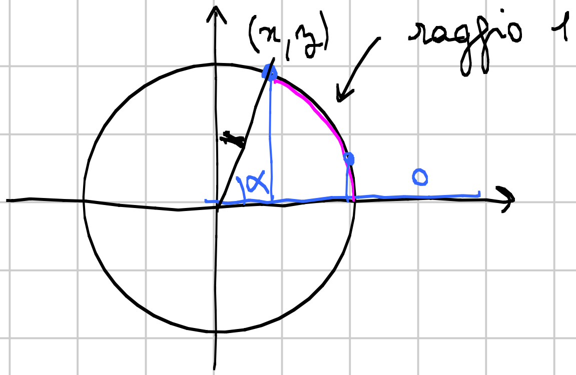
G 1 TRIGONOMETRIA

Morandin / Pernazza

- Sintetico ♡
- Analitico
- Trigonometria
- Vettori
- Complessi

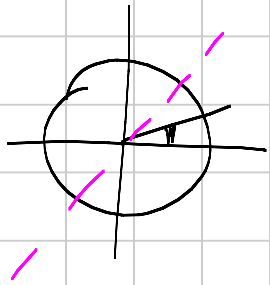
} fare i conti = algebrizzare

GONIOMETRIA



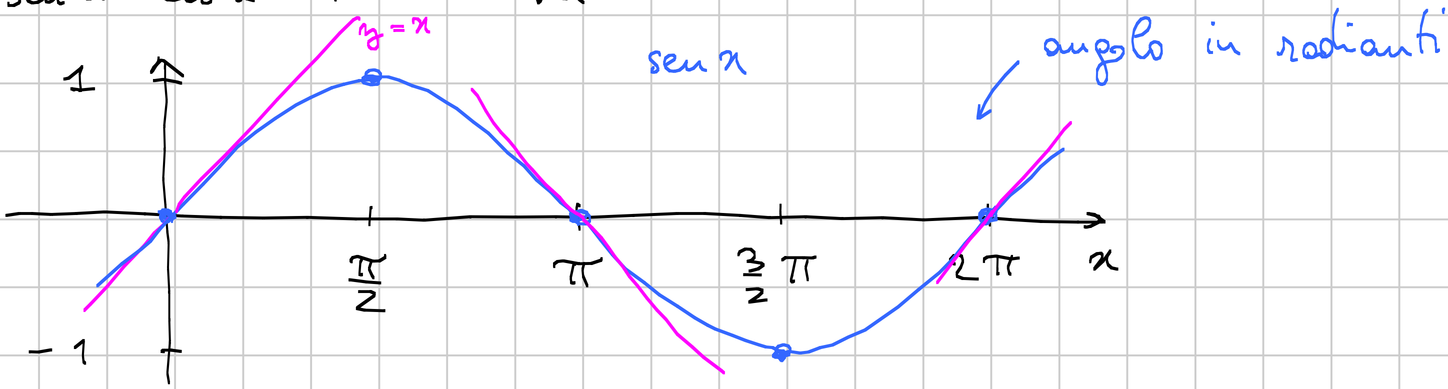
$$x^2 + y^2 = 1$$

$$\begin{cases} x = \cos\alpha \\ y = \sin\alpha \end{cases}$$



$\operatorname{sen}\alpha, \cos\alpha \in [-1, 1] \quad \forall \alpha$

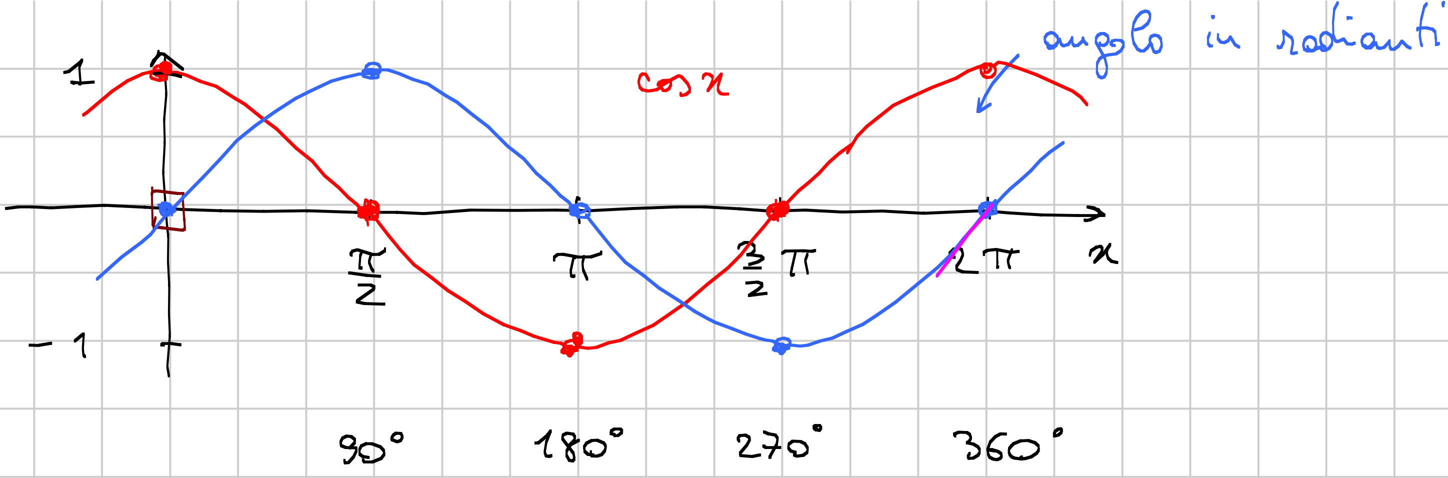
$\operatorname{sen}^2\alpha + \cos^2\alpha = 1 \quad \forall \alpha$



