G3 BASIC

-Maria-

Titolo nota 09/09/2010

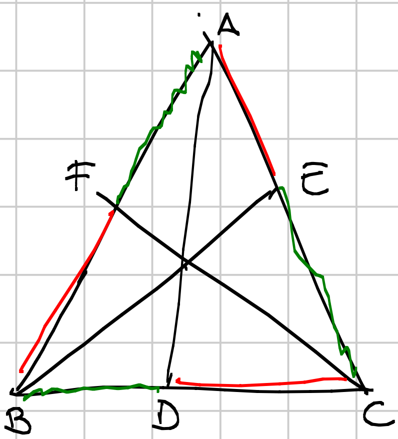
- Allineamenti e concorrenze: Ceva, Menelao, Carnot, ...
- Circonferenze: angoli, potenze...
- Trasformazioni geometriche
- Retta di Eulero, Feuerbach
- Excerchi...

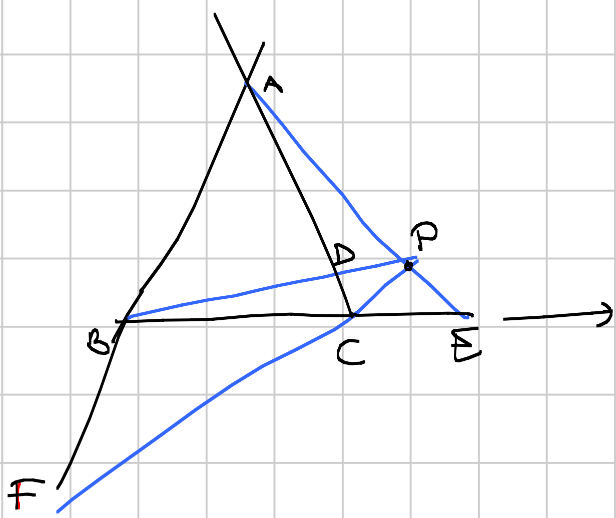
⋮

Teo di Ceva:

AD, BE, CF concorrono

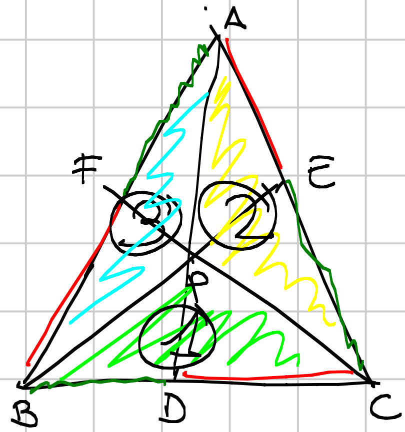
$$\Leftrightarrow \quad \frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$$





$\Rightarrow$) Dim (Aree)

$$\frac{AF}{FB} = \frac{[AFC]}{[BFC]} = \frac{[AFP]}{[BFP]} \overset{?}{=} \frac{[APC]}{[BPC]}$$

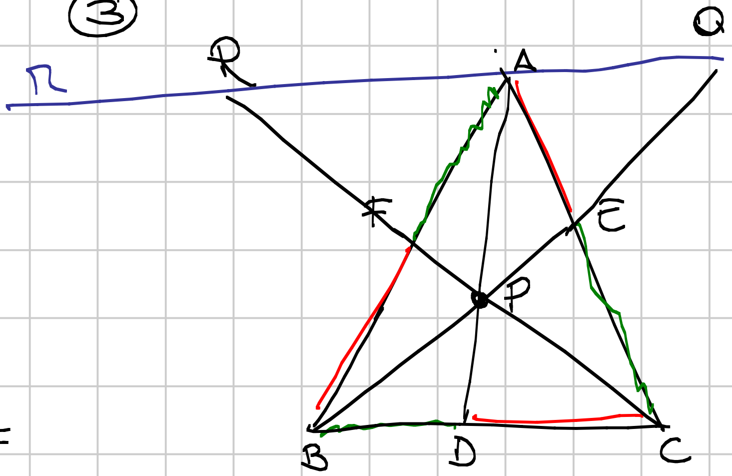


$$\frac{[APC]}{[BPC]} = \frac{[AFC]-[AFP]}{[BCF]-[BFP]}$$

$$= \frac{[AFC]}{[BFC]} \left(\underbrace{\frac{1-\frac{[AFP]}{[AFC]}}{1-\frac{[BFP]}{[BFC]}}}_{=1} \right)$$

$$= \frac{AF}{FB} = \frac{2}{1}$$

$$\frac{BD}{DC} = \frac{3}{2} \qquad \frac{CE}{EA} = \frac{1}{3}$$



Dim:

$r \parallel BC$.

$$\frac{AF}{FB} = \frac{AR}{BC}$$

simili $AFP \sim BCF$

$$\frac{BD}{DC} = \frac{QA}{AR}$$

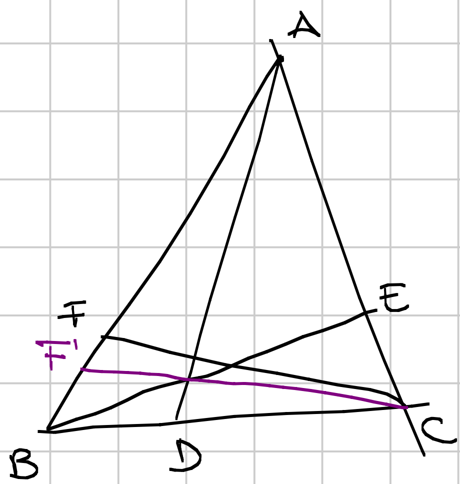
$$\frac{CE}{EA} = \frac{BC}{AQ}$$

$BC\cancel{E} \sim AEQ$

$\Leftarrow$)

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$$

$$\frac{AF'}{F'B} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$$

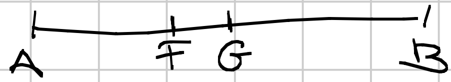


$$\Rightarrow \frac{AF}{FB} = \frac{AF'}{F'B}$$

$$F \longrightarrow \frac{AF}{FB}$$

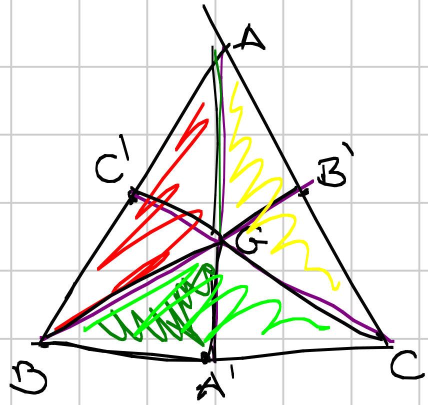
$G > F$

$$\underset{\overset{}{FB\ >\ GB}}{\overset{AF\ <\ AG}{\frac{AF}{FB}}} < \frac{AG}{GB}$$



$\Rightarrow \quad F = F'$

Esempio: baricentro

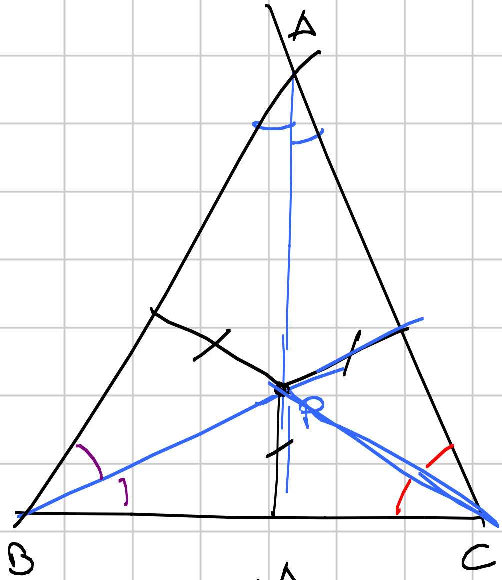


$$\underset{\substack{\shortparallel \\ 1}}{\frac{AC'}{C'B}} \cdot \underset{\substack{\shortparallel \\ 1}}{\frac{BA'}{A'C}} \cdot \underset{\substack{\shortparallel \\ 1}}{\frac{CB'}{B'A}} = 1$$

$$[AGC] = [BCG] = [ABG]$$

$$\frac{AG}{GA'} = \frac{[ABG]}{[A'BG]} = \frac{2 \text{ triangolini}}{1 \text{ triangolino}} = 2$$

Esempio: incentro.

