A2 - base Disuguaglianze

Titolo nota 09/09/2010

no $\mathbb{C}$: $\mathbb{R}, \mathbb{Q}, \mathbb{Z}$ no $\mathbb{Z}/_{n\mathbb{Z}}$

$x^2 \geq 0$ $\quad \frac{a^2+b^2}{2} \geq ab$ $\quad (a-b)^2 \geq 0$

x^2+x+1, x^2-x+1 sempre positivi

$x^2+\lambda x+1 \geq 0$ quando è sempre ≥ 0?

per $|\lambda| \leq 2$

$x^2+1 \geq |\lambda x|$

$$\frac{x^2+1}{2} \geq x$$

$a^2+b^2+c^2 \geq ab+bc+ca$

$(a+b-c)^2 = a^2+b^2+c^2-2ac-2bc+2ab$

$$\frac{a^2+b^2}{2} \geq ab$$

$$\frac{b^2+c^2}{2} \geq bc$$

$$\frac{a^2+c^2}{2} \geq ca$$

$a^2+b^2+c^2+$ $\quad (a+b+c)^2 \geq 0$

$a^2+b^2+c^2+2ab+2bc+2ca \geq 0$

$a^2+b^2+c^2 \geq ab+bc+ca$

$x > 0$

$$x + \frac{1}{x} \geq 2 \qquad \frac{x^2 + 1}{2} \geq \cancel{2}x$$

Riarrangiamento

$$a_1, a_2, \ldots, a_n \xrightarrow{\sigma} a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(n)}$$

$$\sigma: \quad \begin{array}{c|c} 1 & \sigma(1) \\ 2 & \sigma(2) \\ 3 & \vdots \\ \vdots & \vdots \\ n & \sigma(n) \end{array} \qquad \text{bigettiva}$$

Supponiamo di avere $a_1 \geq a_2 \geq \cdots \geq a_n$

e $b_1 \geq b_2 \geq \cdots \geq b_n$ e una permutazione σ

$$\sum_{i=1}^{n} a_i b_{n+1-i} \underset{2}{\leq} \sum_{i=1}^{n} a_i \cdot b_{\sigma(i)} \underset{1}{\leq} \sum_{i=1}^{n} a_i b_i \qquad \begin{array}{c|c} \multicolumn{2}{c}{id} \\ 1 & 1 \\ 2 & 2 \\ 3 & 3 \\ \vdots & \\ n & n \end{array}$$

Dim. 1: Supponiamo che $\sigma \neq id$

dia somma maggiore di tutte

$$\sum_{i=1}^{n} a_i b_{\sigma(i)} > \sum_{h=1} a_i b_i$$

Allora esisterà una coppia j_1, j_2

tale che $\sigma(j_1) > \sigma(j_2)$ $\quad j_1 < j_2$

$$\begin{array}{c|c} \multicolumn{2}{c}{\sigma} \\ j_1 & \sigma(j_1) \\ \wedge & \vee \\ j_2 & \sigma(j_2) \end{array}$$

$$a_1 b_{\sigma(1)} + \cdots + a_{j_1} b_{\sigma(j_1)} + \cdots + a_{j_2} b_{\sigma(j_2)} + \cdots$$

$$a_1 b_{\sigma(1)} + \cdots + a_{j_1} b_{\sigma(j_2)} + \cdots + a_{j_2} b_{\sigma(j_1)} + \cdots$$

$$a_{j_1}\underbrace{\left(b_{\sigma(j_1)} - b_{\sigma(j_2)}\right)}_{\leq 0} + a_{j_2}\left(b_{\sigma(j_2)} - b_{\sigma(j_1)}\right)$$

$$\sigma(j_1) > \sigma(j_2) \Rightarrow b_{\sigma(j_1)} \leq b_{\sigma(j_2)}$$

$$\underbrace{(a_{j_2} - a_{j_1})}_{\geq 0}\underbrace{\left(b_{\sigma(j_2)} - b_{\sigma(j_1)}\right)}_{\geq 0} \leq 0$$

Ma se era il massimo, <0 è assurdo

$=0 \Rightarrow$
- $a_{j_1} = a_{j_2}$ e anche quelli tra j_1 e j_2
- $b_{\sigma(j_2)} = b_{\sigma(j_1)}$ $\sigma(j_2)$ e $\sigma(j_1)$

