G 1 - Medium - TRIGO

Titolo nota 06/09/2010

$$z = a + ib \qquad e^z = e^a(\cos b + i \operatorname{sen} b)$$

$$e^z e^w = e^{z+w} = e^{a+c}\left(\cos(b+d) + i \operatorname{sen}(b+d)\right)$$

$z = a + ib$

$w = c + id$

$$(x+iy)(s+it) =$$

$$= (xs - yt) + i(xt + ys)$$

$a = c = 0$

$$(\cos b + i \operatorname{sen} b)(\cos d + i \operatorname{sen} d) =$$

$$= \cos(b+d) + i \operatorname{sen}(b+d)$$

$$\Rightarrow \cos(b+d) = \cos b \cos d - \operatorname{sen} b \operatorname{sen} d$$

$$\operatorname{sen}(b+d) = \cos b \operatorname{sen} d + \operatorname{sen} b \cos d$$

$$\sum_{n=1}^{2010} \operatorname{sen}(n\theta) = \sum_{n=1}^{2010} \operatorname{Im}\left(e^{in\theta}\right) =$$

$$= \operatorname{Im}\left(\sum_{n=1}^{2010} e^{in\theta}\right) = \operatorname{Im}\left(\sum_{n=1}^{2010}\left(e^{i\theta}\right)^n\right) =$$

$$= \operatorname{Im}\left(\frac{e^{i2011\theta} - 1}{e^{i\theta} - 1} - 1\right) = \operatorname{Im}\left(\frac{e^{i\theta \cdot 2011} - 1}{e^{i\theta} - 1}\right) =$$

$$= \operatorname{Im}\left(\frac{\cos(2011\theta) - 1 + i \operatorname{sen} 2011\theta}{\cos\theta - 1 + i \operatorname{sen}\theta}\right) =$$

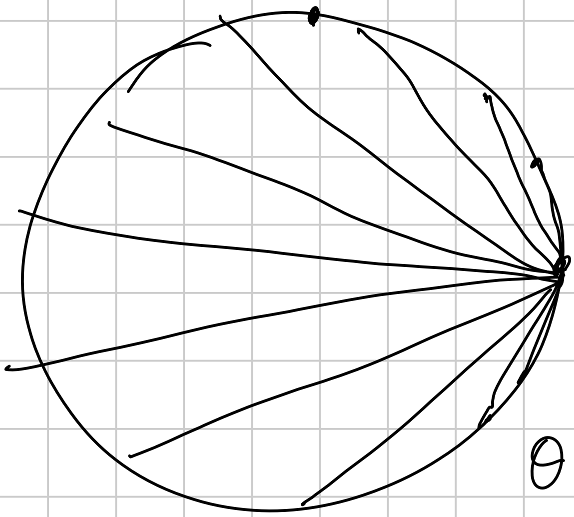
$$= \mathrm{Im}\,\frac{\left[\cos(2011\theta)-1+i\,\mathrm{sen}(2011\theta)\right](\cos\theta-1-i\,\mathrm{sen}\theta)}{(\cos\theta-1)^2+\mathrm{sen}^2\theta} =$$

$$= \frac{\mathrm{sen}(2011\theta)(\cos\theta-1)-\mathrm{sen}\theta\left[\cos(2011\theta)-1\right]}{\cos^2\theta-2\cos\theta+1+\mathrm{sen}^2\theta} =$$

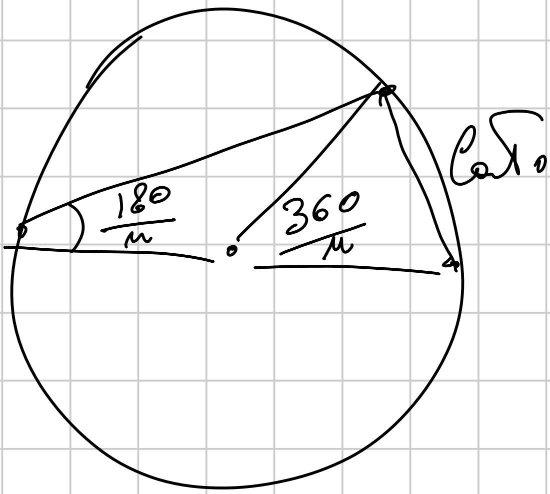
$$= \frac{\mathrm{sen}(2010\cdot\theta)-\mathrm{sen}(2011\cdot\theta)+\mathrm{sen}\,\theta}{2(1-\cos\theta)}$$

Es: n-agono regolare inscritto nella circ. unitaria.

Quanto vale il prodotto di tutti i lati e tutte le diagonali?



$\theta = \frac{180}{n}$



Teo del seno: $\quad$ lato $= 2R\sin\theta = 2\sin\theta$

l-diagonale $= 2\sin 2\theta$

$\sin\alpha = \sin(\pi-\alpha)$ $\qquad ?? = \prod_{k=1}^{n-1} 2\cdot\sin(k\theta) =$

$$= 2^{n-1} \prod_{k=1}^{n-1} \sin(k\theta) = 2^{n-1} \prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{n}\right)$$

$$\prod_{k=1}^{n-1} \left|1 - e^{i\frac{2k\pi}{n}}\right| = \left|\prod_{k=1}^{n-1} (1 - \xi_k)\right| = |p(1)| = n$$

$$\prod_{k=1}^{n-1} (x - \xi_k) = \frac{x^n - 1}{x - 1} = \sum_{k=0}^{n-1} x^k = p(x)$$

$$\prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{n}\right) = \frac{n}{2^{n-1}}$$

Es: $x_{n+1} = \dfrac{1 + x_n}{1 - x_n}$ $\qquad x_0 = 2010$

$x_{2012} = ?$

$x_n = \operatorname{tg} \alpha_n \qquad x_{n+1} = \dfrac{1 + \operatorname{tg} \alpha_n}{1 - \operatorname{tg} \alpha_n} = \operatorname{tag}\left(\alpha_n + \dfrac{\pi}{4}\right)$