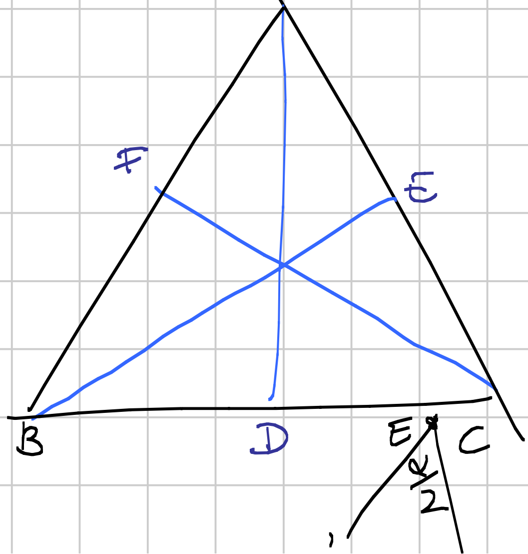


$$\frac{BD}{DC} = \frac{c}{b}$$

$$\frac{CE}{EA} = \frac{a}{c}$$

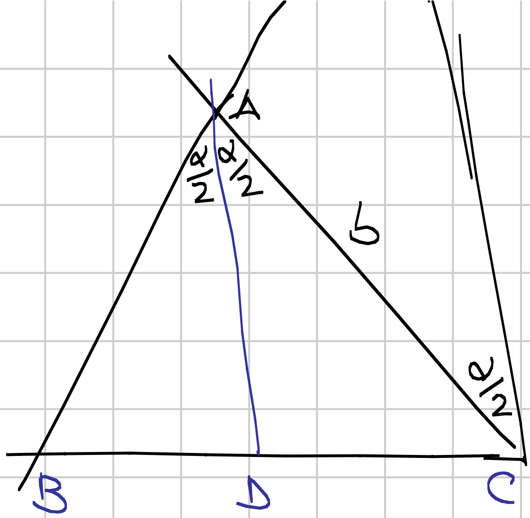
$$\frac{AF}{FB} = \frac{b}{a}$$

$$\frac{\cancel{c}}{\cancel{b}} \cdot \frac{\cancel{a}}{\cancel{c}} \cdot \frac{\cancel{b}}{\cancel{a}} = 1$$



Lemma

$$\frac{BD}{DC} = \frac{c}{b}$$



$$\frac{BD}{DC} = \frac{AB}{AE} = \frac{c}{AE}$$

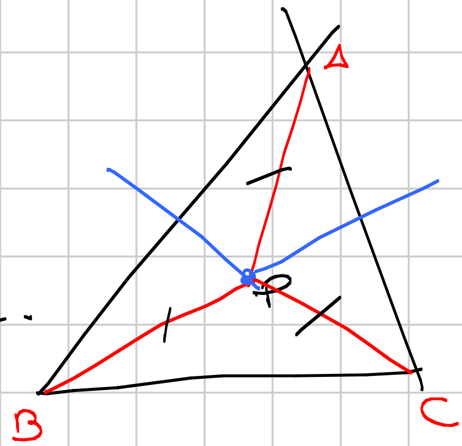
ACE è isoscele $\Rightarrow$ $AE = b$

Circocentro?

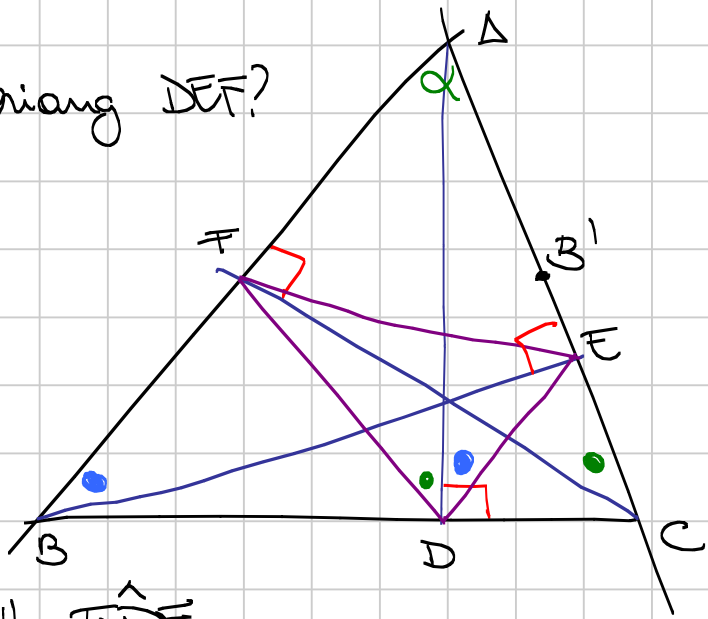
$BP = AP = PC$

$\Downarrow$

P sta sull'asse di BC.



Esempio: ortocentro
(con Ceva, esercizio)
Cos'è AD rispetto al triang DEF?
AD



Oss: AFDC è ciclico.

$\hat{FDA} = F\hat{C}E = 90-\alpha$

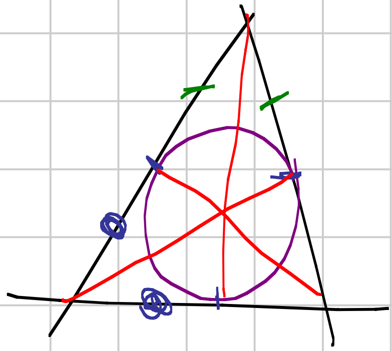
$\hat{ADE} = A\hat{B}E = 90-\alpha$

$\Rightarrow$ AD è bisettrice di $E\hat{D}F$.

Oss: ortocentro di ABC = incentro DEF

Oss: centro della circo circoscr AFDC
= pto medio di AC

Esercizio pto di Gergonne

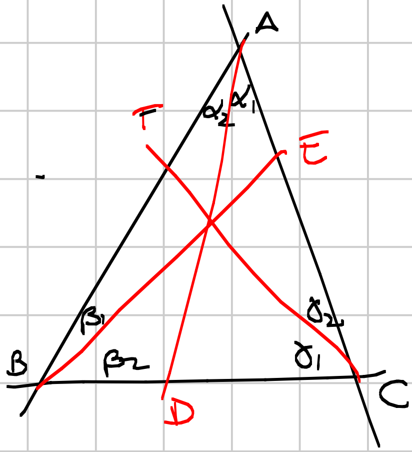


Ceva in vers trigo:

1) AD, BE, CF concorrono

$\Leftrightarrow \quad \frac{\sin\alpha_1}{\sin\alpha_2} \cdot \frac{\sin\beta_1}{\sin\beta_2} \cdot \frac{\sin\gamma_1}{\sin\gamma_2} = 1.$

2)



1) $\Leftrightarrow \quad \frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$

$\frac{DC}{\cancel{\sin\alpha_1}} = \boxed{\frac{AD}{\sin\gamma}} \cdot \sin\alpha_1$ (teo Seni su ACD)

$\frac{BD}{\cancel{\sin\alpha_2}} = \boxed{\frac{AD}{\sin\beta}} \cdot \sin\alpha_2$ (teo Seni ABD)

$AF = \boxed{\frac{CF}{\sin\alpha} \cdot \sin\gamma_2}$ $\qquad BF = \boxed{\frac{CF}{\sin\beta} \cdot \sin\gamma_1}$

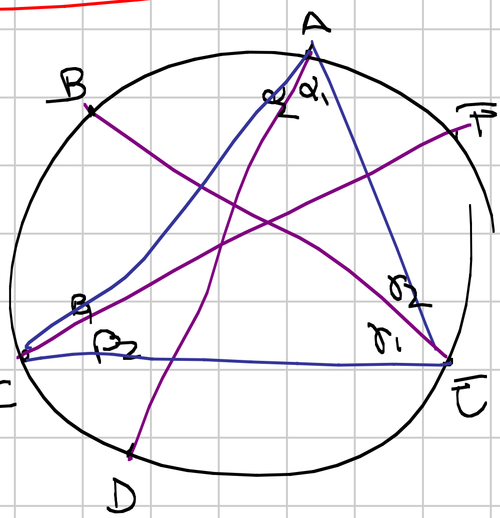
$CE = \boxed{\frac{BE}{\sin\gamma} \sin\beta_2}$ $\qquad AE = \boxed{\frac{BE}{\sin\alpha} \sin\beta_1}$

Esempio

AD, BE, CF concorrono

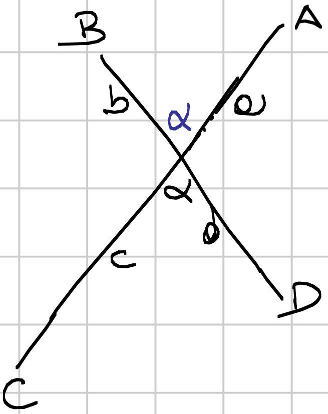
$\Leftrightarrow \quad AB \cdot CD \cdot EF = BC \cdot DE \cdot FA$

$AB = 2R \sin\gamma_2$



Teorema di Carnot.