Quindi
- o σ scambia solo elementi uguali
- o σ non è massimo

Se σ scambia solo elementi uguali allora

$$\sum_{i=1}^{n} a_i b_{\sigma(i)} = \sum_{i=1}^{n} a_i b_i .$$

2 Il caso del minimo è analogo.

$a, b, c > 0$ reali

$a^b b^c c^a \leq a^a b^b c^c$

$b \log a + c \log b + a \log c \leq a \log a + b \log b + c \log c$

a, b e c hanno lo stesso ordine di

$\log a$, $\log b$ e $\log c$

Quindi $a \log a + b \log b + c \log c \geq$ tutte le

somme con le altre permutazioni, quindi

anche $\geq b \log a + c \log b + a \log c$

$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq a+b+c$ $\quad$ a^2 b^2 e c^2 sono

in ordine inverso rispetto a

$\frac{1}{a}$ $\frac{1}{b}$ $\frac{1}{c}$.

Allora $a^2 \cdot \frac{1}{a} + b^2 \cdot \frac{1}{b} + c^2 \cdot \frac{1}{c} \leq$ tutte le somme

tra cui.

$a_1, \ldots, a_n > 0$ reali

$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1} \geq n$

$a_1 \ldots a_n$ e $\frac{1}{a_1} \ldots \frac{1}{a_n}$ inversamente ordinati

tutte le somme $\geq a_1 \cdot \frac{1}{a_1} + a_2 \cdot \frac{1}{a_2} + \cdots = 1+1+\cdots 1$

$= n$

Disuguaglianza di Chebychev

$$a_1 \geq a_2 \geq \dots \geq a_n \qquad b_1 \geq b_2 \geq \dots \geq b_n$$

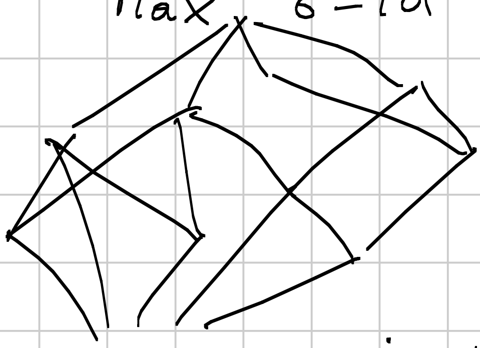
$$\frac{\sum a_i b_{n+1-i}}{n} \leq \frac{\sum_{i=1}^{n} a_i}{n} \cdot \frac{\sum_{i=1}^{n} b_i}{n} \leq \frac{\sum_{i=1}^{n} a_i b_i}{n}$$

Dim

$$\begin{array}{lll}
0 & a_1b_1 + a_2b_2 + \dots + a_nb_n & \leq a_1b_1 + a_2b_2 + \dots + a_nb_n \\
+1 & a_1b_2 + a_2b_3 + \dots + a_nb_1 & \leq a_1b_1 + a_2b_2 + \dots + a_nb_n \\
+2 & a_1b_3 + a_2b_4 + \dots + a_nb_2 & \leq \dots \\
\vdots & & \\
+n-1 & a_1b_n + a_2b_1 + \quad + a_nb_{n-1} & \leq \dots
\end{array}$$

$$\frac{n\sum_{i=1}^{n} a_i b_{n+1-i}}{n^2} \leq \frac{\left(\sum_{i=1}^{n} a_i\right)}{n}\frac{\left(\sum_{i=1}^{n} b_i\right)}{n} \leq \frac{n\sum_{i=1}^{n} a_i b_i}{n^2}$$

Max $\sigma = id$



Min $\sigma = i \mapsto n+1-i$

$$\sigma \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 2 & 1 & 4 \end{array} \qquad \tau \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 1 & 2 & 4 \end{array}$$

σ cambia l'ordine di 3 e 4 $\qquad$ τ lo conserva

per il resto sono uguali

$$\sum_{i=1}^{5} a_i b_{\sigma(i)} \leq \sum_{i=1}^{5} a_i b_{\tau(i)}$$

- variabili solo positive ?
- quando vale $=$?
- è omogenea ?

$$\frac{a^2+b^2}{2} \geq ab \qquad a \to \lambda a \quad b \to \lambda b$$

$$\frac{\cancel{\lambda^2} a^2 + \cancel{\lambda^2} b^2}{2} \geq \cancel{\lambda^2} ab$$ omogenea di grado 2

Riarrang. $\to$ omog. grado 2

Dis. non omogenea vale con difficoltà su tutto $\mathbb{R}^+$

$$x^2 \geq x^3 \qquad 0 \leq x \leq 1$$

$$x^2 \leq x^3 \qquad x \geq 1$$

Chebychev omog. grado 2

Quando $=$? Se esiste $b_i \neq b_j$ prima o poi li scambio con uno "shift" della dim. di Chebychev $\Rightarrow$ tutti i b_i devono essere $=$ perché valga l'$=$ a destra. Ma anche a sinistra è la stessa cosa.