$\tan(\theta + \varphi) = \dfrac{\tan\theta + \tan\varphi}{1 - \tan\theta \tan\varphi} \qquad \alpha_{n+1} = \alpha_n + \dfrac{\pi}{4} \mod 2\pi$

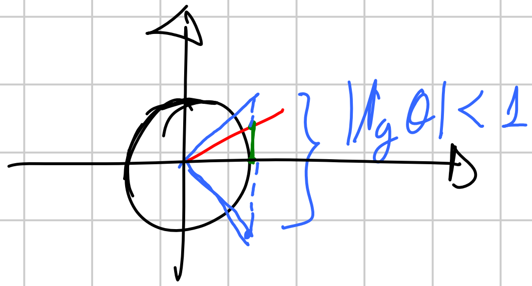
Es: Dati 5 numeri reali $\neq 1, -1$, tra di essi ve ne sono 2, a e b tali che

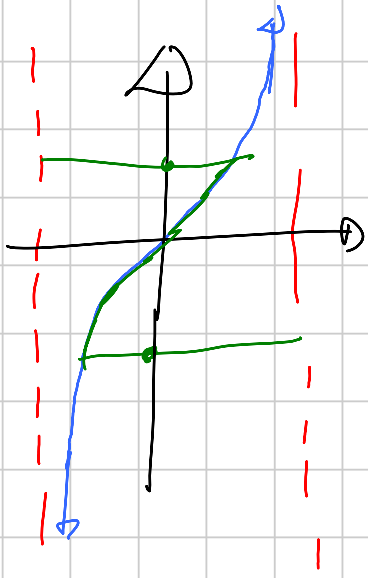
$$|ab + 1| > |a - b|$$

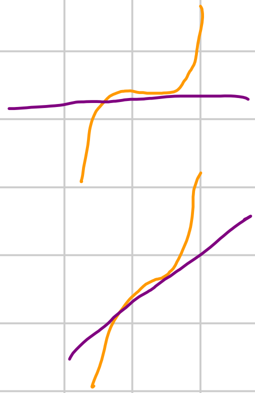
$\left|\dfrac{a-b}{ab+1}\right| < 1 \qquad a = \operatorname{tg}\alpha, \quad b = \operatorname{tg}\beta$

$\qquad \shortparallel$

$|\operatorname{tg}(\alpha - \beta)| < 1 \qquad |\alpha - \beta| < \dfrac{\pi}{4}$







$\underline{\text{Oss}}$: $tg : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$

è bigettiva

$\Downarrow$

Dati 5 numeri reali $\neq -1, 1$ $\exists$ 5 angoli t.c. $x_i = tg\,\theta_i$
x_i

$-\frac{\pi}{2} < \theta_i < \frac{\pi}{2}$

ABC triangolo A, B, C gli angoli $A+B+C = \pi$

•) $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$

$C = \pi - A - B \qquad \frac{C}{2} = \frac{\pi}{2} - \left(\frac{A}{2} + \frac{B}{2}\right)$

$$\tan\frac{C}{2} = \cot\left(\frac{A}{2} + \frac{B}{2}\right) = \frac{1}{tg\left(\frac{A}{2} + \frac{B}{2}\right)}$$

$$\tan\frac{A}{2}\cdot\tan\frac{B}{2} + \tan\frac{B}{2}\cdot\frac{1}{tg\left(\frac{A}{2}+\frac{B}{2}\right)} + tg\frac{A}{2}\,\frac{1}{tg\left(\frac{A}{2}+\frac{B}{2}\right)} =$$

$$tg\left(\frac{A}{2} + \frac{B}{2}\right) = \frac{tg\frac{A}{2} + tg\frac{B}{2}}{1 - tg\frac{A}{2}\,tg\frac{B}{2}}$$

$$= tg\frac{A}{2}\cdot tg\frac{B}{2} + \frac{tg\frac{B}{2}\left(1 - tg\frac{A}{2}\,tg\frac{B}{2}\right)}{\underbrace{tg\frac{A}{2} + tg\frac{B}{2}}} + \frac{tg\frac{A}{2}\left(1 - tg\frac{A}{2}\,tg\frac{B}{2}\right)}{\underbrace{tg\frac{A}{2} + tg\frac{B}{2}}} =$$

$$= \operatorname{tg}\frac{A}{2}\cdot\operatorname{tg}\frac{B}{2} + \left(1-\operatorname{tg}\frac{A}{2}\operatorname{tg}\frac{B}{2}\right) = 1.$$

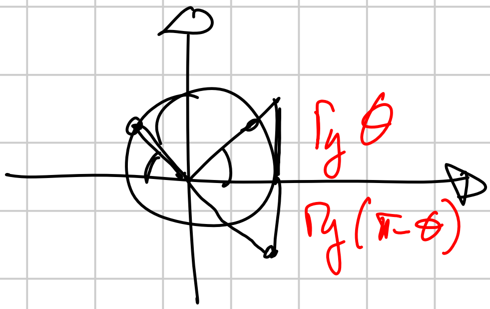
Cor: $\operatorname{tg}\frac{A}{2}\operatorname{tg}\frac{B}{2}\operatorname{tg}\frac{C}{2} \le \frac{\sqrt{3}}{9} = \frac{1}{3\sqrt{3}}$

$$1 = \operatorname{tg}\frac{A}{2}\operatorname{tg}\frac{B}{2} + \operatorname{tg}\frac{B}{2}\operatorname{tg}\frac{C}{2} + \operatorname{tg}\frac{C}{2}\operatorname{tg}\frac{A}{2} \ge 3\sqrt[3]{\left(\operatorname{tg}\frac{A}{2}\operatorname{tg}\frac{B}{2}\operatorname{tg}\frac{C}{2}\right)^2}$$

Cor: $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$

- $\operatorname{tg} A + \operatorname{tg} B + \operatorname{tg} C = \operatorname{tg} A \operatorname{tg} B \operatorname{tg} C$