Lemma (metrica di perp)



$AC \perp BD \iff$
$AB^2 + CD^2 = AD^2 + BC^2$.
① ②

Dim:

$\Rightarrow) = a^2+b^2+c^2+d^2$

$\Leftarrow) \overset{①}{=} a^2+b^2-2ab\cos\alpha +$
$c^2+d^2-2cd\cos\alpha$

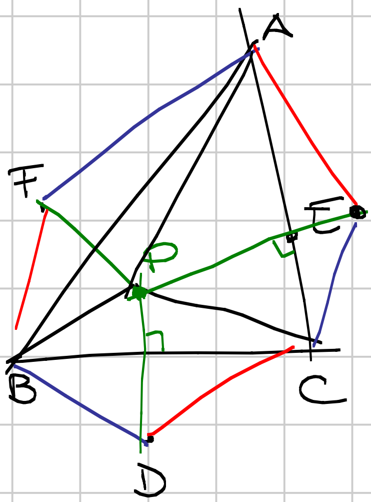
$② = a^2+d^2+2ad\cos\alpha$
$+b^2+c^2+2bc\cos\alpha$

$-2ab-2cd = 2ad+2bc$

Teo:

le perp da F ad AB,
da E a AC,
da D a BC
concorrono $\iff$

$AF^2 + BD^2 + CE^2 = AE^2 + BF^2 + CD^2$



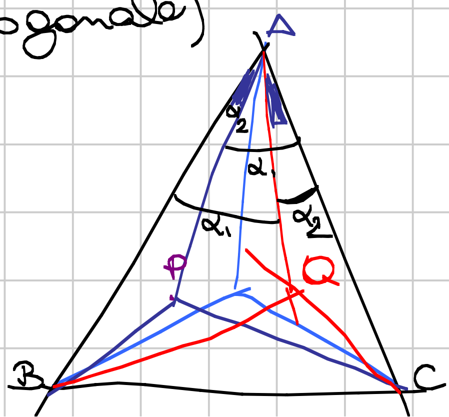
$\Rightarrow)$ $AF^2 + BP^2 = AP^2 + BF^2$
$BD^2 + CP^2 = BP^2 + CD^2$
$CE^2 + AP^2 = CP^2 + AE^3$

Esercizio (esistenza coniugato isogonale)

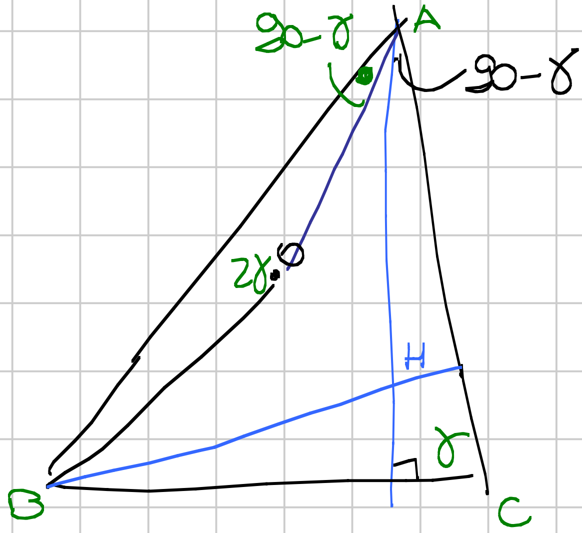
Q è coniugato isog di P.

Dim concorrenza:

$\frac{\sin\alpha_1}{\sin\alpha_2} \cdot \ldots = 1$



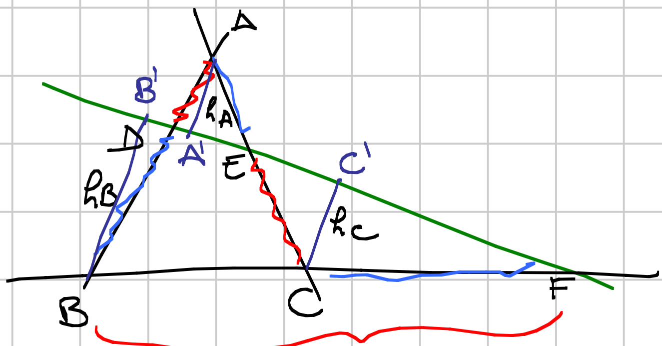
Oss: il coniug isog del circoc è l'ortoc



Teo di Menelao

D, E, F sono all $\Leftrightarrow$

$$\frac{AD}{DB} \cdot \frac{BF}{FC} \cdot \frac{CE}{EA} = -1$$



Dim: esercizio: dedurlo da Ceva.

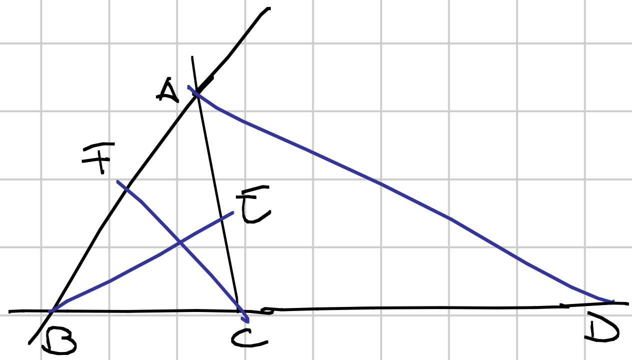
$\Rightarrow$) $\frac{AD}{DB} = \frac{h_A}{h_B}$ sim $AA'D \sim BB'D$

$$\frac{BF}{FC} = -\frac{h_B}{h_C}$$

$$\frac{CE}{EA} = \frac{h_C}{h_A}$$

Sostituisco, ok.

Esercizio:
2 bisettrici interne, 1 esterna.
$\Rightarrow$ D, E, F allineati.



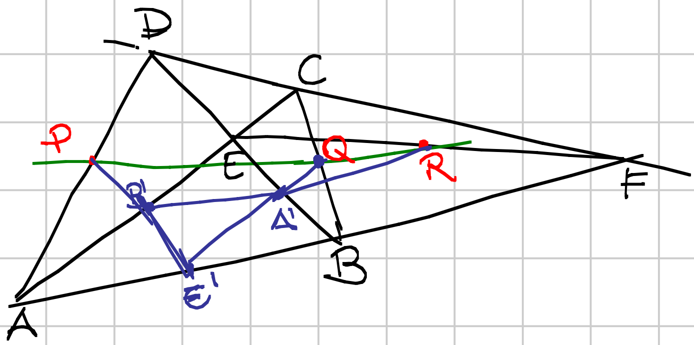
Esempio (linea di Gauss)

ABCD quadril.

R pto medio EF

P, Q di AD, BC.

$\Rightarrow$ P, Q, R allineati.



Dim

Consideriamo $A'E'B'$ mediale di ABE

$\Rightarrow$ E', A', Q sono allineati.

E', B', P

A', B', R

teo di Menelao con $A'B'E'$

P, Q, R sono all $\Longleftrightarrow$ $\dfrac{B'P}{PE'} \cdot \dfrac{E'Q}{QA'} \cdot \dfrac{A'R}{RB'} \overset{!}{=} -1$

$PE' \parallel BD \Rightarrow \dfrac{B'P}{PE'} = \dfrac{ED}{DB}$

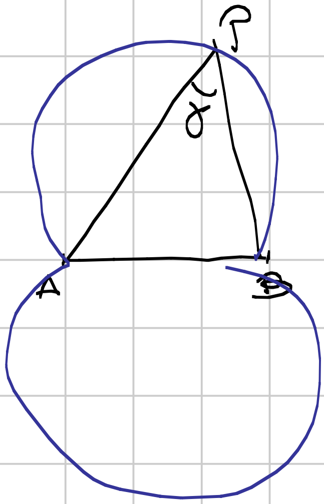
$E'A' \parallel AC \Rightarrow \dfrac{E'Q}{QA'} = \dfrac{AC}{CE}$

$A'B' \parallel AB \Rightarrow \dfrac{A'R}{RB'} = \dfrac{BF}{FA}$

$$\frac{ED}{DB} \cdot \frac{AC}{CE} \cdot \frac{BF}{FA} \overset{?}{=} -1$$

Ok per il teo di Menelao applicato a ABE rispetto alla retta C, D, F.

