geq \frac{1}{2}$ circle $+ \frac{1}{2}$

$=$ full great circle $= \underline{2\pi}$

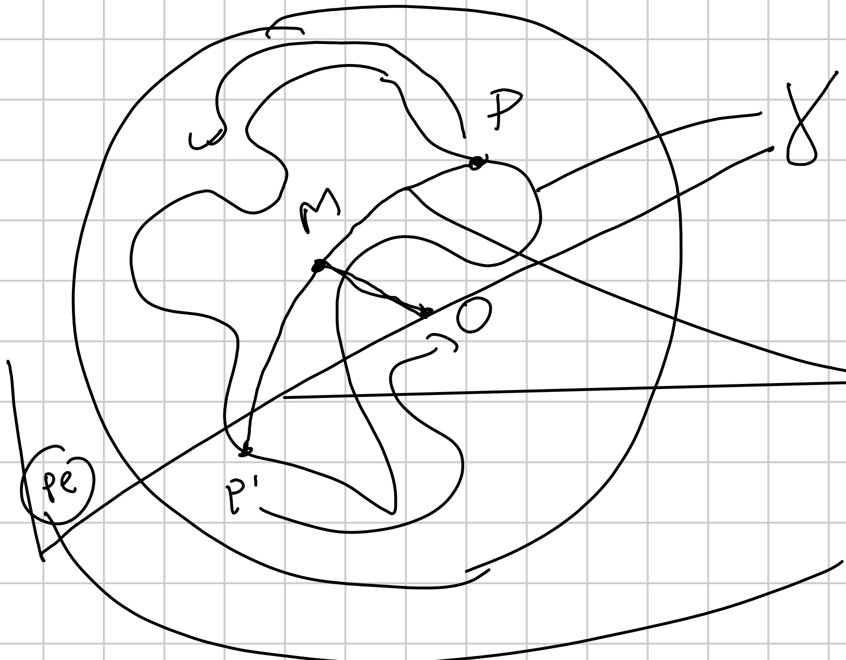
□





$\underline{\lambda(\gamma) < 2\pi}$

$\gamma \subset$ Hemisphere.



$\exists!\ p'$

$\lambda(pp') = \lambda(p'p)$

~~arc of great circle pp'~~

plane $\perp$ OM
"
middle

$\gamma \subset$ Hemispl.

Assume not.

Either pp' $\quad p'p \cap$ "equator"

equat $\cap$? pp' → symmetric with respect to OM

$\times$ → arc $\subset S$

same length $= \frac{1}{2}\lambda(\gamma)$

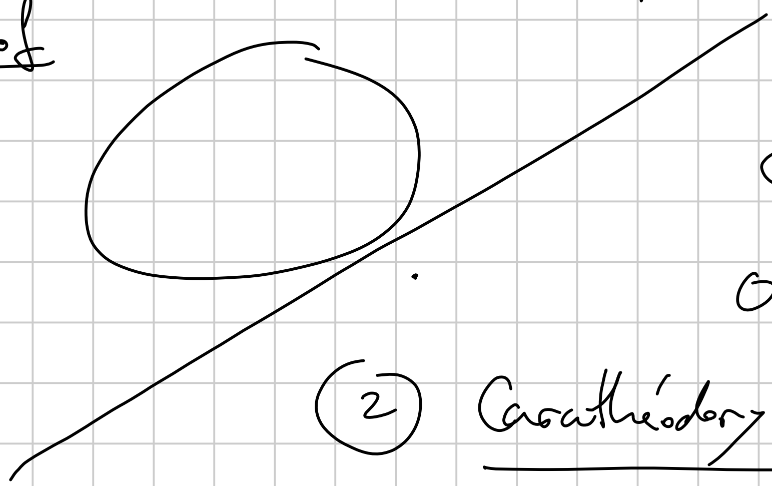
$x' \cup$ also on equator (opposite)

Instead looking at γ : $\underbrace{pp' \cup \text{symmetric arc}}_{= \lambda(\gamma)}$

contains x, x'

<u>My solution</u> : ① The fact that a set $S \subset S^2$ is not contained in ANY hemisphere $\iff \operatorname{Conv}(S) \ni 0$.

<u>Proof</u>



$0 \notin \operatorname{Conv}(S)$

$0 \in \operatorname{Conv}(S)$

② <u>Carathéodory Theorem</u> $\in \gamma$

There must be some $A_1, A_2, A_3, A_4 \ni 0$

③

Monty Python : And now, on a different ...

Problems with hats, names ...

①

Establish an equivalence relation among infinite binary "words"

wwrwwrrrw . . . _ _ _ _ _

$X \sim Y \overset{def}{\iff}$ they differ only in finitely many positions.

Agree on a representative $X \in \hat{X}$ (Axiom of choice)

Each one → write ? the color on the position that he occupies.