90° 180° 270° 360°

$370° \approx 10°$

$x \geq 0 \quad \operatorname{sen} x \leq x$ solo in radianti



- Periodicità e simmetrie

- $\text{sen}(\alpha + 2\pi) = \text{sen}\,\alpha$, $\cos(\alpha + 2\pi) = \cos\alpha = \cos(\alpha + 2k\pi)$ $\forall k \in \mathbb{Z}$

- $\cos(\alpha) = \text{sen}\left(\alpha + \frac{\pi}{2}\right)$

- $\text{sen}(-\alpha) = -\text{sen}\,\alpha$ (dispari) , $\cos(-\alpha) = \cos\alpha$ (pari)

  x^3 $(-x)^3 = -x^3$ x^8 $(-x)^8 = x^8$

- $\text{sen}(\pi - \alpha) = \text{sen}\,\alpha$ angoli supplementari
- $\cos(\pi - \alpha) = -\cos\alpha$
- $\text{sen}\left(\frac{\pi}{2} - \alpha\right) = \cos\alpha$ $\cos\left(\frac{\pi}{2} - \alpha\right) = \text{sen}\,\alpha$

- Valori speciali

$\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$ 45° ...

$\frac{1}{2}, \frac{\sqrt{3}}{2}$ 30°, 60°, ...

- Sviluppo di Taylor

$\text{sen}\, x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$ $\forall x$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

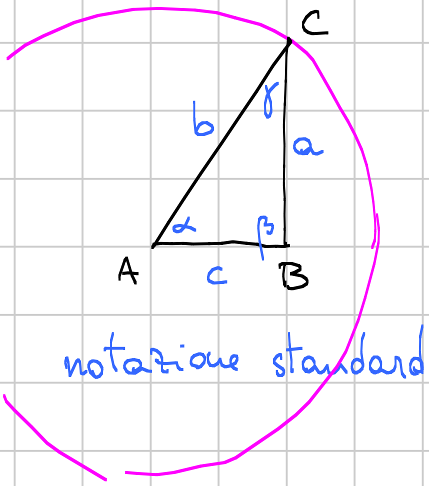
- Altre funzioni trig:

- $\text{tg}\,\alpha := \dfrac{\text{sen}\,\alpha}{\cos\alpha}$ $\qquad \cot\alpha := \dfrac{\cos\alpha}{\text{sen}\,\alpha}$

$$1+\text{tg}^2\alpha = \frac{\text{sen}^2\alpha + \cos^2\alpha}{\cos^2\alpha} = \frac{1}{\cos^2\alpha}$$

$$\boxed{1+\text{tg}^2\alpha = \frac{1}{\cos^2\alpha}}$$

TRIGONOMETRIA BASE



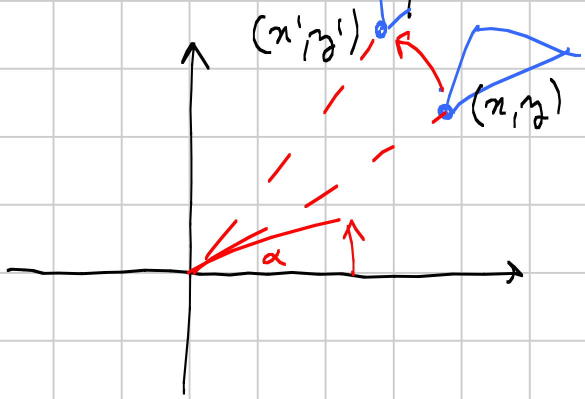
$\beta = \frac{\pi}{2}$

$$\boxed{\begin{aligned} c &= b\cos\alpha \\ a &= b\,\text{sen}\,\alpha \end{aligned}}$$

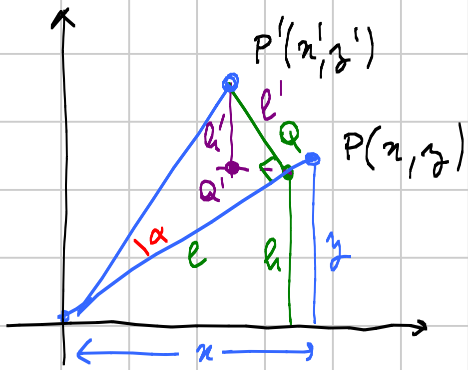
$$\boxed{\text{tg}\,\alpha = \frac{a}{c}}$$

FORMULE

- Rotazioni del piano



$$\boxed{\begin{cases} x' = x\cos\alpha - y\,\text{sen}\,\alpha \\ y' = x\,\text{sen}\,\alpha + y\cos\alpha \end{cases}}$$



Dim (problemi di config) $\qquad Q = (?, h) \qquad l = \overline{OQ} \qquad r = \overline{OP}$