$C = \pi - (A+B)$ $\qquad \operatorname{tg} C = -\operatorname{tg}(A+B) = -\dfrac{\operatorname{tg} A + \operatorname{tg} B}{1 - \operatorname{tg} A \operatorname{tg} B}$



$$\frac{(\operatorname{tg} A + \operatorname{tg} B)(1 - \operatorname{tg} A \operatorname{tg} B) - \operatorname{tg} A - \operatorname{tg} B}{\cancel{1 - \operatorname{tg} A \operatorname{tg} B}} = -\frac{\operatorname{tg} A \operatorname{tg} B(\operatorname{tg} A + \operatorname{tg} B)}{\cancel{1 - \operatorname{tg} A \operatorname{tg} B}}$$

$$\cancel{\operatorname{tg} A + \operatorname{tg} B} - \operatorname{tg} A \operatorname{tg} B(\operatorname{tg} A + \operatorname{tg} B)\cancel{- \operatorname{tg} A - \operatorname{tg} B}$$

Cor: $\operatorname{tg} A \operatorname{tg} B \operatorname{tg} C \ge 3\sqrt{3}$

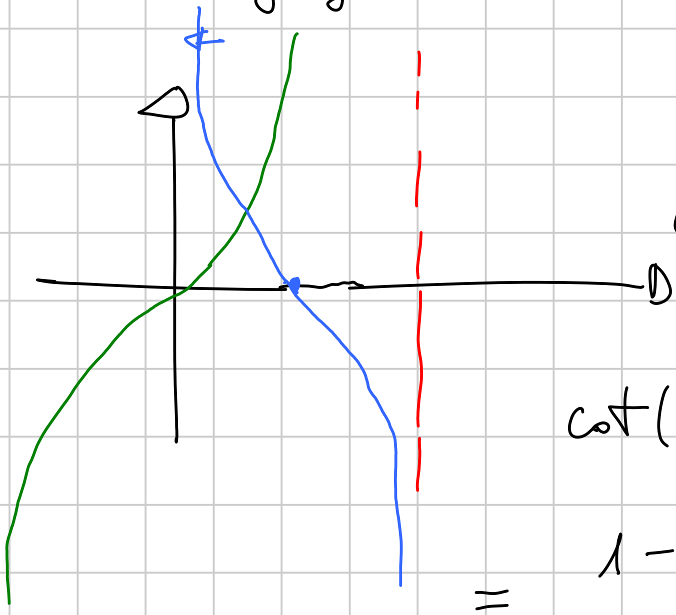
$\sum \operatorname{tg} A \ge 3\sqrt[3]{\prod \operatorname{tg} A} \qquad \prod \operatorname{tg} A \ge 3\sqrt{3}$

$\quad\|$

$\prod \operatorname{tg} A$

• $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$

e se $xy + yz + zx = 1 \Rightarrow \exists\, A, B, C$ angoli di un triangolo tali che $x = \cot A$ etc...



cot biiettiva da $(0, \pi)$ a $\mathbb{R}$

$$\cot(\alpha + \beta) = \frac{1}{\operatorname{tg}(\alpha+\beta)} = $$

$$= \frac{1 - \operatorname{tg}\alpha \operatorname{tg}\beta}{\operatorname{tg}\alpha + \operatorname{tg}\beta} = \frac{\cot\alpha \cot\beta - 1}{\cot\alpha + \cot\beta}$$

$$z = \frac{1 - xy}{x + y} = -\cot(\alpha + \beta)$$

•) $\sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} + \sin^2 \frac{C}{2} + 2 \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2} = 1$

$x^2 + y^2 + z^2 + 2xyz = 1$, $\quad x, y, z > 0$

$\frac{C}{2} = \frac{\pi}{2} - \left(\frac{A}{2} + \frac{B}{2}\right) \qquad \sin^2\frac{C}{2} = \cos^2\left(\frac{A}{2} + \frac{B}{2}\right) = 1 - \sin^2\left(\frac{A}{2} + \frac{B}{2}\right) =$

$$= 1 - \sin^2\frac{A}{2}\cos^2\frac{B}{2} - \sin^2\frac{B}{2}\cos^2\frac{A}{2}$$
$$- 2\sin\frac{A}{2}\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{B}{2}$$

$$2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = 2\sin\frac{A}{2}\sin\frac{B}{2}\cos\left(\frac{A}{2} + \frac{B}{2}\right) =$$

$$= 2\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{A}{2}\cos\frac{B}{2} - 2\sin^2\frac{A}{2}\sin^2\frac{B}{2}$$

$$-\sin^2\frac{A}{2}\cos^2\frac{B}{2}-\sin^2\frac{B}{2}\cos^2\frac{A}{2}\cancel{-2\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{A}{2}\cos\frac{B}{2}}+$$

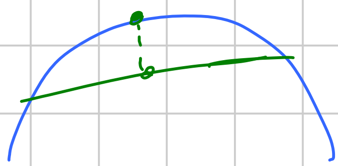
$$\cancel{+2\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{A}{2}\cos\frac{B}{2}}-2\sin^2\frac{A}{2}\sin^2\frac{B}{2}=$$

$$-\sin^2\frac{A}{2}-\sin^2\frac{B}{2}$$

$$-\sin^2\frac{A}{2}\left(\cos^2\frac{B}{2}+\sin^2\frac{B}{2}\right)-\sin^2\frac{B}{2}\left(\sin^2\frac{A}{2}+\cos^2\frac{A}{2}\right)=$$

$$=-\sin^2\frac{A}{2}-\sin^2\frac{B}{2}.$$

$$x=-yz+\sqrt{(1-y^2)(1-z^2)}$$



Es: •) $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\leq\frac{1}{8}$

•) $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}\geq\frac{3}{4}$

•) $\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}\leq\frac{9}{4}$

•) $\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\leq\frac{3\sqrt{3}}{8}$