Media

$$\{a_1, \dots, a_n\} \longrightarrow m$$

$$\min\{a_i\} \leq m \leq \max\{a_i\}$$

$$a_i \longmapsto \lambda a_i \qquad m \longmapsto \lambda m$$

$$a_i \longmapsto a_i + k \qquad m \longmapsto m + k$$

AM $\dfrac{a_1 + a_2 + \dots + a_n}{n}$

GM $\sqrt[n]{a_1 \cdot a_2 \cdot \dots \cdot a_n}$ $\qquad a_i \geq 0$

Disuguaglianza AM-GM

$$\sqrt[n]{a_1 \dots a_n} \leq \frac{a_1 + a_2 + \dots a_n}{n} \qquad \mathbb{R}^+$$

$= \iff a_i$ tutti uguali

$$\frac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2} \qquad x_1 = \sqrt{a_1} \quad x_2 = \sqrt{a_2}$$

omog. grado 1

$$\frac{x_1^2 + x_2^2}{2} \geq x_1 x_2 \qquad \frac{a_1^2 + 2a_1a_2 + a_2^2}{4} \geq a_1 a_2$$

$$(a_1 - a_2)^2 \geq 0$$

$$\frac{a_1^2 + a_2^2}{\cancel{2}\ 4} \geq \frac{a_1 a_2}{2}$$

$$G = \sqrt[n]{a_1 \dots a_n} \qquad x_1 = \left(\frac{a_1}{G}\right)^{-1} \quad x_2 = \left(\frac{a_1 a_2}{G^2}\right)^{-1} \dots x_n = \left(\frac{a_1 a_2 \cdots a_n}{G^n}\right)$$

$$\frac{x_1}{x_2}+\frac{x_2}{x_3}+\cdots \geq n \qquad \frac{x_1}{x_2}=\cancel{\frac{\frac{a_1}{G}}{\frac{a_1a_2}{G^2}}}=\frac{G}{a_2}\quad \frac{a_2}{G}$$

$$\frac{x_2}{x_3}=\cancel{\frac{\frac{a_1a_2}{G^2}}{\frac{a_1a_2a_3}{G^3}}}=\frac{G}{a_3}\quad\cdots\quad \frac{a_3}{G}$$

Anzi voglio $\frac{x_1}{x_2}=\frac{a_2}{G}$ $\quad \frac{x_2}{x_3}=\frac{a_3}{G}\cdots$

$$\frac{x_1}{x_2}+\frac{x_2}{x_3}+\cdots = \frac{a_2}{G}+\frac{a_3}{G}+\cdots+\frac{a_1}{G}\geq n$$

$$\Rightarrow \frac{a_1+a_2+\cdots+a_n}{n}\geq G$$

Trovare il minimo di $x+2y+3z$ se $x^3y^2z=1$

$x>0 \quad y>0 \quad z>0$

$$\frac{x+2y+3z}{3}\geq \sqrt[3]{x\cdot 2y\cdot 3z}=\sqrt[3]{6xyz}$$

$$\frac{\frac{x}{3}+\frac{x}{3}+\frac{x}{3}+y+y+3z}{6}\geq \sqrt[6]{\frac{x^3}{\cancel{27}_{9}}\cdot y^2\cdot \cancel{3}z}=\sqrt[6]{\frac{1}{9}}\,\underbrace{\sqrt[6]{x^3y^2z}}_{1}$$

$$x+2y+3z\geq 6\cdot\sqrt[3]{\frac{1}{3}} \qquad \overset{?}{=} \qquad \frac{x}{3}=y=3z$$