$\qquad\qquad l' = \overline{P'Q}$

$y' = h + h'$

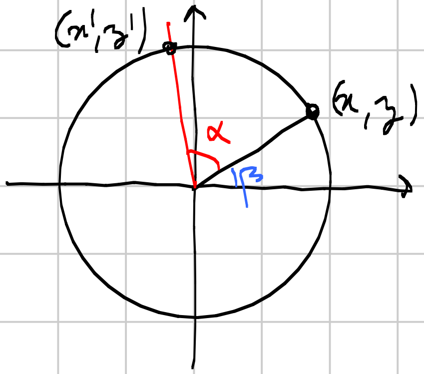
$h : l = y : r \qquad l = r\cos\alpha \qquad h = y\cos\alpha$

$h' : l' = x : r \qquad l' = r\,\text{sen}\,\alpha \qquad h' = x\,\text{sen}\,\alpha$

$y' = x\,\text{sen}\,\alpha + y\cos\alpha$

- Addizione

$$\begin{cases} x' = x\cos\alpha - y\,\mathrm{sen}\,\alpha \\ y' = x\,\mathrm{sen}\,\alpha + y\cos\alpha \end{cases}$$

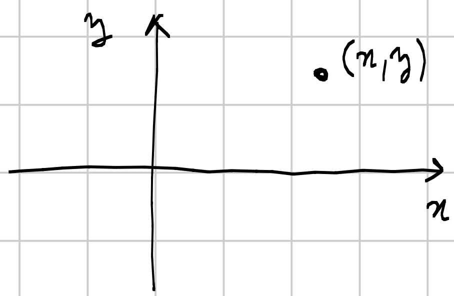


$$(x,y) = (\cos\beta,\ \mathrm{sen}\,\beta)$$

$$(x',y') = (\cos(\alpha+\beta),\ \mathrm{sen}(\alpha+\beta))$$

$$\cos(\alpha+\beta) = x' = \cos\alpha\cos\beta - \mathrm{sen}\,\alpha\,\mathrm{sen}\,\beta$$
$$\mathrm{sen}(\alpha+\beta) = y' = \mathrm{sen}\,\alpha\cos\beta + \mathrm{sen}\,\beta\cos\alpha$$
$$\mathrm{tg}(\alpha+\beta) = y'/x' = \frac{\mathrm{tg}\,\alpha + \mathrm{tg}\,\beta}{1 - \mathrm{tg}\,\alpha\,\mathrm{tg}\,\beta}$$

- Numeri complessi



piano $\mathbb{R}^2$ $\Longrightarrow$ numeri complessi $\mathbb{C}$

$$(x,y) \longmapsto x + iy$$

i tale che $i^2 = -1$

$$(x+iy)(s+it) = xs + ixt + iys + i^2yt = \underbrace{xs - yt}_{\text{"}x\text{"}\ x'} + i\underbrace{(xt+ys)}_{\text{"}y\text{"}\ y'}$$

($i^2 = -1$)

(x,y) e (s,t) sulla circ. goniometrica

$$\begin{cases} x' = xs - yt \\ y' = xt + ys \end{cases}$$

$(x,y) = (\cos\alpha,\ \mathrm{sen}\,\alpha)$ $\quad (s,t) = (\cos\beta,\ \mathrm{sen}\,\beta)$

$$(x',y') = (\cos\alpha\cos\beta - \mathrm{sen}\,\alpha\,\mathrm{sen}\,\beta,\ \mathrm{sen}\,\beta\cos\alpha + \mathrm{sen}\,\alpha\cos\beta)$$
$$= (\cos(\alpha+\beta),\ \mathrm{sen}(\alpha+\beta))$$

"Moltiplicare per un complesso che sta sulla circ. goniom. equivale a ruotare"

$$e^{i\alpha} := (\cos\alpha,\ \mathrm{sen}\,\alpha) =: \cos\alpha + i\,\mathrm{sen}\,\alpha$$
$$e^{i\beta} := (\cos\beta,\ \mathrm{sen}\,\beta) =: \cos\beta + i\,\mathrm{sen}\,\beta$$

$$e^{i\alpha} e^{i\beta} = (\cos\alpha + i\,\text{sen}\,\alpha)(\cos\beta + i\,\text{sen}\,\beta) = \cos(\alpha+\beta) + i\,\text{sen}(\alpha+\beta) = e^{i(\alpha+\beta)}$$

$$e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$\cos\alpha = 1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \ldots$$

$$i\,\text{sen}\,\alpha = i\alpha - i\frac{\alpha^3}{3!} + i\frac{\alpha^5}{5!} - \ldots$$

$$\cos\alpha + i\,\text{sen}\,\alpha = 1 + i\alpha - \frac{\alpha^2}{2!} - i\frac{\alpha^3}{3!} + \frac{\alpha^4}{4!} + i\frac{\alpha^5}{5!} - \ldots$$

$$e^{i\alpha} = 1 + i\alpha + \left(-\frac{\alpha^2}{2!}\right) + \left(-\frac{i\alpha^3}{3!}\right) + \frac{\alpha^4}{4!} \ldots$$

FORMULE DI DUPLICAZIONE, BISEZIONE, TRIPLICAZIONE...