•) $\csc\frac{A}{2}+\csc\frac{B}{2}+\csc\frac{C}{2}\geq 6$

Es: •) $\sum'\sin 2A=4\prod\sin A$

•) $\sum'\cos 2A=-1-4\prod\cos A$

•) $\sum'\sin^2 A=2+2\prod\cos A$

•) $\sum'\cos^2 A+2\prod\cos A=1$

$$\begin{cases} -x + y\cos B + z\cos C = 0 \\ x\cos B - y + z\cos A = 0 \\ x\cos C + y\cos A - z = 0 \end{cases} \qquad (\sin A, \sin C, \sin B)$$

$$\sum \cos^2 A + 2\prod \cos A - 1 = 0.$$

—— ✱ ——

•) $4R = \dfrac{abc}{[ABC]} = \dfrac{abc}{S}$

$$R = \frac{a}{2\sin A} = \frac{abc}{2\sin A\, bc} = \frac{abc}{4S}$$

$$S = \frac{1}{2} bc \sin A$$

•) $2R^2 \sin A \sin B \sin C = S \iff \sin A \sin B \sin C = \dfrac{S}{2R^2}$

$a = 2R\sin A$
$b = 2R\sin B$
$c = 2R\sin C$

$S = \dfrac{abc}{4R}$ $\Downarrow$

$4R \sin A \sin B \sin C = \dfrac{abc}{2R^2}$

•) $2R \sin A \sin B \sin C = r(\sin A + \sin B + \sin C)$

•) $a\cos A + b\cos B + c\cos C = \dfrac{abc}{2R^2}$

$$a\cos A = 2R\sin A\cos A = R\sin 2A$$

$$\cancel{R}(\sin 2A + \sin 2B + \sin 2C) = 4\cancel{R}\sin A \sin B \sin C \quad \text{ok.}$$

•) $r = 4R \sin\dfrac{A}{2} \sin\dfrac{B}{2} \sin\dfrac{C}{2}$

BTW: $r = 4R \ldots \le \frac{1}{8} 4R = \frac{R}{2}$

$r \le \dfrac{R}{2}$ Dis. di Eulero.

$IO^2 = R^2 - 2Rr = R(R-2r) > 0$

$$\sin^2\frac{A}{2} \underset{\text{bisezione}}{=} \frac{1-\cos A}{2} \underset{\text{Carnot}}{=} \frac{1}{2}\left(1-\frac{b^2+c^2-a^2}{2bc}\right) =$$

$$= \frac{1}{2}\left(\frac{a^2-(b-c)^2}{2bc}\right) = \frac{(a-b+c)(a+b-c)}{\cancel{4}bc} =$$

$$= \frac{(s-b)(s-c)}{bc} \qquad s = \frac{a+b+c}{2}$$

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{(s-a)(s-b)(s-c)}{abc} =$$

$$= \frac{s(s-a)(s-b)(s-c)}{sabc} = \frac{[ABC]^2}{sabc} = \frac{[ABC]}{s}\cdot\frac{[ABC]}{abc} = r\,\frac{1}{4R}$$

•) $4R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = s$

$$\parallel$$

$$\frac{1}{2}\,\frac{R\sin A\sin B\sin C}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} = \frac{1}{\cancel{2}}\,\frac{\frac{[ABC]}{\cancel{2}\cancel{R}}}{\frac{\cancel{R}}{4R}} = \frac{[ABC]}{r} = s$$

$$\Rightarrow \quad s \le \frac{3\sqrt{3}}{2}R$$

•) $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = 1 + \frac{r}{R}$

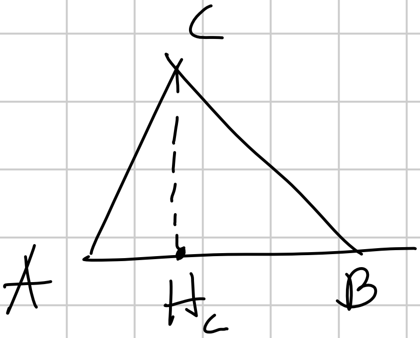
$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) = 2\sin\frac{C}{2}\cos\left(\frac{A-B}{2}\right)$$

$$1-\cos C = 2\sin^2\frac{C}{2} = 2\sin\frac{C}{2}\cos\left(\frac{A+B}{2}\right)$$

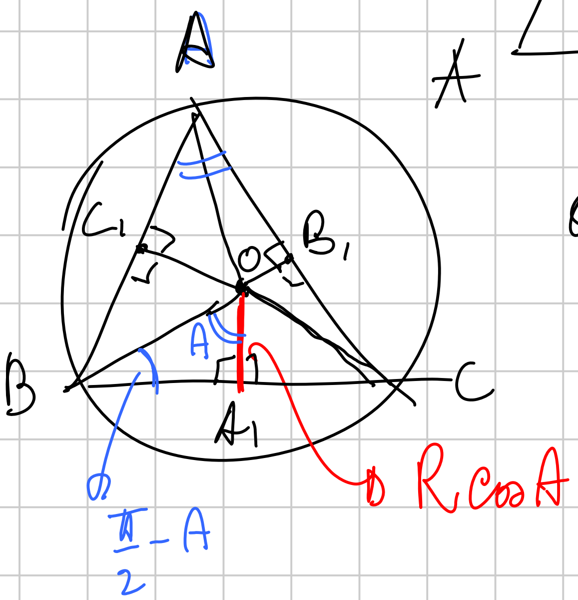
$$\cancel{2\sin\frac{C}{2}}\cos\left(\frac{A-B}{2}\right) = \cancel{2\sin\frac{C}{2}}\cos\left(\frac{A+B}{2}\right) + \cancel{4}^{\,2}\sin\frac{A}{2}\sin\frac{B}{2}\cancel{\sin\frac{C}{2}}$$

$$\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2}\right) = 2\sin\frac{A}{2}\sin\frac{B}{2}$$