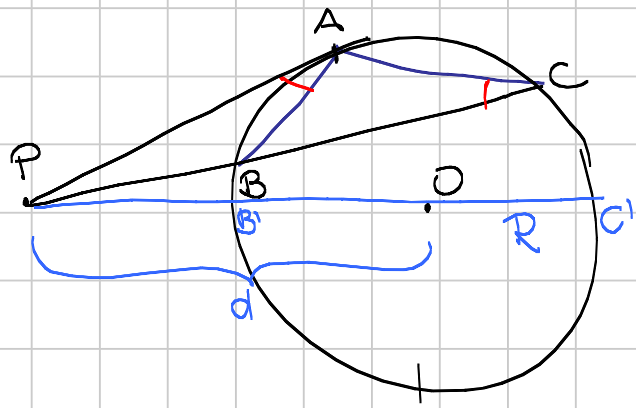
CIRCONFERENZE



$PA^2 = PB \cdot PC$

Cons i triangoli

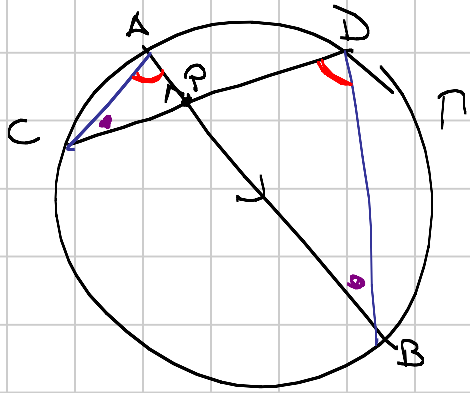
PAB e PCA sono simili.



$$\frac{PA}{PB} = \frac{PC}{PA}$$

$$Pow_\Gamma P = PB \cdot PC = (d-R)(d+R) = d^2 - R^2.$$

$PA \cdot PB = PC \cdot PD$



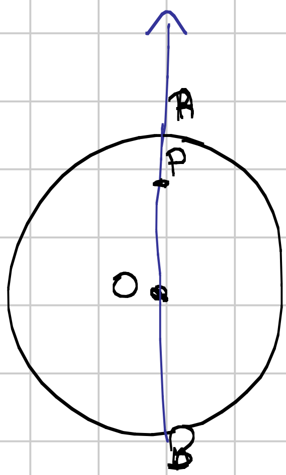
I triangoli APC e BDP sono simili.

$$\frac{PA}{PC} = \frac{PD}{PB}$$ ok.

P interno:

$Pow_\Gamma P = PB \cdot PC < 0$

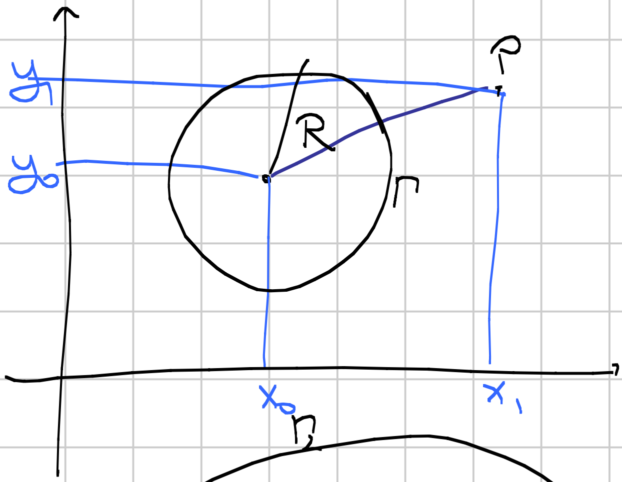
P è interno $\Leftrightarrow$ $Pow_\Gamma P < 0$



$$Pow_\Gamma P = PA \cdot PB = (R-d)(-R-d)$$
$$= d^2 - R^2$$

Oss:

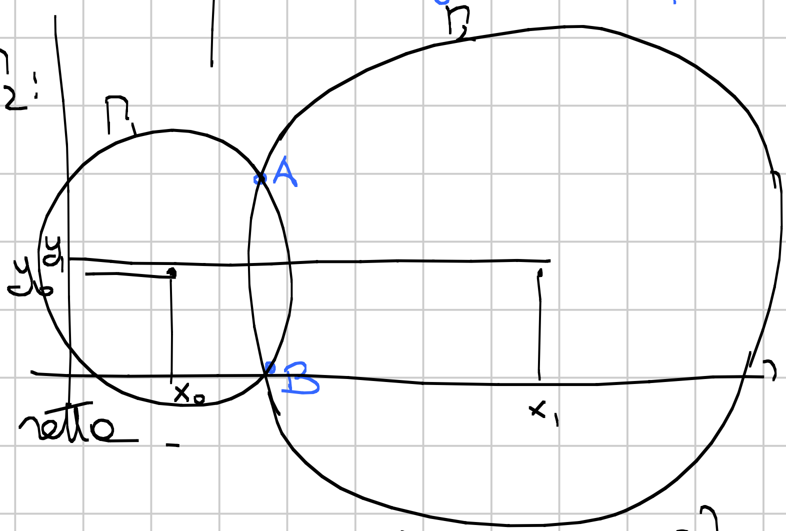
$$\mathrm{Pow}_{\Gamma} P = d^2 - R^2$$
$$= (y_1 - y_0)^2 + (x_1 - x_0)^2 - R^2$$
$$(y - y_0)^2 + (x - x_0)^2 - R^2 = 0$$



Asse radicale di Γ_1 e Γ_2:

$r = \{ P : \mathrm{Pow}_{\Gamma_1} P = \mathrm{Pow}_{\Gamma_2} P \}$

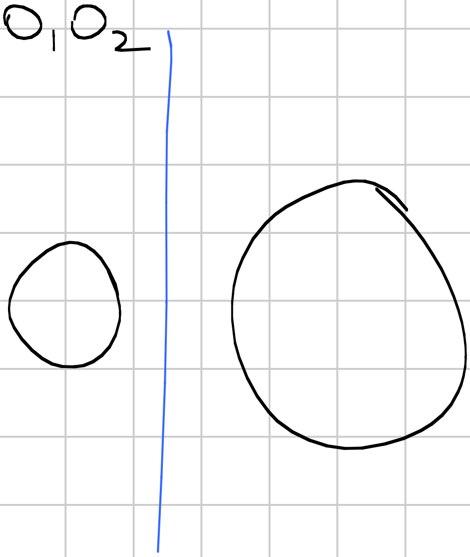
$A, B \in r$.

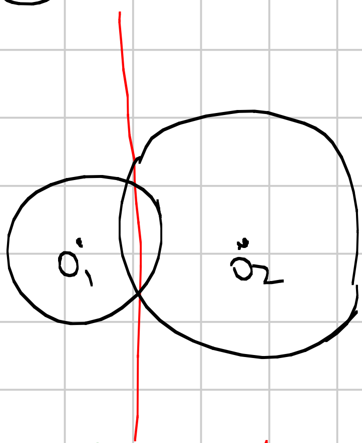


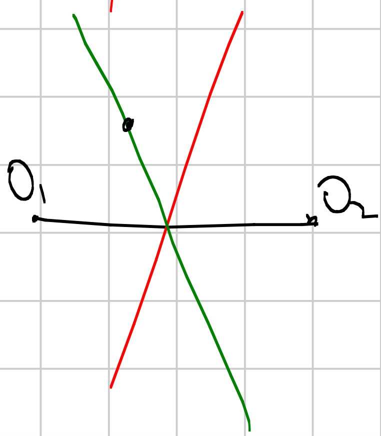
Prop: l'asse rad è una retta.

$$r = \{ (x,y) : (x-x_0)^2 + (y-y_0)^2 - r_1^2 = (x-x_1)^2 + (y-y_1)^2 - r_2^2 \}$$
$$\cancel{dx^2} + \cancel{ey^2} + ax + by + c = 0$$

Oss: $r \perp O_1 O_2$

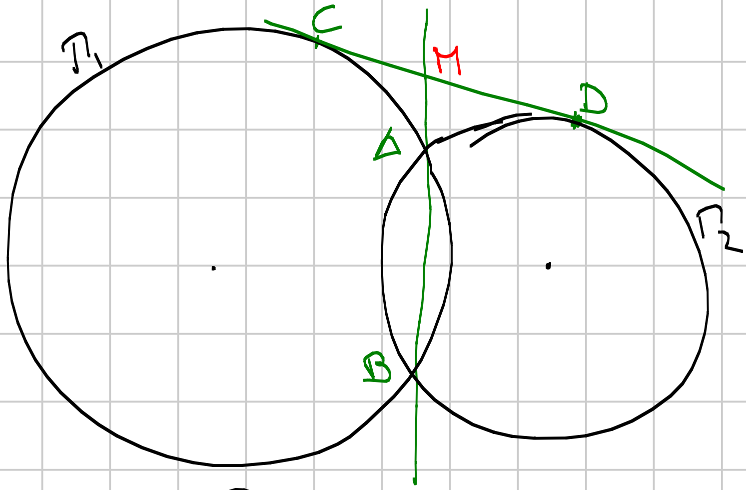






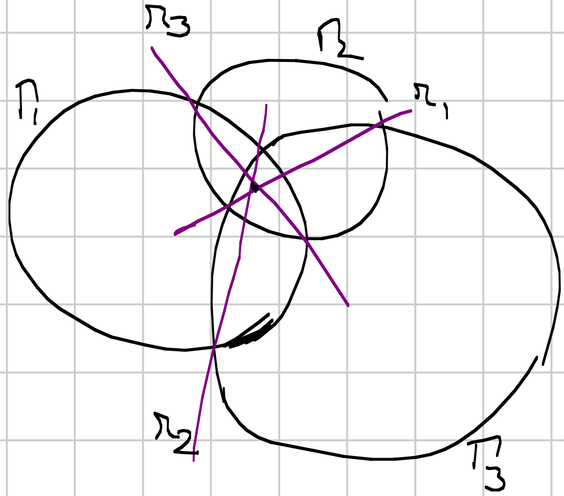
Oss 1:

M è pto medio di CD.