$$\cos 2\alpha = \cos(\alpha+\alpha) = \cos^2\alpha - \text{sen}^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\,\text{sen}^2\alpha$$

$$\text{sen}\,2\alpha = \text{sen}(\alpha+\alpha) = 2\,\text{sen}\,\alpha\cos\alpha$$

$$\text{tg}\,2\alpha = \frac{\text{sen}\,2\alpha}{\cos 2\alpha} = \frac{2\,\text{sen}\,\alpha\cos\alpha}{1-2\,\text{sen}^2\alpha} = \frac{2\,\text{tg}\,\alpha}{\frac{1}{\cos^2\alpha} - 2\,\text{tg}^2\alpha} = \frac{2\,\text{tg}\,\alpha}{1-\text{tg}^2\alpha}$$

☆ Dim form somma tg

$$\cos(\alpha+\beta) = x' = \cos\alpha\cos\beta - \text{sen}\,\alpha\,\text{sen}\,\beta$$

$$\text{sen}(\alpha+\beta) = y' = \text{sen}\,\alpha\cos\beta + \text{sen}\,\beta\cos\alpha$$

$$\text{tg}(\alpha+\beta) = \frac{y'}{x'} = \frac{\text{sen}\,\alpha\cos\beta + \text{sen}\,\beta\cos\alpha}{\cos\alpha\cos\beta - \text{sen}\,\alpha\,\text{sen}\,\beta} = \frac{\text{tg}\,\alpha + \text{tg}\,\beta}{1 - \text{tg}\,\alpha\,\text{tg}\,\beta}$$

$$\cos 2\alpha = 2\cos^2\alpha - 1 = 1 - 2\,\text{sen}^2\alpha$$

$$\cos\alpha = \sqrt{\frac{1+\cos 2\alpha}{2}} \qquad \text{sen}\,\alpha = \sqrt{\frac{1-\cos 2\alpha}{2}}$$

$$\operatorname{tg}\alpha = \sqrt{\frac{1-\cos 2\alpha}{1+\cos 2\alpha}}$$

$$\cos 3\alpha = \cos(\alpha + 2\alpha) = \cos\alpha\cos 2\alpha - \operatorname{sen}\alpha\operatorname{sen}2\alpha$$
$$= \cos\alpha(\cos^2\alpha - \operatorname{sen}^2\alpha) - \operatorname{sen}\alpha(2\operatorname{sen}\alpha\cos\alpha)$$
$$= \cos\alpha(\cos^2\alpha - 3\operatorname{sen}^2\alpha) = \cos\alpha(4\cos^2\alpha - 3) = 4\cos^3\alpha - 3\cos\alpha$$

$$\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$$

$\operatorname{sen}3\alpha =$???

$$\cos(x+y) = \cos x\cos y - \operatorname{sen}x\operatorname{sen}y$$
$$\cos(x-y) = \cos x\cos y + \operatorname{sen}x\operatorname{sen}y$$

$$\cos x\cos y = \frac{\cos(x+y)+\cos(x-y)}{2}$$

$$\operatorname{sen}x\operatorname{sen}y = -\frac{\cos(x+y)-\cos(x-y)}{2}$$

$$\operatorname{sen}(x+y) = \operatorname{sen}x\cos y + \cos x\operatorname{sen}y$$
$$\operatorname{sen}(x-y) = \operatorname{sen}x\cos y - \cos x\operatorname{sen}y$$

$$\operatorname{sen}x\cos y = \frac{\operatorname{sen}(x+y)+\operatorname{sen}(x-y)}{2}$$

$$\cos x\operatorname{sen}y = \frac{\operatorname{sen}(x+y)-\operatorname{sen}(x-y)}{2}$$

$$\cos x\cos y = \frac{\cos(x+y)+\cos(x-y)}{2}$$

$$\operatorname{sen}x\operatorname{sen}y = -\frac{\cos(x+y)-\cos(x-y)}{2}$$

"cento rose sono belle"

Scrittura parametrica

$7\cos\alpha - 8\,\text{sen}\,\alpha = 1 \qquad \alpha = ?$

$7\cos\alpha - 8\sqrt{1-\cos^2\alpha} = 1$

$8\sqrt{1-\cos^2\alpha} = 7\cos\alpha - 1$

?? $64(1-\cos^2\alpha) = 49\cos^2\alpha - 14\cos\alpha + 1$

... eq II grado in $\cos\alpha$

$$t := \text{tg}\,\frac{\alpha}{2} \qquad \cos\alpha = \frac{1-t^2}{1+t^2} \qquad \text{sen}\,\alpha = \frac{2t}{1+t^2}$$

$$\cos 2\beta = 1 - 2\,\text{sen}^2\beta = \cos^2\beta\left(\frac{1}{\cos^2\beta} - 2\,\text{tg}^2\beta\right) = \cos^2\beta\,(1-\text{tg}^2\beta)$$

$$= \frac{1-\text{tg}^2\beta}{1/\cos^2\beta} = \frac{1-\text{tg}^2\beta}{1+\text{tg}^2\beta} = \frac{1-\text{tg}^2\alpha/2}{1+\text{tg}^2\alpha/2} = \frac{1-t^2}{1+t^2} \qquad \beta = \frac{\alpha}{2}$$