Oss: $\cos A$



$AH_c = b\cos A$



$OA_1 + OB_1 + OC_1 = R + r$

T. di Tolomeo.

$OA \cdot C_1B_1 = AC_1 \cdot OB_1 + OC_1 \cdot AB_1$

$R\,\frac{a}{2} = \frac{c}{2}\,OB_1 + \frac{b}{2}\,OC_1$

$$R\left(\frac{a+b+c}{2}\right) = OA_1 \cdot \left(\frac{c+b}{2}\right) + OB_1\left(\frac{c+a}{2}\right) + OC_1\left(\frac{a+b}{2}\right) =$$

$$= OA_1\left(s - \frac{a}{2}\right) + OB_1\left(s - \frac{b}{2}\right) + OC_1\left(s - \frac{c}{2}\right) =$$

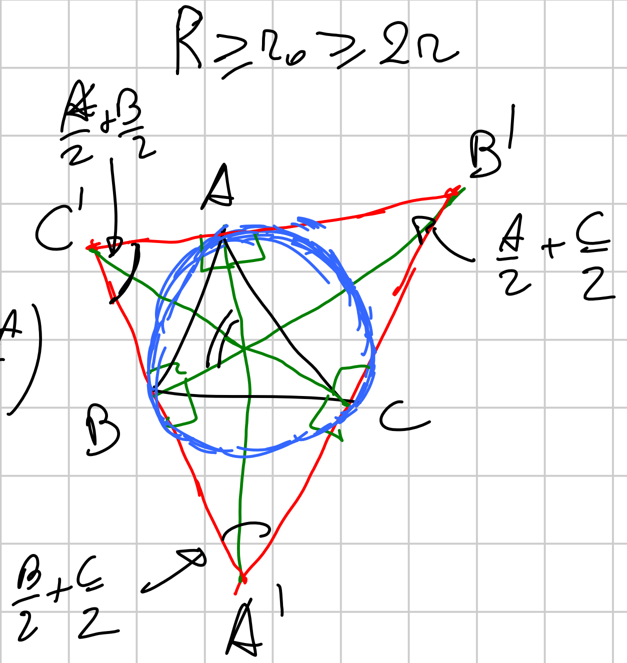
$$= \cancel{s}\left(OA_1 + OB_1 + OC_1\right) - \frac{[ABC]}{s}$$

$OA_1 + OB_1 + OC_1 = R + r$

$\underline{Es}$: ABC triangolo r, R i soliti, r_0 = raggio del cerchio inscritto nel triangolo degli excentri.

$$\Rightarrow \quad r_0 \geq 2r \qquad\qquad R \geq r_0 \geq 2r$$

$$\frac{r}{4R} = \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$



$$\frac{r_0}{\underset{\underset{8R}{\parallel}}{4R_0}} = \sin\left(\frac{A+B}{4}\right)\sin\left(\frac{B+C}{4}\right)\sin\left(\frac{C+A}{4}\right)$$

$$\frac{r_0}{4R} = 2\,\frac{r_0}{4R_0}$$

$$\prod \sin\left(\frac{A+B}{4}\right) \geq \prod \sin\frac{A}{2} \qquad\qquad \sin^2\left(\frac{A+B}{4}\right) \geq \sin\frac{A}{2}\sin\frac{B}{2}$$

$$1 - 2\sin^2\frac{A+B}{4} = \cos\left(\frac{A+B}{2}\right)$$

Jensen su $\log \operatorname{sen}(x)$

$$f' = \frac{\cos x}{\sin x}$$

$$f'' = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x}$$

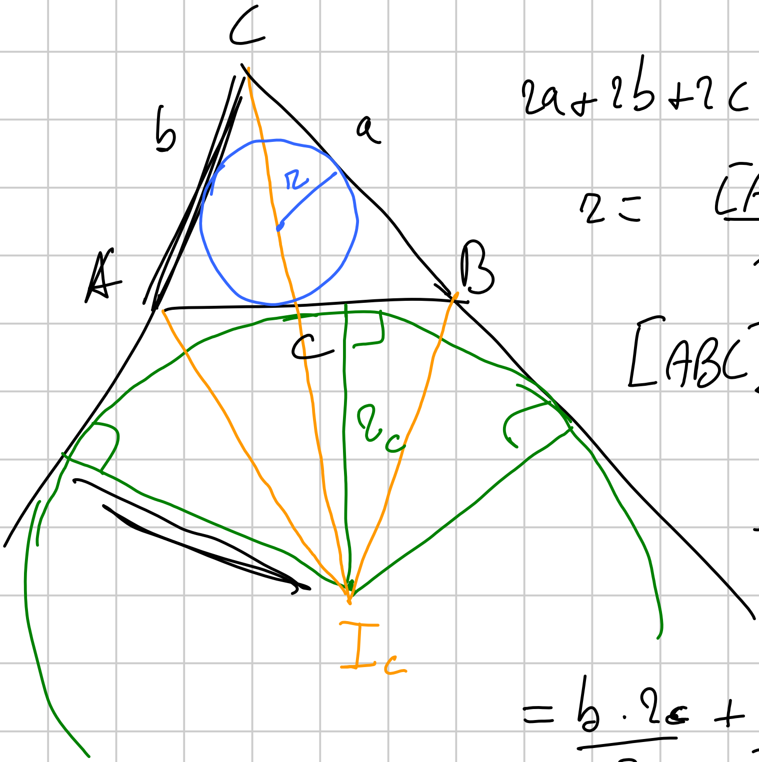
$$\operatorname{sen}^2\frac{A+B}{4} = \frac{1-\cos\left(\frac{A+B}{2}\right)}{2} = \frac{1-\sin\frac{C}{2}}{2}$$

$$= \frac{1}{2}\left(1 - \cos\frac{A}{2}\cos\frac{B}{2} + \sin\frac{A}{2}\sin\frac{B}{2}\right)$$

$$1 - \cos\frac{A}{2}\cos\frac{B}{2} - \sin\frac{A}{2}\sin\frac{B}{2} \geq 0$$

$$1 \geq \cos\left(\frac{A-B}{2}\right) \quad \text{è vera.}$$

•) $\displaystyle\sum \frac{a^2}{r_a} = 4(R+r)$ $\qquad r_a$ = raggio c. exinscritto