$$x=9z \qquad y=3z \qquad 9^3 3^2 z^6=1 \qquad z^6=\frac{1}{9^3 3^2}$$

$$z=\frac{1}{3}\,\frac{1}{\sqrt[3]{3}}$$

$$(a+3b)(b+4c)(c+2a)\geq 60abc$$

$$\begin{array}{cccc} \geq & \geq & \geq & \\ 2\sqrt{3ab} & 2\sqrt{4bc} & 2\sqrt{2ac} & = 8\sqrt{24abc} \end{array}$$

$$\begin{array}{ccc} a+b+b+b & b+c+c+c+c & c+a+a \\ \geq & \geq & \geq \\ 4\sqrt[4]{ab^3} & 5\sqrt[5]{bc^4} & 3\sqrt[3]{ca^2} \end{array}$$

$$\frac{1}{4}+\frac{2}{3} \qquad \frac{4}{5}+\frac{1}{3}$$

$$b \quad \frac{4}{3}c \; \frac{4}{3}c \cdot \frac{4}{3}c \qquad c \; \frac{2}{3}a \; \frac{2}{3}a \; \frac{2}{3}a$$

$$4\sqrt[4]{ab^3} \qquad 4\sqrt[4]{\frac{64}{27}bc^3} \qquad 4\sqrt[4]{\frac{8}{27}ca^3}$$

$$64\sqrt[4]{a^4b^4c^4\frac{512}{729}}$$

$a=b=c$ $\qquad a+3b = x_1+x_2+\dots+x_n$

$$\overbrace{\frac{a}{15} \; \dots}^{15} + \overbrace{\frac{b}{15}}^{45} \qquad a+3b$$

$$\overbrace{\frac{b}{12}}^{12} \quad \overbrace{\frac{c}{12}}^{48} \qquad b+4c$$

$$\overbrace{\frac{a}{20}}^{40} \quad \overbrace{\frac{c}{20}}^{20} \qquad c+2a$$

$$\min\left\{x^2+y^4+z^6 \quad \text{su} \quad xyz=n\right\} = A_n$$

$$\overbrace{\frac{x^2}{6}}^{6} \quad \overbrace{\frac{y^4}{3}}^{3} \quad \overbrace{\frac{z^6}{2}}^{2}$$

$$\frac{x^2+y^4+z^6}{11} \geq \sqrt[11]{(xyz)^{12}\cdot\frac{1}{6^6\cdot3^3\cdot2^2}} = \sqrt[11]{\frac{1}{6^6 3^3 2^2}}\, n^{12/11}$$

$$\frac{z^6}{2}=t \qquad \frac{y^4}{3}=t \qquad \frac{x^2}{6}=t$$

Media quadratica: $\sqrt[2]{\dfrac{a_1^2+a_2^2+\dots+a_n^2}{n}} = QM$

Media armonica: $\dfrac{1}{\dfrac{\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}\dots+\frac{1}{a_n}}{n}} = HM$

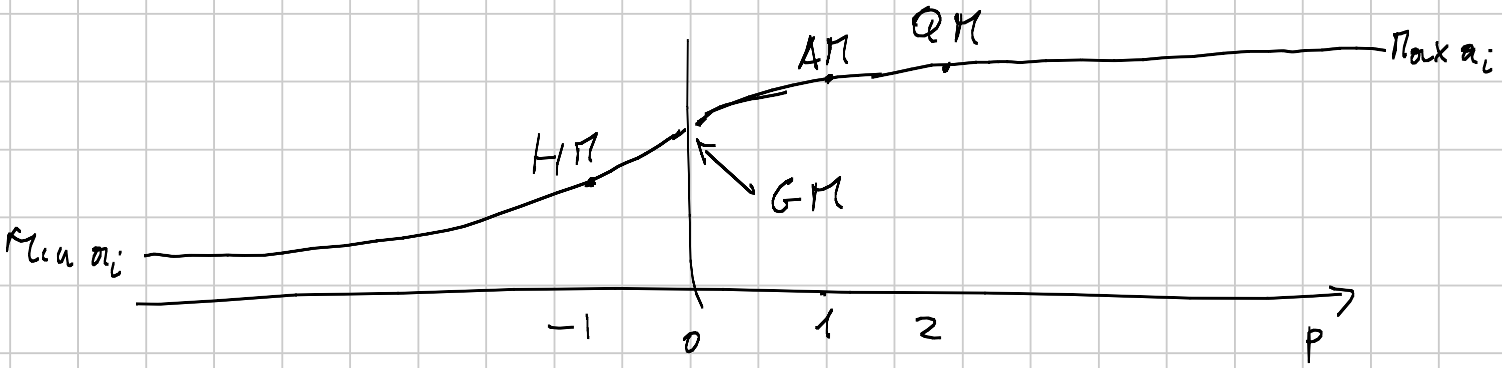
$$M_p = \left(\frac{a_1^p+a_2^p+\dots+a_n^p}{n}\right)^{1/p}$$