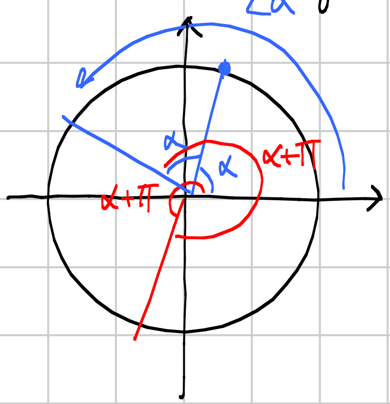
$$\text{sen}\,2\beta = 2\,\text{sen}\,\beta\cos\beta = \cos^2\beta\,(2\,\text{tg}\,\beta) = \frac{2\,\text{tg}\,\beta}{1+\text{tg}^2\beta} = \frac{2\,\text{tg}\,\alpha/2}{1+\text{tg}^2\alpha/2} = \frac{2t}{1+t^2}$$

$7\cos\alpha - 8\,\text{sen}\,\alpha = 1$

$7\,\frac{1-t^2}{1+t^2} - 8\,\frac{2t}{1+t^2} = 1 \qquad 7(1-t^2) - 8\cdot 2t = 1+t^2$

eq II grado più semplice

★ Occhio a dividere gli angoli



$2\alpha = \alpha + \alpha = (\alpha+\pi) + (\alpha+\pi) = 2(\alpha+\pi)$

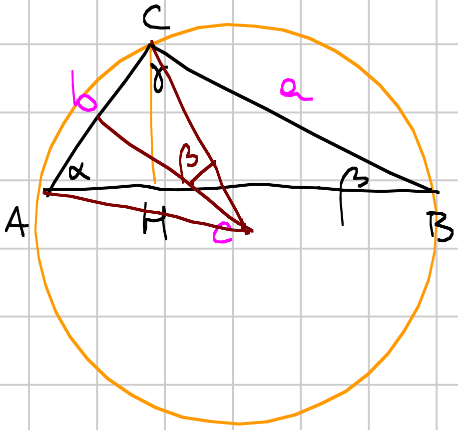
$\frac{\beta}{2}$ non è univoco

$3\alpha = \alpha+\alpha+\alpha = (\alpha + \frac{2}{3}\pi)\cdot 3 = (\alpha + \frac{4}{3}\pi)\cdot 3$

$\text{tg}\,\alpha = \text{tg}\,\alpha+\pi \Rightarrow \text{tg}\,\frac{\beta}{2}$ è univoco $\Rightarrow \beta$ è univoco

TRIGONOMETRIA

- thm dei seni



$b \operatorname{sen} \alpha = HC = a \operatorname{sen} \beta$

$$\frac{a}{\operatorname{sen}\alpha} = \frac{b}{\operatorname{sen}\beta} = \frac{c}{\operatorname{sen}\gamma} = 2R$$

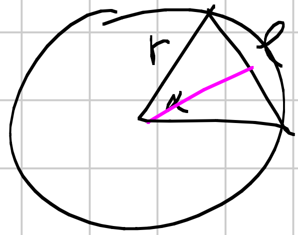
$$\frac{b}{2} = R \operatorname{sen} \beta$$

☆ es: determinare il minimo perimetro di un triangolo con lati tutti interi e due angoli uno il doppio dell'altro

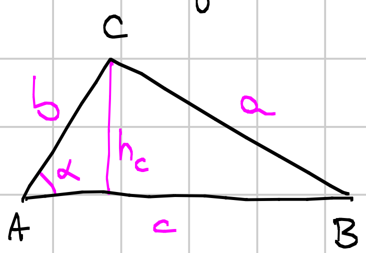
(Bocconi 2010 finale)

- lunghezza corda

$$\ell = 2 \cdot r \operatorname{sen} \frac{\alpha}{2}$$



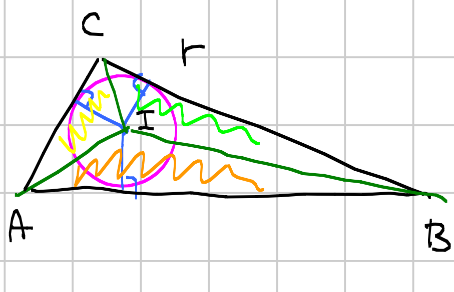
- area trigonometrica



$$\frac{c \cdot h_c}{2} = \frac{c \cdot b \operatorname{sen} \alpha}{2} = [ABC]$$

$$[ABC] = \frac{c \cdot b \cdot \frac{a}{2R}}{2} = \frac{abc}{4R}$$

$$R = \frac{abc}{4[ABC]}$$



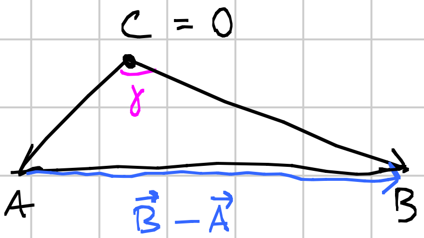
$$[ABC] = \frac{c \cdot r}{2} + \frac{a \cdot r}{2} + \frac{b \cdot r}{2} = r \cdot p$$