$$ra + rb + rc = 2[ABC]$$

$$r = \frac{[ABC]}{s}$$

$$[ABC] = [ACI_c] + [I_cCB] - [BAI_c] = \frac{b \cdot r_c}{2} + \frac{a r_c}{2} - \frac{c r_c}{2}$$

$$r_c = \frac{[ABC]}{\left(\frac{b+a-c}{2}\right)} \qquad r_a = \frac{[ABC]}{\frac{b+c-a}{2}}$$

$$r_b = \frac{[ABC]}{\frac{a+c-b}{2}}$$

a) $$r r_a r_b r_c = \frac{S^4}{s(s-a)(s-b)(s-c)} = \frac{S^4}{S^2} = S^2$$

b) $$\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{[ABC]} \cdot \left(\frac{[ABC]}{r_a} + \frac{[ABC]}{r_b} + \frac{[ABC]}{r_c} \right) =$$

$$= \frac{1}{[ABC]} \left(s-a + s-b + s-c \right) = \frac{s}{[ABC]} = \frac{1}{r}$$

$$\sum \frac{a^2}{r_a} = 4(R+r)$$

$$\sum \frac{a^2}{[ABC]}(s-a) = \frac{abc}{[ABC]} + \frac{4[ABC]}{s}$$

$$s(a^2+b^2+c^2) - \sum a^3 = abc + 4(s-a)(s-b)(s-c)$$

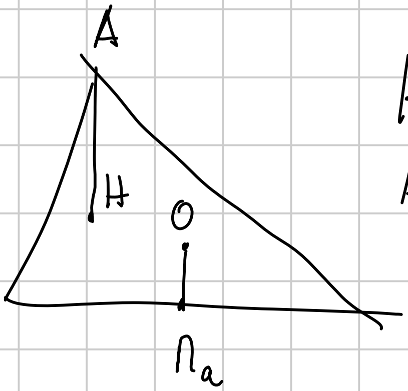
$$\frac{(a+b+c)}{2}(a^2+b^2+c^2) - \cancel{\sum a^3} = \cancel{a}bc + \frac{4}{8}(c+b-a)(a+c-b)(a+b-c)$$

$$[c^2-(b-a)^2](a+b-c)$$

$$\frac{\cancel{a^3+b^3+c^3}}{2} + \frac{ab^2}{2}+\frac{ac^2}{2}+\frac{ba^2}{2}+\frac{bc^2}{2}+\frac{ca^2}{2}+\frac{cb^2}{2}$$

$$\frac{1}{2}\left(c^2(a+b) - \cancel{c^3} - \cancel{b^2a} - \cancel{b^3} - \cancel{a^3} - \cancel{a^2b} + b^2c + a^2c + \cancel{2}a^2b + \cancel{2}ab^2 - \cancel{2abc}\right)$$

$$II_a + II_b + II_c = 4R + 2r_0$$



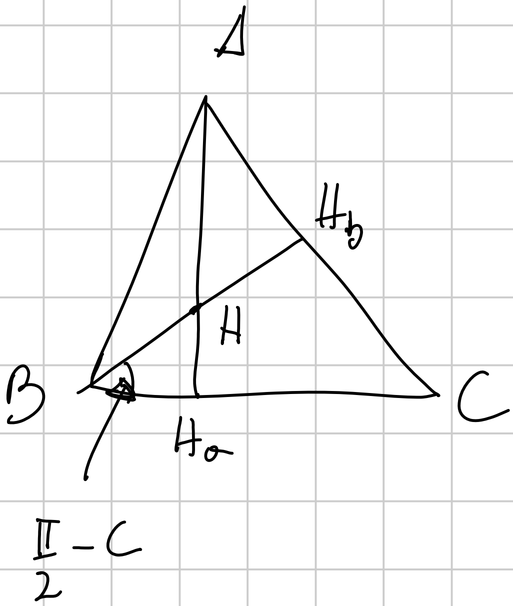
$H \longrightarrow O$

$A \longrightarrow M_a$

omot. di centro G e fatt. $-\frac{1}{2}$

$$HA = 2OM_a$$

$$\sum HA = 2\sum OM_a = 2(R+r)$$



$$AH_a = c\cdot\sin B$$

$$BH_a = c\cos B$$

$$HH_a = BH_a\cdot \operatorname{tg}\left(\frac{\pi}{2}-C\right) = c\cdot\cos B\cdot\frac{\cos C}{\sin C}$$

$$AH = c\left(\sin B - \frac{\cos B\cos C}{\sin C}\right) =$$

$$= 2R(\sin B\sin C - \cos B\cos C) = -2R\cos(B+C) =$$

$$= 2R\cos A.$$

$$II_a + II_b + II_c = 4R + 2r_0 \qquad r_0 \geq 2r$$

$$II_a + II_b + II_c \geq 4(R+r)$$

$$\sum \sin^2 A \qquad 4R^2 \sum \sin^2 A = \sum a^2$$

$$|\vec{A}+\vec{B}+\vec{C}|^2 = \underbrace{|\vec{A}|^2 + |\vec{B}|^2 + |\vec{C}|^2}_{3R^2} + 2\langle \vec{A},\vec{B}\rangle + \dots = (*)$$