Finite variations . $1 \to n$

1, 2, n → as they write down a color, they also pronounce it.

The n persons again not allowed to speak

But instead

(n) people, (n) colors of hats.

Strategy 1st give themselves some number $1 - n$

2nd give the colors some code $1 - n$

Each will write : $\left[\left(\sum \text{colors he sees} \right) - \text{his position} \right] (\text{mod } n)$

$= \sum_{\text{colors}}^{\text{all}} -$ the color of that person hat

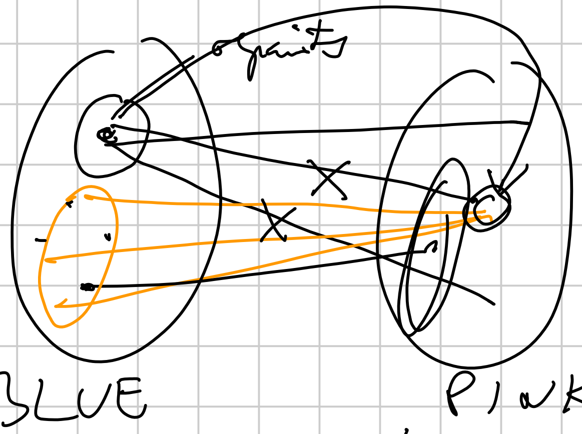
constant

In a forest there n dwarves < blue / pink house

Each one, in turn, visits all his friends.
If strictly > 1/2 his friends have a different color than his → he repaints.

Show, after some time → there is no need to repaint.

Sikinia.



The number of edges between B/P is ↓ strictly

The (2n) dwarves — game :

In cabin there are 2n boxes, each containing the name of one dwarf.

One by one, each enters cabin, allowed to open ≤ n boxes → finding his name.

They win if ALL find their names.
What strategy to maximize the chance to win?

From the "idiot" → $\left(\frac{1}{2}\right)^{2n}$ chance

a $\frac{1}{n}$ chance by improvement

Sketch : k → opens box k — if D
if not opens box ←

$$\sigma = (1, \sigma(1), \sigma(\sigma(1)) \ldots)(3, \sigma(3) \ldots)(k) =$$

$$\Rightarrow \sigma = \prod_{i=1}^{s} \gamma_i \,, \quad \sum \lambda(\gamma_i) = 2n .$$

They win game if $\lambda(\gamma_i) \le n, \ \forall i$.

Estimate $\dfrac{\#\sigma \text{ with cycles} \le n}{\#\text{ perm} = (2n)!}$.

Count N of σ which do have cycles of $\lambda > n$

$n+1$
$n+2$
$\vdots$
$2n$

Let's count $\# N_k$ of σ with a cycle $\lambda = n+k$, $k = 1, 2, \ldots, n$

$(\quad)_{n+k}$ rest arbitrary $(n-k)!$.

$$\binom{2n}{n+k} \cdot (n+k-1)! \cdot (n-k)! = \frac{2n!}{n+k}$$

$$\bar{p} = \frac{\bar{N}}{(2n)!} = \frac{1}{n+1} + \frac{1}{n+2} \ldots \frac{1}{2n} \approx \ln 2$$

$$p = 1 - \ln 2 = 0.7 \approx 30\%$$

$\boxed{\text{LAST}} \rightarrow$ (Martin GARDNER)

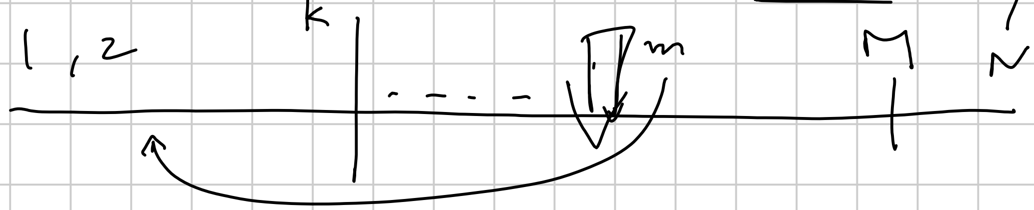
You are given N folded papers, each a real number, distinct.

You pick the $\boxed{\text{largest}}$

The "idiot" — pick one $1/N$

the only possible strategy is:

— pick $1 \leq k \leq N-1$
— open k papers — put them aside.
— start opening, until hit one larger than those rejected



$k \longmapsto P_k$ = probability of winning.

find which k max

$k \simeq \frac{N}{e}$, $P_k \simeq \frac{1}{e}$.

1 . . . 2 . . 3, SCAPPATE!!!! XD

MA MENO MALE CHE CI SONO I CANNONI

"PUÒ BACIARE LA SPOSA!"

"MA È UN LUPO!!!"

"NO! PRENDIAMO X, E MÒ NON LA FAI LA PROIETTIVITÀ CHE MANDA P ALL'INFINITO!