(p = semiperimetro)

$$r = \frac{[ABC]}{p}$$

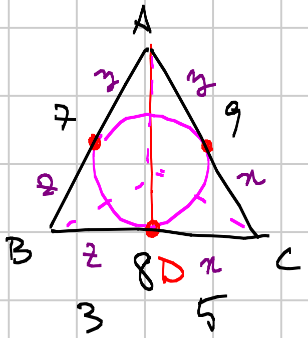
- Carnot (o coseno)

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

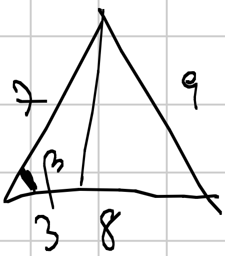


$$c^2 = \overline{AB}^2 = |\vec{B} - \vec{A}|^2 = (\vec{B} - \vec{A}) \cdot (\vec{B} - \vec{A})$$
$$= \vec{B} \cdot \vec{B} + \vec{A} \cdot \vec{A} - \vec{B} \cdot \vec{A} - \vec{A} \cdot \vec{B}$$
$$= |\vec{B}|^2 + |\vec{A}|^2 - 2\vec{A} \cdot \vec{B} = a^2 + b^2 - 2ab\cos\gamma$$

☆ Es 9 :



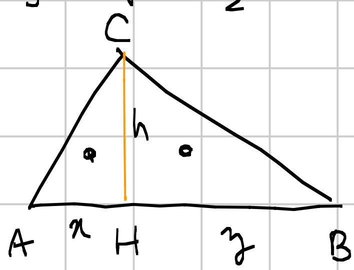
$$x + y + z = 12$$
$$\underbrace{y + z}_{7}$$
$$x = 5$$



$$9^2 = 7^2 + 8^2 - 2 \cdot 7 \cdot 8 \cdot \cos\beta$$
$$AD^2 = 3^2 + 7^2 - 2 \cdot 3 \cdot 7 \cdot \cos\beta$$
$$AD^2 = 46$$

- Formula di Erone

$$[ABC] = \sqrt{\frac{a+b+c}{2} \cdot \frac{a+b-c}{2} \cdot \frac{a-b+c}{2} \cdot \frac{-a+b+c}{2}}$$



$$h^2 + x^2 = b^2$$
$$h^2 + y^2 = a^2$$
$$y^2 - x^2 = a^2 - b^2$$
$$(y - x)\,c = a^2 - b^2$$
$$\begin{cases} y - x = \dfrac{a^2 - b^2}{c} \\ y + x = c \end{cases}$$

$$y = \frac{a^2 - b^2 + c^2}{2c}$$

$$h^2 = a^2 - y^2 = \frac{4a^2c^2 - (a^2 - b^2 + c^2)^2}{4c^2}$$

$$[ABC] = \frac{ch}{2} = \sqrt{\frac{4a^2c^2 - (a^2-b^2+c^2)^2}{4}}$$

$$4a^2c^2 - a^4 - b^4 - c^4 + 2a^2b^2 + 2b^2c^2 - 2a^2c^2 = \sum_{cyc} a^2b^2 - \sum_{cyc} a^4$$

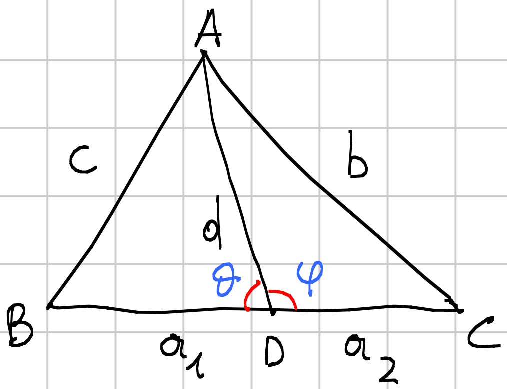
$$[ABC] = \sqrt{\frac{\sum_{cyc} a^2b^2 - \sum_{cyc} a^4}{4}}$$

$$\left((a+b)^2 - c^2\right)\left(c^2 - (a-b)^2\right)$$ sviluppo e viene ...

★ α, β, γ angoli di un triangolo

$$tg\,\alpha + tg\,\beta + tg\,\gamma = tg\,\alpha\, tg\,\beta\, tg\,\gamma$$

Teorema di Stewart



$$a(a_1 a_2 + d^2) = a_1 b^2 + a_2 c^2$$

$$\cos\vartheta = \frac{a_1^2 + d^2 - c^2}{2a_1 d} \qquad \cos\varphi = \frac{a_2^2 + d^2 - b^2}{2a_2 d}$$

$$a_2(a_1^2 + d^2 - c^2) + a_1(a_2^2 + d^2 - b^2) = 0$$

$$a_1 a_2 (a_1 + a_2) + (a_1 + a_2) d^2 = a_2 c^2 + a_1 b^2$$

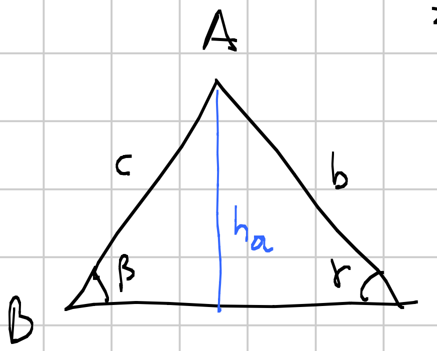
$$a(a_1 a_2 + d^2) = a_2 c^2 + a_1 b^2$$

$$a_1 = a_2 = \frac{a}{2}$$

$$\not{a}\left(\frac{a}{2}\cdot\frac{a}{2} + d^2\right) = \frac{\not{a}}{2}\left(c^2+b^2\right)$$