$\vec{A}+\vec{B}+\vec{C} = \vec{H}$

Origine = circocentro

$$\langle \vec{A}-\vec{B}, \vec{A}-\vec{B}\rangle = c^2$$

$$\|A\|^2 + \|B\|^2 - 2\langle \vec{A},\vec{B}\rangle$$

$$2R^2$$

$$2\langle A,B\rangle = 2R^2 - c^2$$

$$(*) = 9R^2 - a^2 - b^2 - c^2 = 9R^2 - \sum a^2 = OH^2$$

$$9R^2 - \sum a^2 \geq 0 \qquad \sum a^2 \leq 9R^2$$

$$\sum \sin^2 A \leq \frac{9}{4}$$

•) $IH^2 = 4R^2 + 4Rr + 3r^2 - s^2 \qquad \dfrac{a\vec{A}+b\vec{B}+c\vec{C}}{a+b+c} = \vec{I}$

$$IH^2 \geq 0$$

$$s^2 \leq 4R^2 + 4Rr + 3r^2$$

$$s^2 + r^2 \leq 4(R^2 + Rr + r^2)$$

$$ab+bc+ca = s^2 + r^2 + 4Rr$$

$$\| $$

$$\frac{1}{4}(a+b+c)^2 + \frac{[ABC]^2}{s^2} + \frac{abc}{\cancel{[ABC]}} \cdot \frac{\cancel{[ABC]}}{s}$$

$$s^2(ab+bc+ca) = s^4 + [ABC]^2 + abcs$$

$$[ABC]^2 = s(s-a)(s-b)(s-c) = s\big(\overbrace{s^3 - (a+b+c)s^2}^{-s^3} + (ab+bc+ca)s - abc\big) =$$

$$= -s^4 + s^2(ab+bc+ca) - sabc$$

Teo di Tolomeo: $AC \cdot BD = AB \cdot BC + AD \cdot BC$

$\Longleftrightarrow A, B, C, D$ conciclici.

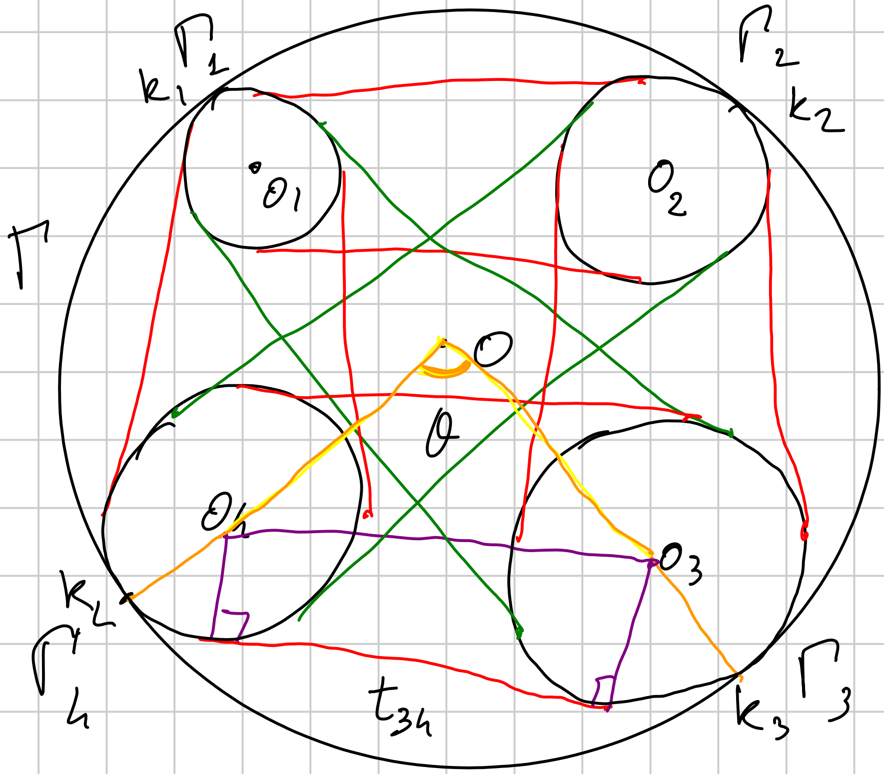
$\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ $\qquad t_{ij}$ = lungh. della tg est. comune di Γ_i e Γ_j

Teo di Casey (o Tol. generalizzato)

$$t_{13}t_{24} \pm t_{12}t_{34} \mp t_{14}t_{23} = 0$$

$$\Updownarrow$$

$\exists\, \Gamma$ che tange $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$



$$k_3k_4^2 = OK_3^2 + OK_4^2 - 2OK_3OK_4\cos\theta = 2R^2(1-\cos\theta)$$

$$\cos\theta = 1 - \frac{k_3k_4^2}{2R^2}$$

$$t_{34}^2 = O_4O_3^2 - (R_3-R_4)^2 =$$

$$= OO_4^2 + OO_3^2 - 2\,OO_3 \cdot OO_4 \cos(O_3OO_4) - (R_3-R_4)^2$$

$$\cos(O_3OO_4) = \cos(K_3OK_4) = \cos(2\,K_3CK_4) \qquad C \in \Gamma$$

$$= \underbrace{(R-R_4)^2}_{} + \underbrace{(R-R_3)^2}_{} - \underbrace{2(R-R_3)(R-R_4)\Big(1}_{} - \frac{K_3K_4^2}{2R^2}\Big) -$$

$$-(R_3-R_4)^2 =$$

$$= \underbrace{\Big[(R-R_4) - (R-R_3)\Big]^2}_{(R_3-R_4)^2}$$

$$(R_3-R_4)^2 + \frac{K_3K_4^2}{R^2}(\cancel{R} - R_3)(R-R_4) - (R_3-R_4)^2 =$$

$$= K_3K_4^2\,\frac{(R-R_3)(R-R_4)}{R^2}$$

$$t_{34} = \frac{K_3 K_4}{R} \sqrt{(R-R_3)(R-R_4)}$$

$$t_{12} t_{34} = \frac{K_1 K_2 \cdot K_3 K_4}{R^2} \sqrt{(R-R_1)(R-R_2)(R-R_3)(R-R_4)}$$