$p=2$ QM
$p=1$ AM
$p=-1$ HM

$p = 0$? se $p \leq q$ $M_p \leq M_q$

se $p < q$ e $M_p = M_q \iff a_i$ tutti uguali

se faccio un grafico al variare di p $a_1 \dots a_n$ fissati



HM - GM ?
$$\frac{n}{\frac{1}{a_1} + \dots + \frac{1}{a_n}} \leq \sqrt[n]{a_1 \cdots a_n}$$

$$b_i = \frac{1}{a_i} \qquad \frac{n}{b_1 + \dots + b_n} \leq \sqrt[n]{\frac{1}{b_1 \cdots b_n}}$$

$$\sqrt[n]{b_1 \cdots b_n} \leq \frac{b_1 + \dots + b_n}{n}$$

Dim sia $b_i = \frac{1}{a_i}$ per AM - GM

$$\sqrt[n]{b_1 \cdots b_n} \leq \frac{b_1 + \dots + b_n}{n}$$ Allora

$$\frac{n}{b_1 + \dots + b_n} \leq \sqrt[n]{\frac{1}{b_1 \cdots b_n}}$$ quindi sost.

$$\frac{n}{\frac{1}{a_1} + \dots + \frac{1}{a_n}} \leq \sqrt[n]{a_1 \cdots a_n}$$ cioè la tesi.

Disuguaglianza di Cauchy-Schwarz

$a_1 \dots a_n \qquad b_1 \dots b_n \qquad \in \mathbb{R}$

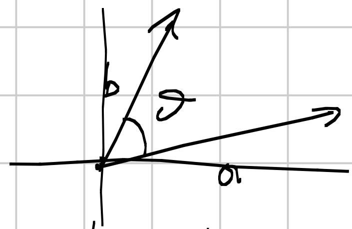
$$\sqrt{\sum_{i=1}^n a_i^2} \cdot \sqrt{\sum_{i=1}^n b_i^2} \geq \left| \sum_{i=1}^n a_i \cdot b_i \right|$$

$v = (a_1, a_2, \dots a_n) \qquad w = (b_1, \dots, b_n)$

$\|v\| = \sqrt{a_1^2 + a_2^2 + \dots a_n^2}$

$$\|v\| \cdot \|w\| \geq \left| \sum_{i=1}^n a_i b_i \right| =$$

prodotto scalare tra v e w



$a \cdot b = ab \cos\vartheta$

$\|v\| \geq 0$

$\|v + \lambda w\| \geq 0 \qquad \forall \lambda$

$$(a_1 + \lambda b_1)^2 + (a_2 + \lambda b_2)^2 + \dots + (a_n + \lambda b_n)^2 \geq 0$$

$$a_1^2 + a_2^2 + \dots + a_n^2 + 2(a_1 b_1 + a_2 b_2 + \dots + a_n b_n)\lambda + \qquad \forall \lambda$$
$$+ (b_1^2 + \dots + b_n^2)\lambda^2 \geq 0$$

$$(a_1 b_1 + \dots + a_n b_n)^2 - (a_1^2 + \dots + a_n^2)(b_1^2 + \dots + b_n^2) \leq 0$$

$$\left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right) \geq \left(\sum_{i=1}^n a_i b_i\right)^2$$

1) $\in \mathbb{R}$

2) $=$ se $a_i = t b_i \quad \forall i$

$a_1 \dots a_n \qquad b_1 = 1 \quad b_2 = 1 \dots b_n = 1$

$$\left(\sum_{i=1}^n a_i^2\right) \cdot n \geq \left(\sum a_i\right)^2 \qquad \sqrt{\frac{\sum_{i=1}^n a_i^2}{n}} \geq \frac{\sum_{i=1}^n a_i}{\cancel{\sqrt{n}}\; n}$$