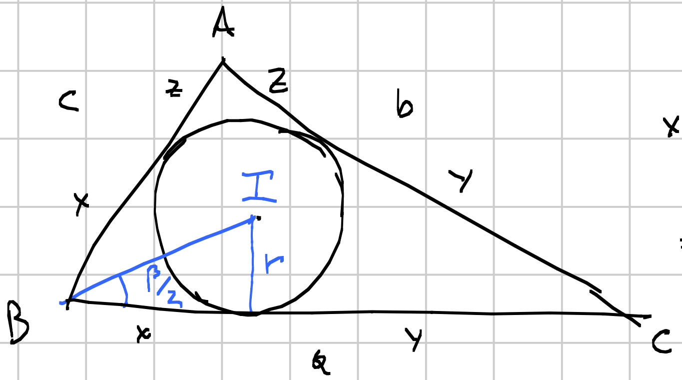
$$d^2 = \frac{b^2}{2} + \frac{c^2}{2} - \frac{a^2}{4}$$

$$d = \sqrt{\frac{2b^2+2c^2-a^2}{4}} = m_a$$

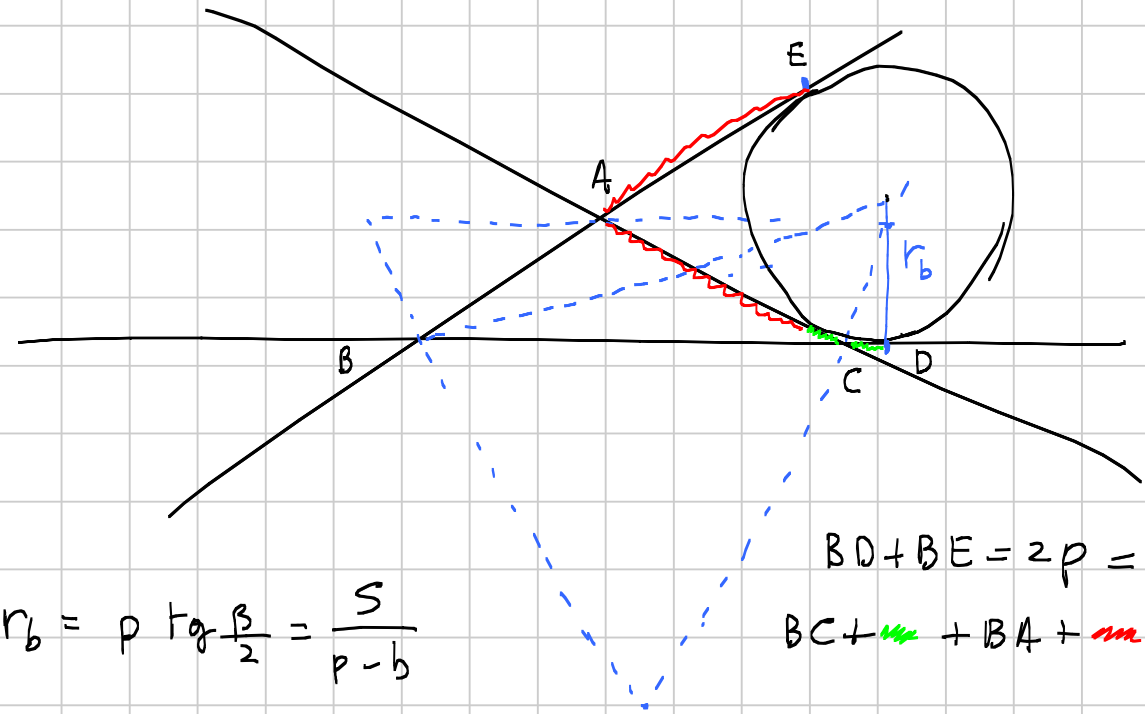


$$h_a = \frac{2S}{a} = c \operatorname{sen}\beta = b \operatorname{sen}\gamma = \frac{2}{a}\sqrt{p(p-a)(p-b)(p-c)}$$

$$r = \frac{S}{p} = (p-a)\operatorname{tg}\frac{\alpha}{2} = (p-b)\operatorname{tg}\frac{\beta}{2} = (p-c)\operatorname{tg}\frac{\gamma}{2}$$



$$x = \frac{a+c-b}{2} = \frac{a+c+b-2b}{2} = p-b$$



$$BD + BE = 2p = BC + \ldots + BA + \ldots$$

$$r_b = p\operatorname{tg}\frac{\beta}{2} = \frac{S}{p-b}$$

$$S = \sqrt{r\cdot r_a\cdot r_b\cdot r_c}$$

$$\frac{1}{r} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}$$

$$\frac{p}{S} = \frac{p-a}{S} + \frac{p-b}{S} + \frac{p-c}{S}$$

$r_a + r_b + r_c - r = 4R$

d_a bisettrice di α

$a_1 : b = a_2 : c$

$$d_\alpha = \frac{2bc}{b+c} \cos\frac{\alpha}{2} = \sqrt{bc - a_1 a_2} =$$

$a_1 : b = a : (b+c)$

$$= \sqrt{bc\left(1 - \frac{a^2}{(b+c)^2}\right)}$$

$$\cos\alpha = \frac{b^2+c^2-a^2}{2bc} = \frac{b^2+c^2-2bc+2bc-a^2}{2bc} =$$

$$= \frac{2bc+(b-c)^2-a^2}{2bc} = \frac{2bc+(b-c-a)(b-c+a)}{2bc} =$$

($2p - 2c$)