$= D$

$$t_{13} t_{24} = K_1 K_3 \cdot K_2 K_4 \cdot D$$

$$t_{14} t_{23} = K_1 K_4 \cdot K_2 K_3 \cdot D$$

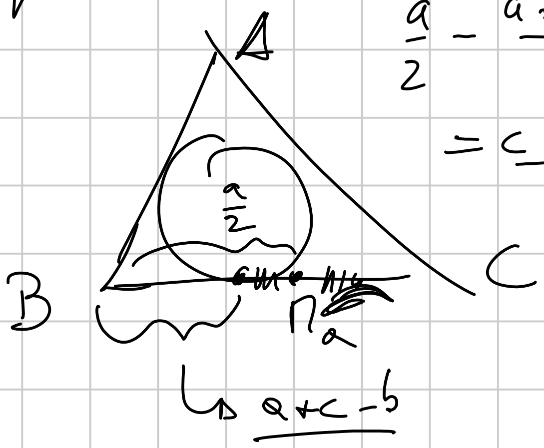
$$t_{12} t_{34} \pm t_{13} t_{24} \mp t_{14} t_{23} = D\left(K_1 K_2 \cdot K_3 K_4 \pm K_1 K_3 \cdot K_2 K_4 \mp K_1 K_4 \cdot K_2 K_3 \right)$$

__Cor__: Feuerbach tange le ex- inscritte

$\Gamma_a, \Gamma_b, \Gamma_c, \omega$

$\frac{a}{2}$ $\frac{b}{2}$ $\frac{c}{2}$ $\frac{c-b}{2}$ $\frac{b-a}{2}$ $\frac{a-c}{2}$

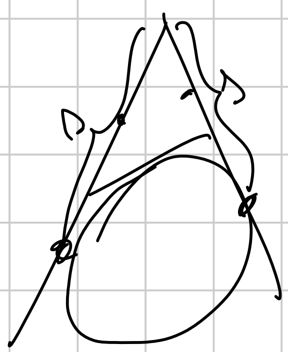
$\downarrow$ $\Gamma_b \Gamma_c$ $\downarrow$ $\Gamma_a \omega$



$$\frac{a}{2} - \frac{a+c-b}{2} = \frac{c-b}{2}$$

$$\frac{a}{2}\left(\frac{c-b}{2}\right) \pm \frac{b}{2}\left(\frac{a-c}{2}\right) \pm \frac{c}{2}\left(\frac{a-b}{2}\right)$$

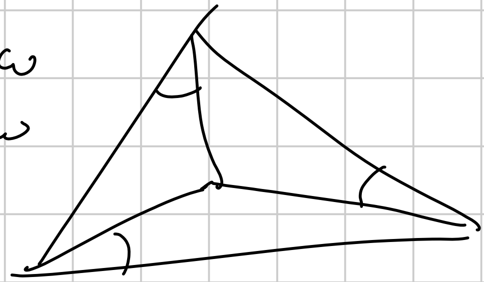
$$a(c-b) + b(a-c) - c(a-b) = 0.$$

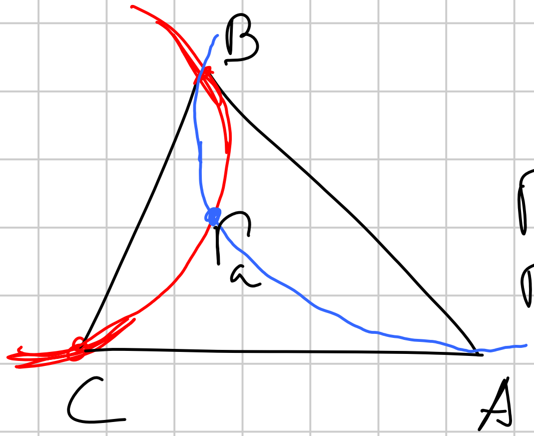


Punto di Brocard

1. $\exists$ due punti Ω e Ω' tali che

$$\Omega\hat{A}B = \Omega\hat{B}C = \Omega\hat{C}A = \omega$$
$$\Omega'\hat{A}B = \Omega'\hat{B}C = \Omega'\hat{C}A = \omega$$





Γ_a per B, C tang. a AC.
Γ_b per AB tg a CB
Γ_c per A, C tg a AB.

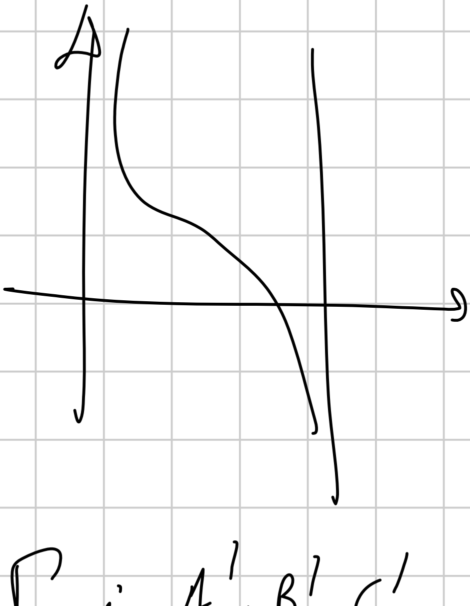
2) il tri pedale di Ω è simile ad ABC

3) Ω e Ω' sono coniug. isogonali

4) $\cot\omega = \cot A + \cot B + \cot C$

$$\left[\cot A + \cot B + \cot C \geq \underset{\cot 30^\circ}{\underset{\shortparallel}{\sqrt{3}}}\right]$$

$$\omega \leq 30^\circ$$



5) $A\Omega, B\Omega, C\Omega$ incontrano Γ in A', B', C'

$\Rightarrow A'B'C' \equiv ABC \qquad AOA' = 2\omega$

6) $O\Omega = O\Omega'$ e $\Omega\hat{O}\Omega' = 2\omega$

7) la cf. $\overset{= \Gamma_\omega}{\text{per}}$ O, Ω, Ω' passa per K = coniug. isog. di G
(pt di Lemoine)

e OK è diametro,

8) Γ_ω è l'inverso in Γ dell'asse $\overset{= \text{asse di Lemoine}}{}$ per i centri delle cf. di Apollonio (= diam LL', L = piede della bisett. int.
L' = " " " est.)

9) O, K, J_1, J_2 sono all. e $\perp$ all'asse di Lemoine.

J_1: pti comuni delle 3 cf. di Apoll

10) $$\cot \Gamma_\omega = \frac{a^2+b^2+c^2}{4[ABC]}$$