QM $\qquad$ AM

$$a^2+b^2+c^2 \geq ab+bc+ca$$

$$v=(a,b,c) \qquad w=(b,c,a)$$

$$\sqrt{a^2+b^2+c^2}\cdot\sqrt{b^2+c^2+a^2} \geq |ab+bc+ca|.$$

Nesbitt: $\dfrac{a}{b+c}+\dfrac{b}{c+a}+\dfrac{c}{a+b} \geq \dfrac{3}{2}$

$a+b=y$
$b+c=y$
$c+a=z$

riarrang.

$$\left(\sqrt{\frac{a}{b+c}},\sqrt{\frac{b}{c+a}},\sqrt{\frac{c}{a+b}}\right)$$

$$\left(\sqrt{a}\sqrt{b+c},\sqrt{b}\sqrt{c+a}\ \sqrt{c}\sqrt{a+b}\right)$$

$$\left(\frac{a}{b+c}+\cdots\right)\left(2(ab+bc+ca)\right) \geq (a+b+c)^2$$

$$\frac{a}{b+c}+\cdots \geq \underbrace{\frac{a^2+b^2+c^2}{2(ab+bc+ca)}}_{\geq \frac{1}{2}}+\underbrace{\frac{2(ab+bc+ca)}{2(ab+bc+ca)}}_{=1}=\frac{3}{2}$$

≥ 1

Dis. raggruppamento o bunching o Muirhead.

$a_1 \cdots a_n \qquad \sum_{cyc} a_i^2 a_{i+1} \qquad \sum_{cyc}^{a,b,c} a^2b=a^2b+b^2c+c^2a$

$$\sum_{sym}^{a,b,c} a^2b = a^2b+b^2c+c^2a+ab^2+bc^2+ca^2$$

2, 1, 0	2 1 0	2 1 1	
a b c	a b c	a b c	a^2bc
b c a	a c b	a c b	a^2cb
c a b	b a c		
	b c a		
	c a b		
	c b a		

1 1 1
a b c
b a c

$$\sum_{sym} abc = 6abc$$

su $\mathbb{R}^+$

Le somme simmetriche più grandi sono quelle con gli esponenti più concentrati.

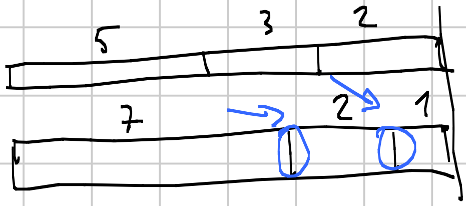
$\bar{k}$ $\quad k_1 \geq \dots \geq k_n$

$\bar{h}$ $\quad h_1 \geq \dots \geq h_n$

$\bar{k} \geq \bar{h}$ se

$k_1 \geq h_1$ e $k_1+k_2 \geq h_1+h_2$ e

$k_1+k_2+k_3 \geq h_1+h_2+h_3$ e.. $\sum k_i = \sum h_i$



$$\sum_{sym} x^4 y^2 z \geq \sum_{sym} x^3 y^2 z^2$$

$4 > 3$
$4+2 > 3+2$
$4+2+1 = 3+2+2$

$\mathbb{R}^+$

$\geq$

$=$ se e solo se tutte le variabili uguali

omog. di vari gradi

$$\sum_{sym} x^3 + \sum_{sym} xyz \geq 2 \sum_{sym} x^2 y \qquad \text{Schur}$$

$$a\underbrace{(a-b)(a-c)}_{\geq 0} + b\underbrace{(b-c)(b-a)}_{\leq 0} + c\underbrace{(c-b)(c-a)}_{\geq 0} \geq 0$$

$a \geq b \geq c$

$a(a-c) - b(b-c) \geq 0$? Ovviamente sì

$\mathbb{R}^+$

$\geq$

$=$ due var. $=$ e 1 è 0

$$a^k(a-b)(a-c) + b^k(b-c)(b-a) + c^k(c-b)(c-a) \geq 0.$$

$$\frac{1}{a^3+b^3+1}+\frac{1}{b^3+c^3+1}+\frac{1}{c^3+a^3+1}\leq 1$$

$$abc=1$$

$$\frac{1}{a^3+b^3+abc}+\frac{}{+abc}+\frac{}{+abc}\leq\frac{1}{abc}$$

$$\sum a^6b^3\geq\sum a^5b^2c^2$$