$M \in AB \Rightarrow \underset{\overset{\shortparallel}{MC^2}}{Pow_{\Gamma_1} M} = \underset{\overset{\shortparallel}{MD^2}}{Pow_{\Gamma_2} M}$

Oss 2: $\Gamma_1, \Gamma_2, \Gamma_3$ circo $\Rightarrow$ gli assi radicali concorrono



$r_2 \cap r_3 = P$: $Pow_{\Gamma_1} P = Pow_{\Gamma_3} P$

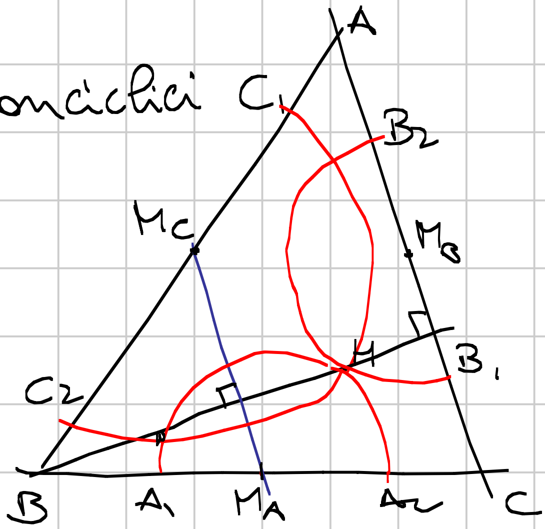
$\shortparallel$ $\quad$ $\shortparallel$

$Pow_{\Gamma_2} P$

$\Rightarrow P \in r_1$

Es: IMO 2008 - 1

Tesi: A_i, B_i, C_i, $i = 1, 2$, sono conciclici



Oss: se fossero conciclici, O sarebbe il centro.

BH • $\perp$ $M_A M_C$

• passa per H

BH è asse radicale

$BC_1 \cdot BC_2 = BA_1 \cdot BA_2$

$\Rightarrow A_1 A_2 C_1 C_2$ è ciclico.

il centro della circ. per $A_1 A_2 C_1 C_2$ è O

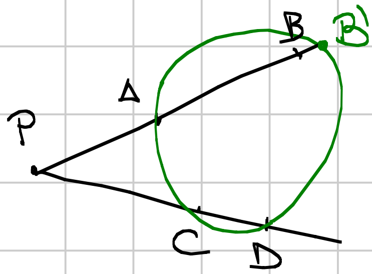
(sta sull'asse di $A_1 A_2$, cioè AB, e di $C_1 C_2$, cioè BC)

Oss: ABCD è ciclico $\iff PA \cdot PB = PC \cdot PD$

$\Rightarrow$) ok

$\Leftarrow$) $PA \cdot PB' = PC \cdot PD$

$PB' = PB \Rightarrow B = B'$.



Oss:
I 3 cerchi concorrono.

$P = \Gamma_1 \cap \Gamma_2$

$P \in \Gamma_3 \iff$

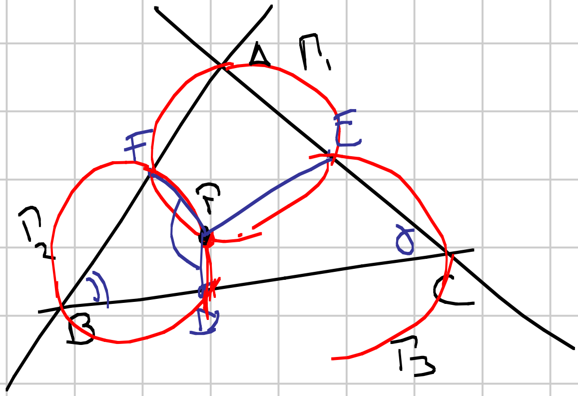
$\hat{DPE} + \gamma = 180$

$\hat{DPF} = 180 - \beta$

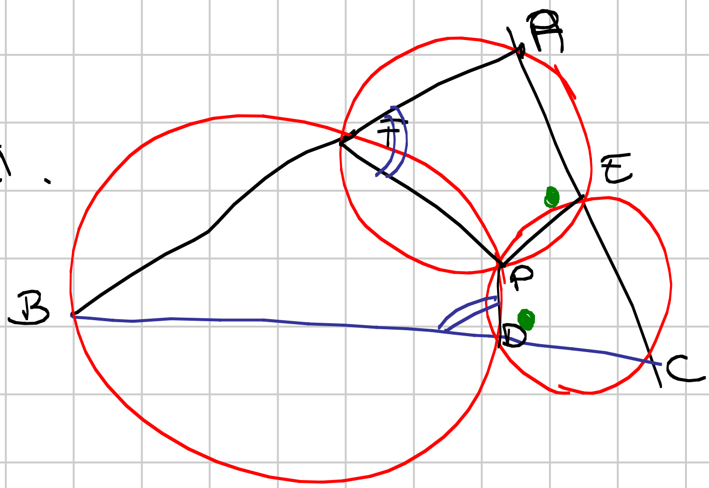
$\hat{FPE} = 180 - \alpha$

$\Rightarrow \hat{DPE} = 360 - \hat{DPF} - \hat{FPE}$

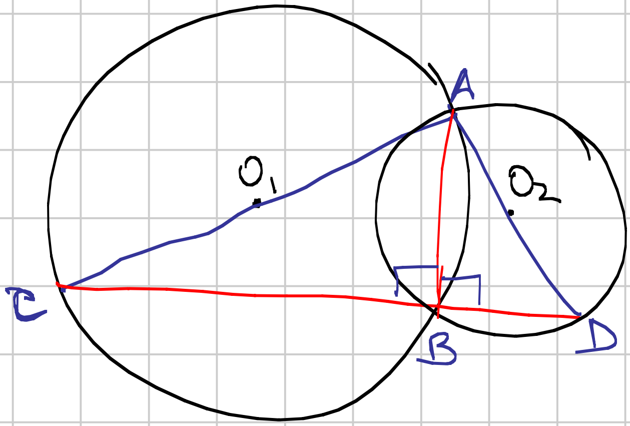
$= \alpha + \beta = 180 - \gamma$



Oss:
$\Rightarrow$ B, D, C allineati.

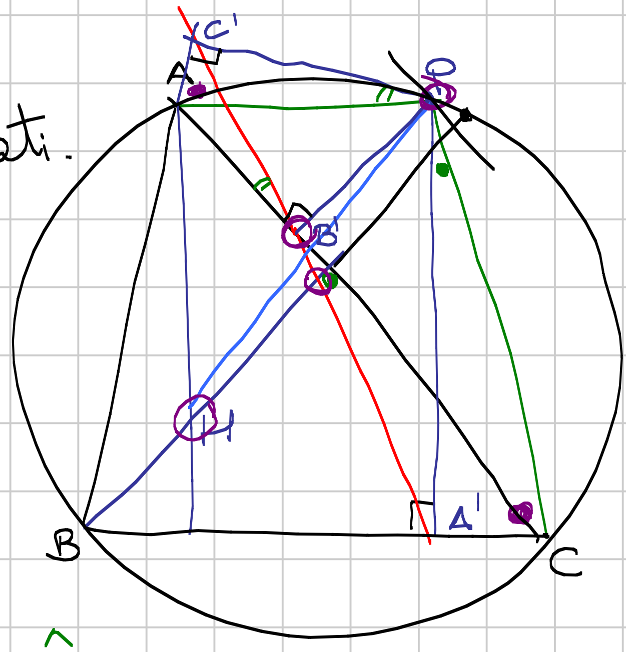


Oss:
C, B, D all.