$$= \frac{bc-2(p-b)(p-c)}{bc} = 1 - 2\frac{(p-b)(p-c)}{bc}$$

$$\cos\alpha = 1 - 2\,\text{sen}^2\frac{\alpha}{2}$$

$$\text{sen}^2\frac{\alpha}{2} = \frac{(p-b)(p-c)}{bc}$$

Formula di Briggs

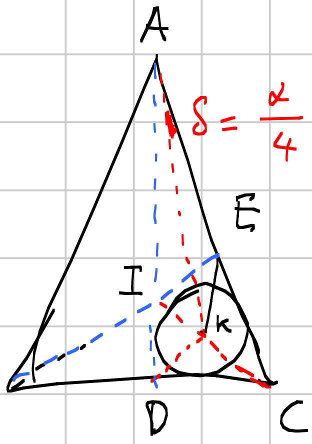
$$\text{sen}\frac{\alpha}{2} = \sqrt{\frac{(p-b)(p-c)}{bc}}$$

$$\cos\frac{\alpha}{2} = \sqrt{\frac{bc-(p-b)(p-c)}{bc}} = \sqrt{\frac{p(p-a)}{bc}}$$

$$S = \frac{1}{2}bc\,\text{sen}\,\alpha = \sqrt{p(p-a)(p-b)(p-c)} =$$

$$\text{sen}\,\alpha = 2\sqrt{\frac{p(p-a)}{bc}\;\frac{(p-b)(p-c)}{bc}}$$

IMO 4/2009

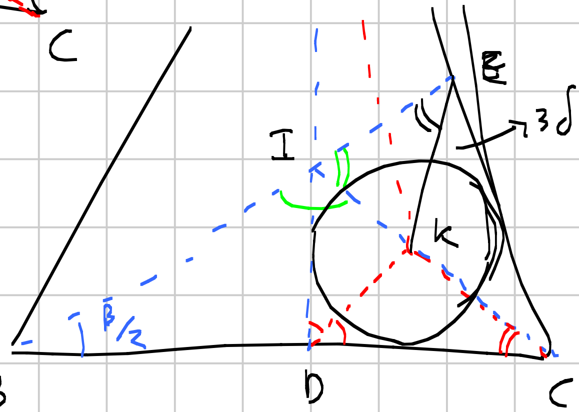


AB = AC

$\widehat{BEK} = 45°$

Quali valori può assumere $\widehat{BAC}$?

C K I allineati.



$$\frac{\beta}{2} = \frac{180-\alpha}{4} = 45-\delta$$

$$\widehat{CDK} = 45°$$

$$\widehat{BEC} = 180 - 3\frac{\beta}{2} = 45+3\delta$$

$$\beta = 90 - \frac{\alpha}{2}$$

$$\frac{IK}{KC} = \frac{ID}{DC} = tg(45-\delta)$$

Teorema dei seni su IEK $\frac{IK}{KE} = \frac{sen\,45}{sen\,E\hat{I}K} =$

$$\frac{sen\,45}{sen\,\beta} = \frac{sen\,45}{sen(90-2\delta)} = \frac{sen\,45}{cos\,2\delta}$$

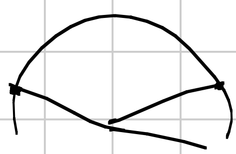
Teorema dei seni su EKC $\frac{KC}{KE} = \frac{sen\,3\delta}{sen(45-\delta)}$

$$\frac{IK}{KC} = \frac{\frac{IK}{KE}}{\frac{KC}{KE}} \quad \frac{\frac{sen\,45}{cos\,2\delta}}{\frac{sen\,3\delta}{sen(45-\delta)}} = \frac{sen(45-\delta)}{cos(45-\delta)}$$

$$sen\,45 \cdot cos(45-\delta) = cos\,2\delta\ sen\,3\delta$$

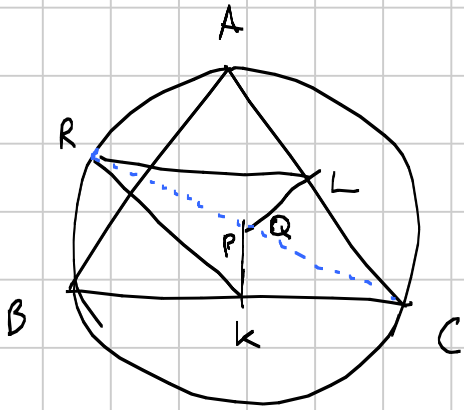
$$\frac{sen(90-\delta)+sen\,\delta}{2} = \frac{sen\,5\delta + sen\,\delta}{2}$$

$\delta = 15° + k30°$ $\quad\quad$ $\delta = 22{,}5°$



IMO
4/2007

$\alpha = 60°$ $\alpha = 90°$



Area RPK = Area RQL