Es: SIMSON LINE

$P \in \Gamma \Rightarrow A', B', C'$ allineati.



Sono allineati se
$A\hat{B'}C' = A'\hat{B'}C$

$AB'PC'$ è ciclico $\Rightarrow$
$A\hat{B'}C' = A\hat{P}C'$

$A'CPB'$ ciclico $\Rightarrow$ $A'\hat{B'}C = A'\hat{P}C$

$\| = 90 - \bullet$

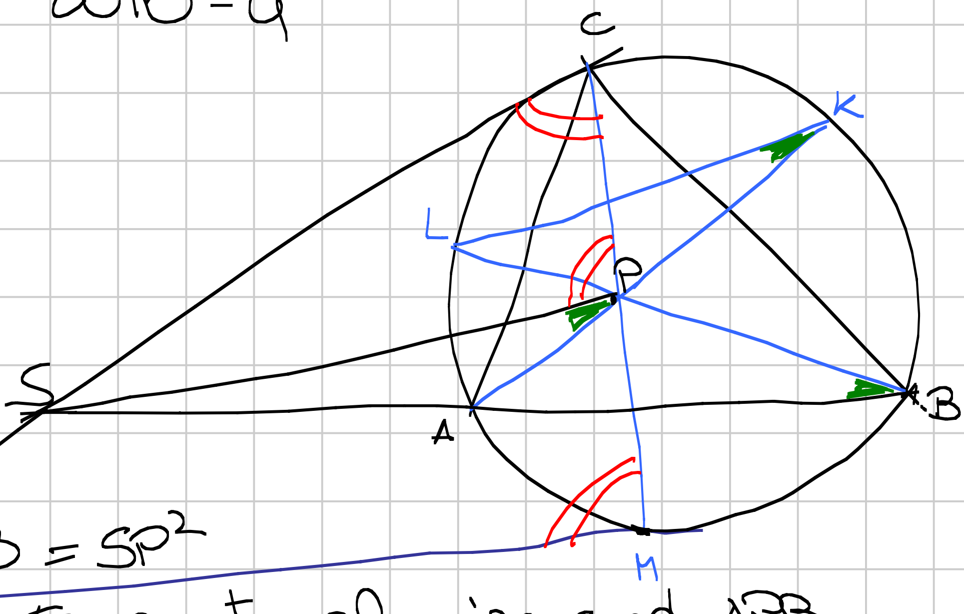
$ABCP$ ciclico $\Rightarrow$ $C'\hat{A}P = A'\hat{C}P$

$\bullet = 90 - \bullet$

Esercizio: la SE di P passa per il pto medio di PH.

Esercizio IMO 2010-4

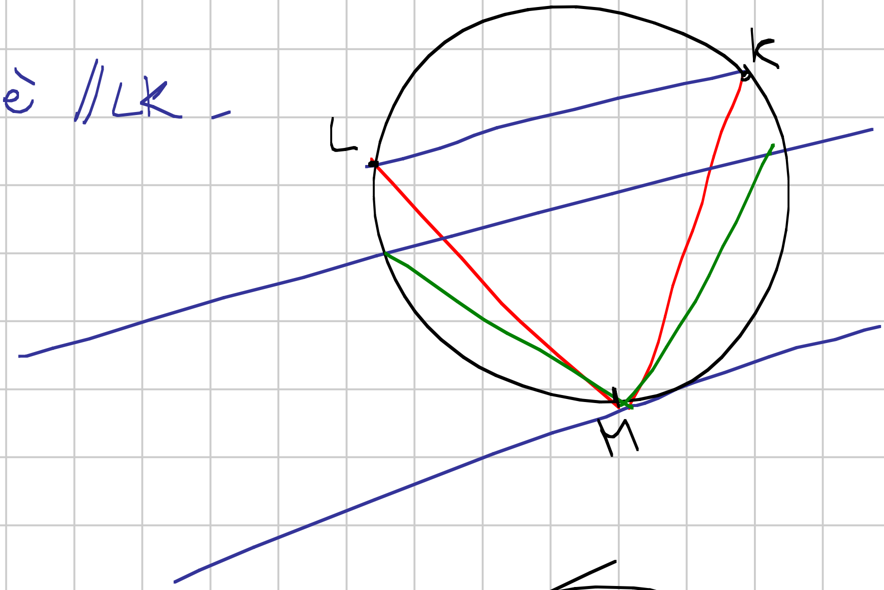
$SC = SP$
$\Rightarrow MK = ML$

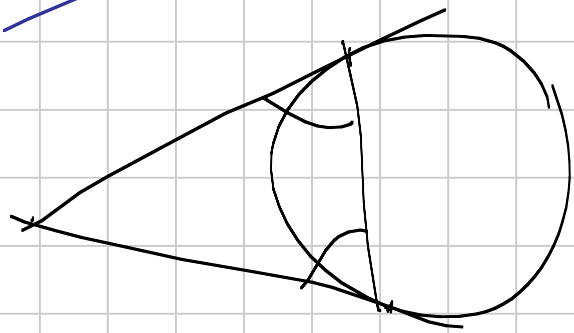


$SC^2 = SA \cdot SB = SP^2$
$\Rightarrow$ SP è tangente al circoc ad APB

$\Rightarrow SP \parallel LK$
Manca: $TM \parallel SP$.

$LM = MK \iff$
la tang in M è $/\!/ LK$.



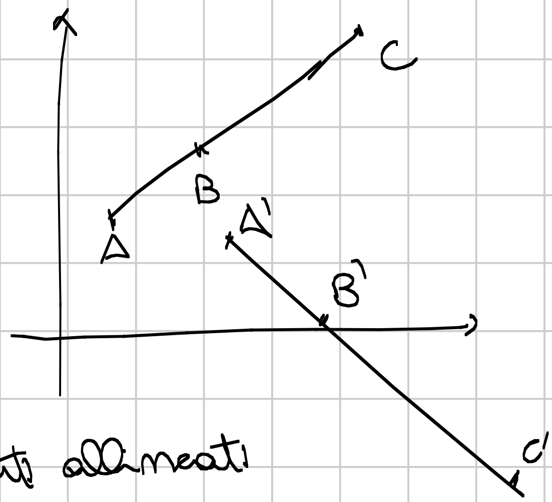


TRASFORMAZIONI GEOMETRICHE

AFFINITÀ:

$$\begin{cases} x \longrightarrow ax+by+c = x' \\ y \longrightarrow dx+ey+f = y' \end{cases}$$

(con $ae-db \neq 0$, invertibile)



- conserva: • concorr, parall,
  • rapporti di segmenti allineati
- non conserva: - similitudine
  - angoli
  - circonferenze

Oss A, B, C non all, A', B', C' non all

$\Rightarrow$ esiste affinità che manda $A \to A'$

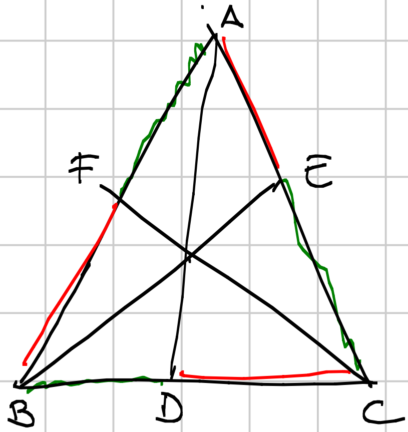
$B \to B'$

$C \to C'$

Teo di Ceva:

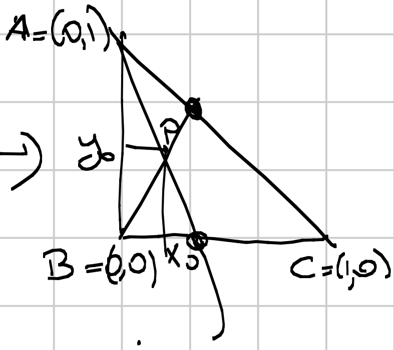
AD, BE, CF concorrono

$\Leftrightarrow \quad \dfrac{AF}{FB} \cdot \dfrac{BD}{DC} \cdot \dfrac{CE}{EA} = 1$



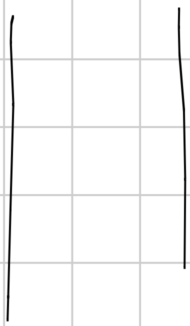
Dim:

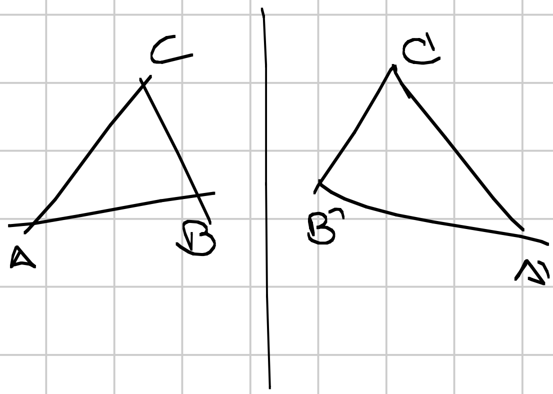
A meno di affinità possiamo supporre



Riflessioni, rotazioni, traslazioni

Oss: rifl + rifl = trasl (assi paralle) / rotazione (non paralle)



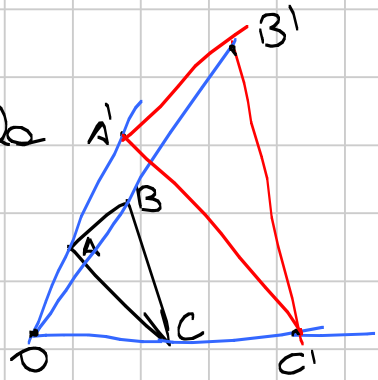


Omotetie:

Conserva gli angoli,

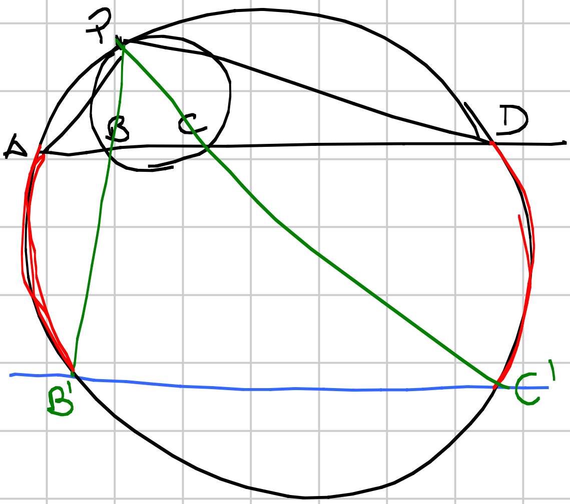
parall (e una retta è parall alla sua immagine)

rapporti di lunghezze



Esempio

$\hat{APC} = \hat{BPD}$.



Omotetia di centro P.

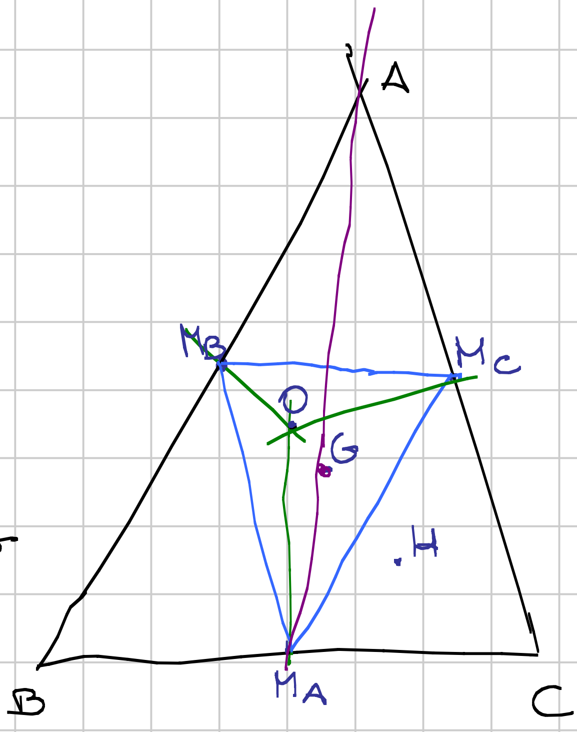
$B'C' \parallel AD$

Tesi $\iff \hat{APB} = \hat{CPD}$

Retta di Eulero.

O, G, H sono all,

$\vec{OG} = \frac{1}{3}\vec{OH}$



Dim:

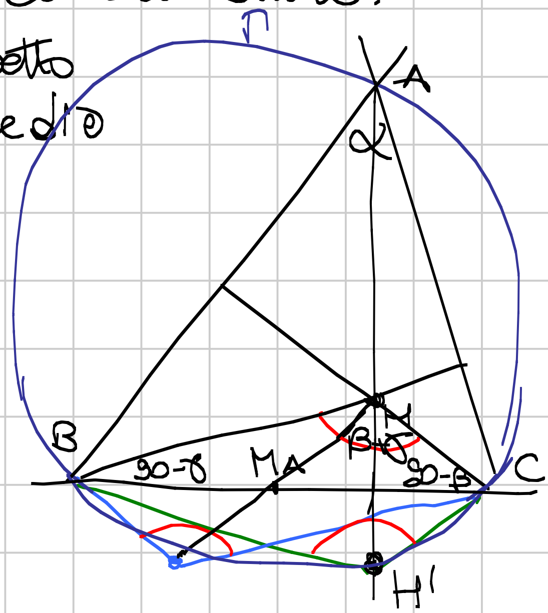
Omotetia di centro G e rapporto $-\frac{1}{2}$.

$A \longrightarrow M_A$

$H \longrightarrow$ ortoc del triangolo $M_A M_B M_C$ = circoc di ABC (altezza M_A su $M_B M_C$ è asse BC).

O $\quad G = \frac{A+B+C}{3} \quad H = A+B+C$

Lemma del simmetrico dell'ortocentro:
il simmetrico dell'ortoc rispetto a un lato o a un pto medio sta sulla circo circoscritta



Dim:
Caso Acutangolo
Devo verif
$B\hat{A}C + B\hat{H'}C = 180°$

$B\hat{H'}C = B\hat{H}C = \beta+\gamma$

X i punti medi, quasi uguale.
Dim 2 pti medi

$X \longrightarrow 2M_A - X$

$H \longrightarrow 2 \cdot \frac{B+C}{2} - (A+B+C) = -A$

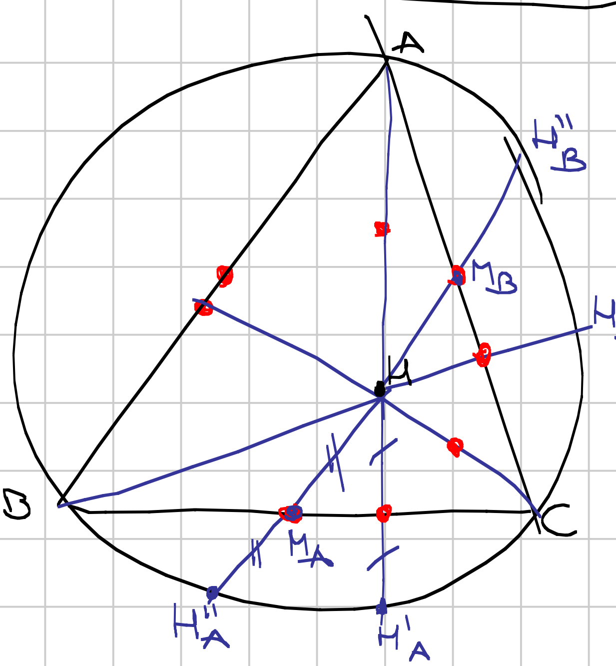
Omotetia di centro H e rapporto $\frac{1}{2}$.

$H'_A \rightarrow$ piede altezza

$H''_A \rightarrow M_A$

$A \rightarrow$ pto medio di AH

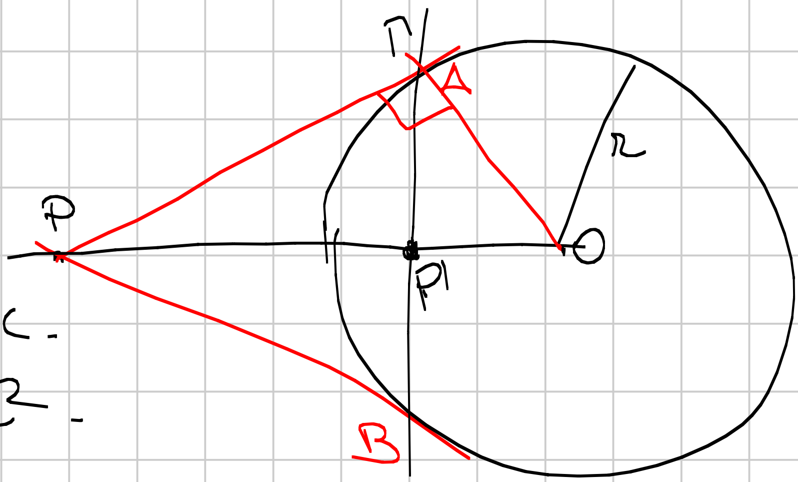
M_A, M_B, M_C, piedi delle alt



Sono conciclici

E' detto CIRCONFERENZA DI FEUERBACH!

<u>Inversione</u>



$P \to P'$ su OP t.c.

$OP \cdot OP' = r^2$.

P' si ottiene $AB \cap OP$

Dim: chiamiamo "P''" e verifichiamo

$OP \cdot OP'' = r^2$.

E' il 1° teo di Euclide sul triangolo AOP.

Punti fissi: Π

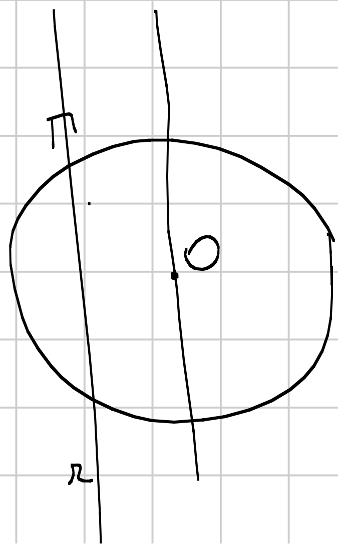
Retta per O → sè

Retta non per O →

Circonferenza per O

$a \operatorname{Re} z + b \operatorname{Im} z + c = 0$

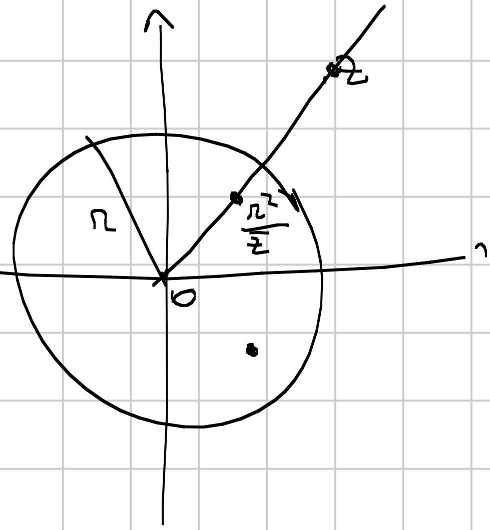
$z \to \frac{r^2}{\bar{z}}$



$$\frac{r^2}{|z|} \qquad |z|\cdot|w| = r^2$$

$$\frac{r^2}{\bar z} \qquad z \to \frac{1}{\bar z}$$

$$e^{i\theta} \qquad e^{i\theta} = e^{i(-\theta)}$$



$$a \operatorname{Re} \frac{r^2}{\bar z} + b \operatorname{Im} \frac{r^2}{\bar z} + c = 0$$

$$a \operatorname{Re} \frac{r^2 z}{|z|^2} + b \operatorname{Im} \frac{r^2 z}{|z|^2} + c = 0$$

$$a\, r^2 \operatorname{Re} z + b\, r^2 \operatorname{Im} z + c|z|^2 = 0$$

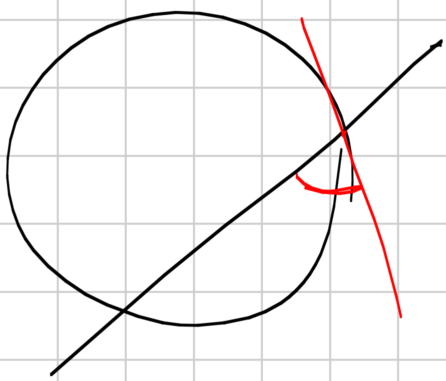
$$\operatorname{Re} z = x \qquad \operatorname{Im} z = y$$

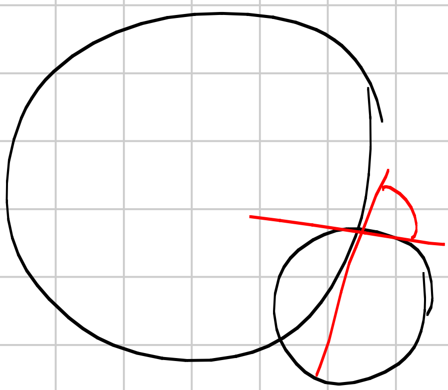
$$a r^2 x + b r^2 y + c(x^2+y^2) = 0$$

Circonferenze non per O $\longrightarrow$ circonferenze non per O.

Attenzione: i centri non vanno necess. nei centri.

Oss: conservo gli angoli tra rette e circonf.



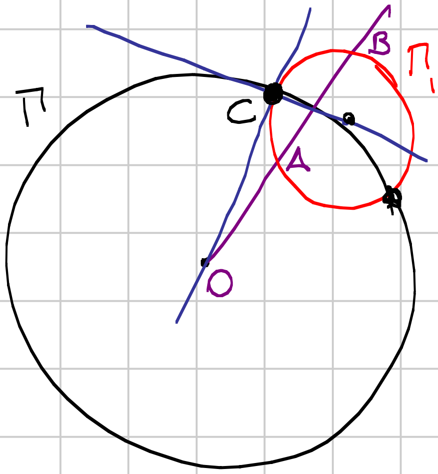


Ossi:

Se inverto rispetto a Γ,

Γ_1 resta fermo

Immagine di Γ_1



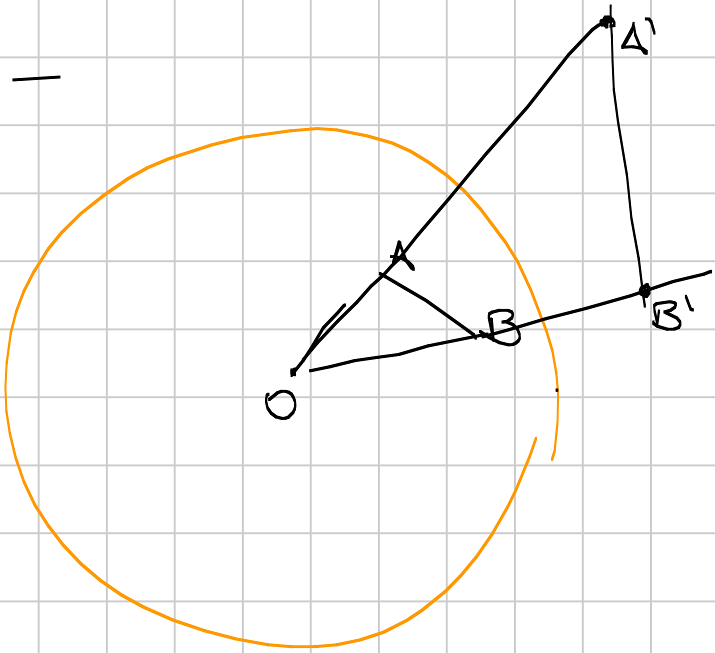
B e A si scambiano?

$$OA \cdot OB = r^2$$

$$\text{Pow}_{\Gamma_1} O = OC^2$$

Teorema di Tolomeo —

$A'B' = ?$



$$OA \cdot OA' = r^2$$

$$OB \cdot OB' = r^2$$

$$\frac{OA}{OB} = \frac{OA'}{OB'}$$

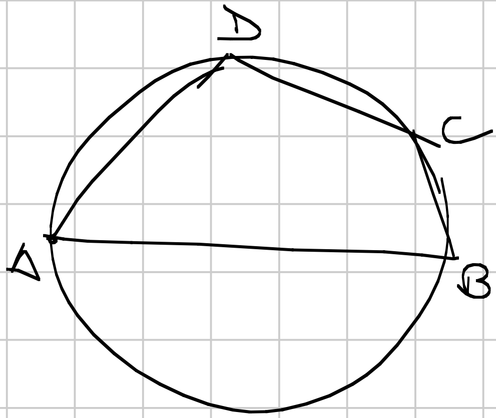
$\Rightarrow$ similitudine di OAB e $OB'A'$

$$\boxed{\frac{A'B'}{AB} =} \frac{OA'}{OB} = \boxed{\frac{r^2}{OB \cdot OA}}$$

Teo di Tolomeo:

$AB \cdot CD + BC \cdot AD \geq AC \cdot AD$,

con uguaglianza $(\Leftrightarrow)$ A, B, C, D sono conciclici.



Dim: Invertiamo rispetto ad A con raggio 1

$$B'C' + C'D' \geq B'D'$$

con $=$ $\Leftrightarrow$

B', C', D' sono alle

$\Leftrightarrow$ A, B, C, D conciclici A'

$$\frac{BC}{AB \cdot AC} + \frac{CD}{AC \cdot AD} \geq \frac{BD}{AB \cdot AD}$$